# Discrete Optimization Comprehensive Exam — Spring 2008

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**Instructions:** Unless otherwise stated, do not use results without proof. If you are asked to state or prove a result in a part of a question, you may use it without proof in subsequent parts of the question. The reference CCPS refers to the Cook-Cunningham-Pulleyblank-Schrijver book.

## Problem 1: Matroid Theory (22 marks)

(a) Let $M$ be a matroid on $S$ let $J$ be an independent set of $M$, and let $e \in S$. Prove that $J \cup \{e\}$ contains at most one circuit (that is, minimal dependent subset) of $M$.

(b) In the *coloured spanning tree problem*, we are given an undirected graph $G$ where each edge $e$ has a unique colour in $\{1, \ldots, k\}$. Each colour $i$ has an integer $\pi_i \geq 0$ associated with it. Let $C_i$ denote the set of edges having colour $i$. A feasible solution to the problem is a spanning tree $T$ of $G$ such that $|T \cap C_i| \leq \pi_i$ for all $i = 1, \ldots, k$.

Derive a combinatorial necessary and sufficient condition for the coloured spanning tree problem to have a feasible solution. You may use results from CCPS.

## Problem 2: Matching (22 marks)

(a) Use the Tutte-Berge formula to prove that, if $G$ is a bipartite graph having no cover of size less than $k$, then $G$ has a matching of size $k$.

(b) Let $G = (V, E)$ be an undirected graph, let $p, q$ be integers with $0 \leq p \leq q$, and let $x \in \mathbb{R}^E$ satisfy

$$\begin{aligned} x(\delta(v)) &\leq 1 \text{ for all } v \in V \\ x(\gamma(S)) &\leq (|S| - 1)/2 \text{ for all } S \subseteq V \text{ with } |S| \text{ odd and at least } 3 \\ p \leq x(E) &\leq q \\ x_e &\geq 0, \text{ for all } e \in E. \end{aligned}$$

Recall that $\gamma(S)$ is the set of edges with both ends in $S$. Prove that $x$ is a convex combination of matchings of $G$, each having cardinality at least $p$ and at most $q$. You may use the matching polyhedron theorem, as well as elementary results about matching.

## Problem 3: Polyhedral Theory (22 marks)

For this question, you may use any result from the CCPS book (except the one you are asked to prove).

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a non-empty rational polyhedron, where $A$ is an $m \times n$ matrix, such that for every $c \in \mathbb{R}^n$, the linear program: $\max c^T x$ s.t. $x \in P$, is bounded. Let $P_I$ denote the convex hull of the integral vectors in $P$.

(a) Define the term *Chvátal-Gomory cut* for $P$. Let $P'$ be the set of all $x \in P$ that satisfy all the Chvátal-Gomory cuts for $P$. Prove that $P'$ is a polyhedron.

(b) Show that if $x^*$ is a vertex of $P$ and $x^* \notin P_I$, then there exists a Chvátal-Gomory cut for $P$ that cuts off $x^*$ (i.e., $x^*$ does not satisfy the Chvátal-Gomory inequality).

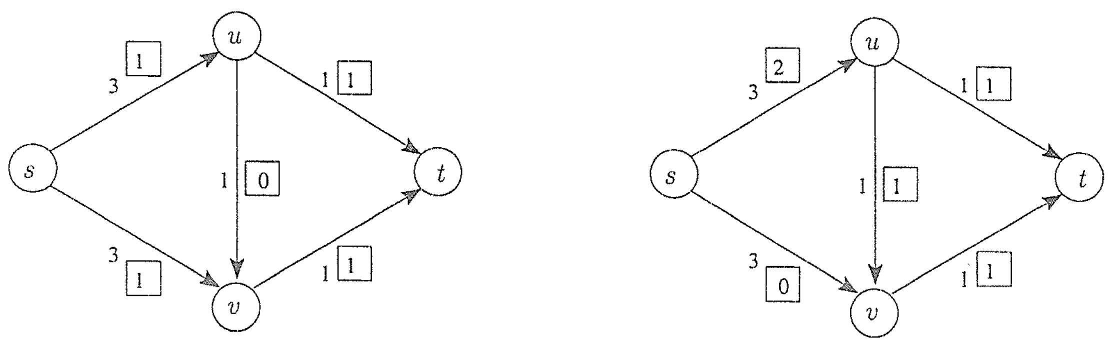
(c) Define $P^{(0)} = P$, and $P^{(i+1)} = \left(P^{(i)}\right)'$ for $i \geq 0$. Show that if $P^{(k+1)} = P^{(k)}$ for some integer $k$, then $P^{(k)} = P_I$.

**Problem 4: Network Flows** **(22 marks)**

Let $G = (V, E)$ be a directed graph with source $s \in V$, sink $t \in V$, and integer capacities $u_e \geq 0$ on the edges with $u_e = k$ for every edge $e$ out of $s$. There are no edges entering the source $s$. For an integer $i \geq 0$, let $u^{(i)}$ denote the capacity-vector where $u_e^{(i)} = i$ for every edge $e$ out of $s$, and $u_e^{(i)} = u_e$ otherwise. Call an $s$-$t$ flow $f$ a *prefix-maximal flow* if for every integer $i \in [0, k]$, there exists an $s$-$t$ flow $h \leq f$ that is (feasible and) a maximum $s$-$t$ flow for the instance with capacity-vector $u^{(i)}$.

Consider the flow-network in Figure 1 for example. Figures 1a) and 1b) show two maximum $s$-$t$ flows in this network. The capacity of each edge appears as a label next to the edge and the boxed numbers give the flow on each edge. The flow in Fig. 1a) is a prefix-maximal flow, as can be easily verified. The flow in Fig. 1b) is *not* prefix maximal: if one decreases the capacities of the edges leaving $s$ to 1, then the value of the maximum $s$-$t$ flow for this reduced-capacity instance is 2 (the flow in Fig. 1a) continues to be a max-flow), whereas every $s$-$t$ flow $h \leq f$ must have $h_{s,u} \leq 1$, $h_{s,v} = 0$ and thus have value at most 1.

You may use standard results about flows to answer the following questions.




(a) (b)

Figure 1: (a) a prefix-maximal flow, and (b) a flow that is not prefix maximal.

(a) Let $f$ be a maximum $s$-$t$ flow (with capacity-vector $u$). Argue that $f$ is a prefix-maximal flow iff for every integer $i \in [0, k]$, the value of the maximum $s$-$t$ flow for the instance with capacity-vector $u^{(i)}$ is equal to $\sum_{e \in E: e=(s,u)} \min(i, f_e)$.

(b) Prove that a prefix-maximal flow always exists.

**Problem 5: Complexity** **(12 marks)**

For each of the following problems, either indicate that the problem is *NP*-hard, or that it admits a polynomial-time algorithm. In the former case indicate a reduction from a well-known *NP*-hard problem; in the latter case, indicate briefly how the problem can be solved using results from CCPS.

(a) Given a graph with integer weights on the edges, determine whether there is a circuit of negative weight.

(b) Given a graph with integer weights on the edges, and distinct vertices $r, s$, determine whether there is a simple $(r, s)$-path of negative weight.

(c) Given a graph with integer weights on the edges, determine whether there is a spanning connected subgraph of negative weight.