

## Discrete Optimization

### Comprehensive Examination

9 a.m. to 12 noon, Wednesday, July 3, 2002

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**Note.** The total number of marks is 100. The presentation of your solutions is important in assigning marks. Do not use results without proof unless the question allows it.

1. (a) Let  $S \subseteq \mathcal{R}^n$  be a finite set and let  $w \in \mathbf{R}^n$ . Show that  $\max\{w^T x : x \in S\} = \max\{w^T x : x \in \text{conv.hull}(S)\}$  where  $\text{conv.hull}(S)$  denotes the set of all convex combinations of  $S$ . [7]
- (b) Let  $S \subseteq \mathbf{R}^n$  be a finite set, and let  $v \in \mathbf{R}^n \setminus \text{conv.hull}(S)$ . Show that there exists an inequality  $w^T x \leq t$  that separates  $v$  from  $\text{conv.hull}(S)$ , that is,  $w^T s \leq t$  for all  $s \in S$  but  $w^T v > t$ . You may use Farkas' Lemma. [11]
2. (a) State the Tutte-Berge min-max formula. Define any notation that you use. Prove that the max is at most the min. [5]
- (b) Let  $G = (V, E)$  be a graph and let  $c \in \mathbf{R}^E$ . Suppose that  $M$  is a matching of size  $k$  and that  $M$  has maximum  $c$ -weight among all matchings of size  $k$ . Let  $P$  be the edge-set of an  $M$ -augmenting path such that  $c(P \setminus M) - c(P \cap M)$  is maximum. Show that  $(M \setminus P) \cup (P \setminus M)$  has maximum  $c$ -weight among all matchings of size  $k + 1$ . [12]
- (c) Let  $G = (V, E)$  be a bipartite graph and let  $k$  be a positive integer. Let  $P$  denote the set of all  $x \in \mathbf{R}^E$  satisfying [12]

$$\begin{aligned}x(\delta(v)) &\leq 1, \text{ for all } v \in V \\x(E) &= k \\x_e &\geq 0, \text{ for all } e \in E.\end{aligned}$$

Prove that  $P$  is the convex hull of incidence vectors of matchings of size  $k$ .  
**Hint:** Construct a bipartite graph  $G'$  in which perfect matchings correspond to matchings of size  $k$  in  $G$ , and use a result on perfect matchings.

3. (a) State the matroid intersection theorem, and prove that the maximum is at most the minimum. [5]
- (b) Let  $M_1 = (S, \mathcal{I}_1)$ ,  $M_2 = (S, \mathcal{I}_2)$  be matroids, and let  $J$  be a set that is independent in both. Define the auxiliary digraph  $G = G(M_1, M_2, J)$ , that is used in the matroid intersection algorithm. Prove that if  $A \subseteq S$  such that  $\delta(A \cup \{r\}) = \emptyset$ , then  $A$  is a minimizer in the matroid intersection theorem. [8]
- (c) Let  $G = (V, E)$  be a connected graph, and suppose that each edge of  $G$  is assigned a colour. Prove that  $G$  has a spanning tree in which all edges have different colours if and only if, for every subset  $A$  of  $E$ , [10]

$$\kappa(G - A) \leq \chi(A) + 1.$$

Here  $\kappa(H)$  denotes the number of components  $H$ , and  $\chi(B)$  denotes the number of different colours occurring on edges in  $B$ . You may use results on matroids.

4. (a) Given a digraph  $G = (V, E)$ ,  $l \in (\mathbb{Z} \cup \{-\infty\})^E$ , and  $u \in (\mathbb{Z} \cup \{\infty\})^E$ , with  $l \leq u$ , state a necessary and sufficient condition for the existence of an integral circulation  $x$  with  $l \leq x \leq u$ . (This is Hoffman's circulation theorem.) [4]
- (b) We are given a digraph  $G = (V, E)$  and  $l \in (\mathbb{Z} \cup \{-\infty\})^E$  such that every arc  $e$  with  $l_e > 0$  is in some  $(r, s)$ -dipath, and there is no  $(s, r)$ -dipath. Using the previous result, show that there exists an integral  $(r, s)$ -flow  $x$  such that  $x \geq l$  and  $f_x(s) \leq k$  if and only if for every  $R \subseteq V$  with  $r \in R, s \notin R$ ,  $\delta(\bar{R}) = \emptyset$ , we have  $l(\delta(R)) \leq k$ . [10]
- (c) Consider an acyclic digraph  $G$  where certain arcs are colored blue, while the others are colored red. Consider the problem of covering the blue arcs by directed paths, which can start and end at any node (these paths can contain arcs of any color). Show that the minimum number of directed paths needed to cover the blue arcs is equal to the maximum number of blue arcs which satisfy the property that no two of these arcs belong to the same directed path. You may use the results in previous parts of this question. [8]
- (d) You are now given a general digraph  $G$  (not necessarily acyclic) where certain arcs are colored blue, while the others are colored red. Consider the problem of finding a minimum number of simple directed paths which cover all the blue arcs. Is this problem polynomial-time solvable, or NP-hard? In either case, give an appropriate reduction. (A list of NP-hard problems is given at the end of this paper.) [8]

## NP-complete problems

### Exact Cover.

*INSTANCE:* Collection  $\mathcal{F}$  of subsets of a finite set  $X$ .

*QUESTION:* Is there a subcollection of  $\mathcal{F}$  that forms a partition of  $X$ ?

### Edge Colouring

*INSTANCE:* Graph  $G = (V, E)$  and integer  $k \leq |V|$ .

*QUESTION:* Can each edge be assigned one of  $k$  different colours so that edges incident to the same node get distinct colours?

### Directed Hamiltonian Path.

*INSTANCE:* Directed graph  $G = (V, E)$ .

*QUESTION:* Is there a simple directed path of  $G$  which contains every node in  $V$ ?

### Feedback Arc Set.

*INSTANCE:* Directed graph  $G = (V, E)$ , positive integer  $k \leq |E|$ .

*QUESTION:* Is there a subset  $E' \subseteq E$  with  $|E'| \leq k$  such that  $E'$  contains at least one arc from every directed circuit in  $G$ ?

### $k$ -Closure.

*INSTANCE:* Directed graph  $G = (V, E)$  and integer  $k \leq |V|$ .

*QUESTION:* Does there exist a nonempty subset  $X$  of  $V$ , with  $|X| \leq k$ , such that there are no arcs  $uv$  of  $G$  with  $u \in X$  and  $v \notin X$ ?

