Comprehensive Exam – Discrete Optimization – Spring 2004

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Instructions: Do not use results without proof unless the question allows it (if you are asked to state a result in part of a question you can use it in the other parts). The total number of points is 100. Feel free to ask questions if any notation is unclear.

1. [25 marks]

(i) Consider a directed graph $G = (V, E)$. We say that a family of circuits $\mathcal{S}$ is a *good family* if every node r is in at least two circuits of $\mathcal{S}$ and no arc is in more than one circuit of $\mathcal{S}$. Note by circuit we mean a connected edge induced subgraph where every vertex has indegree and outdegree one (so circuits are directed). Formulate the problem of whether G admits a good family as the problem of checking whether a circulation exists (recall $x \in \Re^E$ is a circulation if $f_x(v) = 0$ for all $v \in V$ and $l \leq x \leq u$ where $l \in \mathcal{R} \cup \{-\infty\}$,$u \in \mathcal{R} \cup \{+\infty\}$).

(ii) For subsets S, S' of V, we define $d(S, S') := |\{uv \in E : u \in S, v \in S'\}|$. Show that there exists a good family if and only if for all partitions S_1, S_2, S_3 of V (where the sets need not be all non-empty) we have

$$|S_1| \leq \frac{1}{2}\left[(d(S_1, S_1) + d(S_3, S_1) + d(S_1, S_2) + d(S_3, S_2)\right].$$

For this question you may use any result from the book.

(iii) You are given a directed graph G. Consider the following two problems:

- Does G have a good family ?
- Does G have a good family with at most k circuits (k is part of the input).

For these two problems either find a good algorithm (polynomial) or indicate why it is likely that there is none. A list of NP-complete problems is given at the end of the section.

2. [25 marks] Let $G = (V, E)$ be a graph. Let $\nu(G)$ denote the size of a maximum matching in G, let $\text{def}(G) = |V| - 2\nu(G)$, and let $\text{odd}(G)$ denote the number of odd components of G. A vertex $v \in V$ is *avoidable* if $\nu(G-v) = \nu(G)$.

(i) Let $u, v, w \in V$ be avoidable vertices in G. Prove that, if $\nu(G - u - v) < \nu(G)$ and $\nu(G - v - w) < \nu(G)$, then $\nu(G - u - w) < \nu(G)$.

(ii) Prove Gallai's Lemma that: *If G is a connected graph and each vertex of G is avoidable, then $def(G) = 1$.*

(iii) Prove the Tutte-Berge Formula: *For any graph $G = (V, E)$,*

$$def(G) = \max_{A \subseteq V} odd(G - A) - |A|.$$

3. [25 marks]

(i) State the Matroid Intersection Theorem.

(ii) Let $M = (S, \mathcal{I})$ be a matroid. Using the Matroid Intersection Theorem, prove that S can be partitioned into k independent sets if and only if $|A| \leq k r_M(A)$ for each $A \subseteq S$.

(iii) Let P be the set of all vectors $x \in \mathbb{R}^S$ satisfying:

$$\begin{aligned} x(A) &\leq r_M(A) \quad A \subseteq S \\ x &\geq 0. \end{aligned}$$

Using the result stated in Part (ii), prove that P is the convex hull of characteristic vectors of independent sets of M. (Hint: Scale a rational vector $x \in P$ to make it integral.)

4. [25 marks] For this question you may use any result from the book.

(i) Using linear programming duality prove the following version of Farkas' Lemma: "For a system $Ax \leq b$, either $Ax \leq b$ is feasible or there exists a vector $y \geq 0$ such that $yA = 0$ and $yb < 0$, but not both".

(ii) Given a vector $x \in \Re^n$ we denote $\{j \in \{1, \ldots, n\} : x_j \neq 0\}$ by $support(x)$. Show that x^* is an extreme point of $P = \{x : Ax = b, x \geq 0\}$ if and only if there exist no $x' \in P$ such that $support(x') \subset support(x^*)$.

(iii) A system of inequalities is *minimal infeasible* if it does not have a solution but all subsystems have a solution. Consider a system $Ax \leq b$ which does not have a solution. Show that there is a one-to-one correspondence between the minimal infeasible subsystems of $Ax \leq b$ and the support of the vertices of the polyhedron $\{y : yA = 0, yb = -1, y \geq 0\}$.

NP-complete problems

Exact Cover.

INSTANCE: Collection $\mathcal{F}$ of subsets of a finite set X.

QUESTION: Is there a subcollection of $\mathcal{F}$ that forms a partition of X ?

Edge Colouring

INSTANCE: Graph $G = (V, E)$ and $K \leq |V|$.

QUESTION: Can each edge be assigned one of K different colours so that edges incident to the same vertex get distinct colours ?

Knapsack.

INSTANCE: A finite set U, values $s_u, v_u \in Z_+$ for all $u \in U$ and $b, K \in Z_+$.

QUESTION: Is there a subset $U' \subseteq U$ such that

$$\sum_{u \in U'} s_u \leq B \qquad \sum_{u \in U'} v_u \geq K.$$

Minimum cover.

INSTANCE: Collection $\mathcal{C}$ of subsets of a set S, positive integer K.

QUESTION: Does $\mathcal{C}$ contain a *cover* for S of size K or less, that is, a subset $\mathcal{C}' \subseteq \mathcal{C}$ with $|\mathcal{C}'| \leq K$ and such that $\cup_{c \in \mathcal{C}'} c = S$?

Disjoint connecting paths.

INSTANCE: Graph G, collection of disjoint vertex pairs $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$.

QUESTION: Does G contains k mutually vertex disjoint paths, connecting s_i and t_i for each $i = 1, \ldots, k$?

Directed Hamiltonian circuit.

INSTANCE: Directed graph $G = (V, E)$.

QUESTION: Is there a directed circuit of G which contains every vertex in V ?

Feedback Arc Set.

INSTANCE: Directed graph $G = (V, E)$, positive integer $k \leq |E|$.

QUESTION: Is there a subset $E' \subseteq E$ with $|E'| \leq k$ such that E' contains at least one arc from every directed circuit in G ?