Discrete Optimization Comprehensive Exam — Spring 2009

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Instructions: Unless otherwise stated, do not use results without proofs. If you are asked to state or prove a result in a part of a question, you may use it without proof in subsequent parts of the question. The reference CCPS refers to the Cook-Cunningham-Pulleyblank-Schrijver book.

Problem 1: **Network Flows** **(22 marks)**

Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a family of sets over some ground set U. Suppose that each element $i \in U$ has an integer capacity $u_i \geq 0$ and that each set F_j has an integer requirement $b_j \geq 0$. A $\mathcal{F}$-*cover* is a collection $(F'_1, \ldots, F'_k)$ of subsets of U such that

(P1) $F'_j \subseteq F_j$ and $|F'_j| \geq b_j$ for all $j \in \{1, \ldots, k\}$;

(P2) every element $i \in U$ is in at most u_i of the sets $F'_1, \ldots, F'_k$.

(a) Formulate the problem of determining whether a $\mathcal{F}$-cover exists as a network-flow problem.

(b) Using the construction in part (a), show that a $\mathcal{F}$-cover exists if and only if

$$\sum_{i \in S} u_i + \sum_{j=1}^{k} \min\{b_j, |F_j \setminus S|\} \geq \sum_{j=1}^{k} b_j \qquad \text{for all subsets } S \subseteq U.$$

You may use standard results about flows.

Problem 2: **Matroid Theory** **(22 marks)**

(a) Let U be a ground set. A collection $\mathcal{L}$ of subsets of U is called a *laminar collection*, if for any two sets $S, T \in \mathcal{L}$, either $S \subseteq T$ or $S \cap T = \emptyset$. Given such a laminar collection $\mathcal{L}$, and a nonnegative integer b_S for every $S \in \mathcal{L}$, let $\mathcal{I} = \{A \subseteq U : |S \cap A| \leq b_S \text{ for all } S \in \mathcal{L}\}$. Prove that $M = (U, \mathcal{I})$ is a matroid.

(b) Let $G = (V, E)$ be an undirected graph, $\mathcal{S}$ denote a laminar collection of subsets of V, and let b_S be a nonnegative integer associated with each $S \in \mathcal{S}$. Given an orientation of G and $S \subseteq V$, let $\gamma(S)$ denote the set of all arcs with both ends in S, and $\delta^{\text{in}}(S)$ denote the incoming arcs of S. We wish to determine if there exists an orientation of G such that $|\gamma(S) \cup \delta^{\text{in}}(S)| \leq b_S$ for every set $S \in \mathcal{S}$. Formulate this problem as a matroid intersection problem.

You *do not* need to give a min-max formula.

Problem 3: Matching **(22 marks)**

Let $G = (V, E)$ be a graph and let $A \subseteq V$. Let $\text{oc}(H)$ denote the number of components of H with an odd number of vertices and let $\text{def}_G(A)$ be defined as $\text{oc}(G \setminus A) - |A|$.

(a) Consider a matching M and $A \subseteq V$. Show that there are at least $\text{def}_G(A)$ vertices that are M-exposed. Recall that v is an M-exposed vertex if there is no edge of M incident to v.

Call a set A a *Tutte set* if there exists a matching M with exactly $\text{def}_G(A)$ M-exposed vertices. The Tutte-Berge formula states the existence of a Tutte set. You may use this fact for the remainder of this question. Among all Tutte sets, let S be the one that minimizes the number of vertices in the odd components of $G \setminus S$. Suppose $\text{def}_G(S) \geq 1$ and let $C_1, \ldots, C_k$ denote the odd components of $G \setminus S$.

(b) Show that for every i and every vertex $v \in C_i$, there is a matching of C_i such that v is the only exposed vertex. (Hint: If not, then $C_i \setminus \{v\}$ has a non-empty Tutte set.)

(c) Construct a bipartite graph H with vertices S and $\{1, \ldots, k\}$ such that $s \in S$ and i are connected whenever there is an edge of G joining s to some vertex of C_i. Show that for every $S' \subseteq S$ where $S' \neq \emptyset$, $|\delta_H(S')| \geq |S'| + 1$. Deduce that for every $i \in \{1, \ldots, k\}$, the bipartite graph $H \setminus \{i\}$ has a matching of size $|S|$.

You may use standard results about bipartite matching.

(d) Using the results of parts (b) and (c), show that for every i and every vertex $v \in C_i$ there is a maximum matching of G such that v is exposed.

Problem 4: Polyhedral Theory **(22 marks)**

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a non-empty polytope. Given $c \in \mathbb{R}^n$, let (LP$_c$) denote the linear program: $\max c^T x$ s.t. $x \in P$. You may use any result from the CCPS book (except the one you are asked to prove) to answer the following parts.

(a) Show that $\hat{x}$ is an extreme point of P iff there is some vector $c \in \mathbb{R}^n$ such that $\hat{x}$ is the unique optimum solution to (LP$_c$).

(b) Let $C \subseteq \mathbb{R}^n$ be a set of vectors such that for every extreme point $\hat{x}$ of P, there is some $c \in C$ for which $\hat{x}$ is the unique optimum solution to (LP$_c$). Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a function such that (i) $f(x) \in P$ for all $x \in P$; and (ii) $c^T x \leq c^T f(x)$ for all $x \in P,\ c \in C$. Prove that P is the convex hull of $\{x \in P : x = f(x)\}$.

Problem 5: Complexity **(12 marks)**

For each of the following problems, either indicate that the problem is *NP*-hard, or that it admits a polynomial-time algorithm. In the former case indicate a reduction from a well-known *NP*-hard problem; in the latter case, indicate briefly how the problem can be solved using results from CCPS.

(a) Given a graph G and an integer K, determine if G has a 2-edge connected spanning subgraph with at most K edges.

(b) Given a graph $G = (V, E)$ with integer edge-weights w_e for every edge e, find an optimal solution to the following LP-relaxation of (a weighted version of) the problem in part (a):

$$\min \sum_{e \in E} w_e x_e \quad \text{s.t.} \quad x(\delta(S)) \geq 2 \quad \text{for all } \emptyset \subsetneq S \subsetneq V, \qquad 0 \leq x_e \leq 1 \quad \text{for all } e \in E.$$