Comprehensive Exam – Discrete Optimization – June 2006

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Instructions: All questions have the same number of points. If you are pressed for time it is better to answer some questions completely than to give partial answers to all questions. Feel free to ask questions if any notation is unclear. Do not use results without proof unless the question allows it (if you are asked to state a result in part of a question you can use it in the other parts).

(Q1: Network flows)

In this question, you may use the max-flow min-cut theorem without proof.

(i) Let $G = (V, E)$ be a digraph, let $u \in \mathbf{R}_+^E$ assign a nonnegative capacity to each arc, and let $b \in \mathbf{R}^V$ assign a demand to each node. State and prove necessary and sufficient conditions for the existence of a flow $x \in \mathbf{R}^E$ such that

$$\begin{aligned} f_x(v) &= b_v, && \text{for all } v \in V \\ 0 \le x_e &\le u_e && \text{for all } e \in E. \end{aligned}$$

(Recall that for $x \in \mathbf{R}^E$, $f_x(v)$ means $\sum_{e\in\delta(V\setminus v)} x_e - \sum_{e\in\delta(v)} x_e$.)

(ii) Now, suppose that $\ell \in \mathbf{R}^E$ assigns a lower bound to each arc, and moreover, $\ell \le u$. State and prove necessary and sufficient conditions for the existence of a flow $x \in \mathbf{R}^E$ such that

$$\begin{aligned} f_x(v) &= b_v, && \text{for all } v \in V \\ \ell_e \le x_e &\le u_e && \text{for all } e \in E. \end{aligned}$$

(iii) Are the following two decision problems polynomial-time solvable or NP-complete? Give brief and convincing explanations; detailed proofs are not needed.

Let $G = (V, E)$ be a graph, let $\alpha \in \mathbf{Z}_+^E$ assign a nonnegative integer to each edge, and let $\beta \in \mathbf{Z}^V$ assign an integer to each node. Let $f : 2^V \to \mathbf{Z}$ be the function $f(S) = \alpha(\delta(S)) - \beta(S)$.

(a) Given an integer k, does there exist a set of nodes S with $f(S) \le k$?

(b) Given an integer k, does there exist a set of nodes S with $f(S) \ge k$?

(Q2: Matroids)

(i) Show that partition matroids are indeed matroids. Show that the direct sum of two matroids is also a matroid.

(ii) Given a graph $G = (V, E)$ and an integer k, the *forest partitioning* problem is the problem of finding a partition of E into k forests. Show that if the forest partitioning problem has a solution then

$$\text{for all } B \subseteq V: \quad k(|B| - 1) \geq \gamma(B) \tag{1}$$

where $\gamma(B) = |\{uv \in E : u, v \in B\}|$, i.e., $\gamma(B)$ is the number of edges with both end nodes in B.

(iii) Using part (i) and a theorem on matroids, show that if (1) holds then the forest partitioning problem has a solution.

(iv) Are the following two decision problems polynomial-time solvable or NP-complete? Give brief and convincing explanations; detailed proofs are not needed.

(a) Given $G = (V, E)$ and integer k, is there a partition of E into k forests,

(b) Given $G = (V, E)$ and integers k, ℓ, is there a partition of E into k forests $F_1, \ldots, F_k$ where each non-leaf vertex of each forest F_i has degree ℓ.

(Q3: Matching theory)

(i) Write down a linear programming formulation (P) for the minimum-weight perfect matching problem, and write down the dual (D) of (P). State the Perfect Matching Polytope Theorem.

(ii) Prove: if $G = (V, E)$ is a d-regular graph that is d-edge connected, and $|V|$ is even, then G has a perfect matching.
(Note: d is a positive integer, G is called d-regular if $|\delta(v)| = d, \forall v \in V$, and G is called d-edge connected if $|\delta(S)| \geq d$, $\forall S \subseteq V, S \neq \emptyset$ and $S \neq V$.)

Now, suppose that $G = (V, E)$ is d-regular, $(d-1)$-edge connected, and $|V|$ is even. Does G have a perfect matching?

(iii) Describe the Blossom algorithm for finding a perfect matching. (An informal but precise overview is acceptable; make sure to define your notation; you may use standard notions from matching theory.)

(iv) State the Tutte-Berge formula, and prove the formula using the termination conditions of the Blossom algorithm for maximum matching. Make sure to state the relevant conditions that hold at the termination of the algorithm.

(Q4: Polyhedral theory)

(i) Show that a rational polytope Q is integral if and only if for all integral vectors w the optimal value of $\max\{w^T x : x \in Q\}$ is an integer.

(ii) Consider a graph $G = (V, E)$ with distinct vertices s and t. Show that the length (given by the number of edges) of the shortest st-path is equal to the maximum number of pairwise disjoint st-cuts.

(iii) Using (i) and (ii) show that the polytope

$$Q = \{x \in \Re^E : \mathbf{0} \leq x \leq \mathbf{1}, x(\delta(S)) \geq 1, \ \forall S \subseteq V \text{ where } s \in S, t \notin S\}$$

is integral.

NP-Complete problems

Directed Hamiltonian circuit

INSTANCE: Directed graph $G = (V, E)$.

QUESTION: Is there a directed circuit of G containing every vertex in V?

Disjoint connecting paths

INSTANCE: Graph G, collection of disjoint vertex pairs $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$.

QUESTION: Does G contain k mutually vertex disjoint paths, connecting s_i and t_i for each $i = 1, \ldots, k$?

Feedback arc set

INSTANCE: Directed graph $G = (V, E)$ and $k \in Z_+$.

QUESTION: Is there a subset $E' \subseteq E$ with $|E'| \leq k$ such that E' contains at least one arc from every directed circuit in G?

Edge colouring

INSTANCE: Graph $G = (V, E)$ and $K \in Z_+$.

QUESTION: Can each edge be assigned one of K different colours so that edges incident to the same vertex get distinct colours? Remains hard for cubic graphs and $K = 3$.

Exact cover

INSTANCE: Collection $\mathcal{F}$ of subsets of a finite set X.

QUESTION: Is there a sub-collection of $\mathcal{F}$ that forms a partition of X.

Knapsack

INSTANCE: A finite set U, values $s_u, v_u \in Z_+$ for all $u \in U$ and $B, K \in Z_+$.

QUESTION: Is there a subset $U' \subseteq U$ such that

$$\sum_{u \in U'} s_u \leq B \qquad \sum_{u \in U'} v_u \geq K$$

Maximum clique

INSTANCE: Graph G and $K \in Z_+$.

QUESTION: Does G contain a clique (induced complete subgraph) of size K?

Maximum cut

INSTANCE: A graph $G = (V, E)$ and $K \in Z_+$.

QUESTION: Is there $S \subseteq V$ such that $|\delta(S)| \geq K$?

Minimum cover

INSTANCE: Collection $\mathcal{C}$ of subsets of a set $\mathcal{S}$, positive integer K.

QUESTION: Does $\mathcal{C}$ contain a *cover* for $\mathcal{S}$ of size K or less, that is, a subset $\mathcal{C}' \subseteq \mathcal{C}$ with $|\mathcal{C}'| \leq K$ and such that $\cup_{c \in \mathcal{C}'} C = S$?

k-closure

INSTANCE: Directed graph $G = (V, E)$ and integer $k \leq |V|$.

QUESTION: Does there exists a nonempty subset X of V, with $|X| \leq k$, such that there are no arcs uv of G with $u \in X$ and $v \notin X$?