Comprehensive Exam – Discrete Optimization

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Note:

The exam has five problems, and each is worth 20 points.

If you are pressed for time, then it is better to answer some questions completely than to give partial answers to all questions.

Do not use results without proof unless the question allows it.

If you are asked to state or prove a result in one part of a question, then you may use it to answer the other parts.

Feel free to ask questions if any notation is unclear.

Problem 1: Network Flows

(a) State and prove Hoffman's Circulation Theorem.
HINT: Recall that the theorem gives necessary and sufficient conditions for a circulation x satisfying $\ell \leq x \leq u$. You may use (without proof) the result on necessary and sufficient conditions for a flow x satisfying $0 \leq x \leq u$, and $f_x(v) = b_v, \forall v$.

(b) Briefly sketch a proof of the following statement; a detailed proof is NOT required.
If x is an (s,t)-flow then there exists a collection of dicircuits $C_1, \ldots, C_r$ and (s,t)-dipaths $P_1, \ldots, P_q$, together with non-negative values $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_q$ such that $x = \alpha_1 x^{C_1} + \ldots + \alpha_r x^{C_r} + \beta_1 x^{P_1} + \ldots + \beta_q x^{P_q}$.

(c) Consider a minimum-cost flow problem on a digraph $D = (N, A)$, with arc capacities $u \in \mathbb{R}^A_+$, arc costs $c \in \mathbb{R}^A$ and demands $b \in \mathbb{R}^N$. Assume that (i) there is no dicircuit of negative cost, and (ii) the problem has a feasible solution.

Prove: There exists an optimal solution x such that every arc ij has $x_{ij} \leq B$, where B denotes the sum of the positive demands, that is $B = \sum\{b_v \mid v \in N, b_v > 0\}$.

Problem 2: Polyhedral Theory

Let $Ax \leq b$ be a system of inequalities, where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$.

(a) What is meant by the statement "$Ax \leq b$ is TDI (totally dual integral)?"

(b) Prove that if A is rational, b is integral, $Ax \leq b$ is TDI, and $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is a polytope, then every extreme point of P is integral.
HINT: You may use the following result, but then you should give a proof: A rational polytope P is integral if and only if for all integral vectors w the optimal value of $\max\{w^T x : x \in P\}$ is an integer.
You may use other results without proof.

(c) Write down a TDI system for the minimum spanning tree (MST) problem (that is, for the convex hull of characteristic vectors of spanning trees). Briefly explain why the system is TDI; a detailed proof is NOT required.

Problem 3: Matroid Theory

For the following question, recall that a function r over subsets of S is *submodular* if $r(A) + r(B) \geq r(A \cap B) + r(A \cup B)$, for all $A, B \subseteq S$; it is *monotone* if $r(A) \leq r(B)$ for all $A \subseteq B \subseteq S$.

(a) Let $(S, \mathcal{I})$ be a matroid with rank function r. Show that r is submodular.

(b) Suppose r is a monotone, submodular function defined on the subsets of S such that $0 \leq r(A) \leq |A|$ for all $A \subseteq S$. Show that $M_r := (S, \{A \subseteq S : |A| = r(A)\})$ is a matroid.

You may use the following fact about submodular functions without proof.

Fact 1. *Suppose r is a monotone, submodular function defined on the subsets of S with $0 \leq r(A) \leq |A|$ for all $A \subseteq S$. Let $A \subseteq S$, and $e_1, \ldots, e_p \in S \setminus A$ such that $r(A \cup \{e_i\}) = r(A)$ for all i. Then, we also have $r(A \cup \{e_1, \ldots, e_p\}) = r(A)$.*

(c) Use (a) and (b) to show the following. Let $M = (S, \mathcal{I})$ be a matroid with rank function r. Define function r^* by letting

$$r^*(A) = |A| + r(S \setminus A) - r(S),$$

for all $A \subseteq S$. Define M_{r^*} as in (b), and show that it is a matroid.

You may use (without proof) that $r(A \cup \{e\}) \leq r(A) + 1$ for any $A \subseteq S$, and $e \in S$, and that r is monotone.

Problem 4: Matchings

Recall the Tutte-Berge formula:

$$\max\{|M| : M \text{ is a matching}\} = \min\left\{\frac{1}{2}(|V| - \text{oc}(G \setminus A) + |A|) : A \subseteq V\right\},$$

where $\text{oc}(G \setminus A)$ denotes the number of odd components in $G \setminus A$.

(a) Recall that a vertex is essential if it is covered by all maximum matchings. Suppose that $A^* \subseteq V$ is a minimizer of the Tutte-Berge formula. Show that each vertex $v \in A^*$ must be essential.

(b) A graph $G = (V, E)$ is called factor critical if $G - v$ has a perfect matching for all $v \in V$.

Suppose that $G = (V, E)$ is connected, and has the property that every node v is inessential. Show that the size of a maximum matching is precisely $(|V| - 1)/2$; i.e., G is factor critical.

Hint: Show that the unique minimizer of the Tutte-Berge formula is the empty set.

(c) Let $G = (V, E)$ be a 3-regular, connected graph with at most two bridges (where an edge e is a bridge if $G - e$ is disconnected). Then use the Tutte-Berge formula to show that G has a perfect matching.

Hint: Use the fact that G has an even number of vertices, and if you do, say why.

Problem 5: Complexity Theory

Are the following decision problems polynomial-time solvable or NP-complete? Give brief and convincing explanations; detailed proofs are not needed.

(a) Let $D = (N, A)$ be a digraph with arc costs $c \in \mathbb{Z}^A$. The arc costs may be negative or nonnegative. There exists a dicircuit of zero cost.

(b) Let $D = (N, A)$ be a digraph with arc costs $c \in \mathbb{Z}^A$ that has no dicircuit of negative cost. The arc costs may be negative or nonnegative. There exists a dicircuit of zero cost.

(c) Consider a connected graph G that has a color $c(e)$ assigned to each edge e (you may assume that each "color" is an integer in $\{1, \ldots, k\}$). Let ℓ be a positive integer. There exists a spanning tree of G that has edges of at least ℓ distinct colors.

Appendix: NP-Complete problems

Directed Hamiltonian circuit

INSTANCE: Directed graph $G = (V, E)$.

QUESTION: Is there a directed circuit of G containing every vertex in V?

Directed Hamiltonian path

INSTANCE: Directed graph $G = (V, E)$, and a pair of vertices s, t.

QUESTION: Is there a directed (s, t)-path of G containing every vertex in V?

Disjoint connecting paths

INSTANCE: Graph G, collection of disjoint vertex pairs $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$.

QUESTION: Does G contain k mutually vertex disjoint paths, connecting s_i and t_i for each $i = 1, \ldots, k$?

Feedback arc set

INSTANCE: Directed graph $G = (V, E)$ and $k \in \mathbb{Z}_+$.

QUESTION: Is there a subset $E' \subseteq E$ with $|E'| \leq k$ such that E' contains at least one arc from every directed circuit in G?

Edge colouring

INSTANCE: Graph $G = (V, E)$ and $K \in \mathbb{Z}_+$.

QUESTION: Can each edge be assigned one of K different colours so that edges incident to the same vertex get distinct colours? Remains hard for cubic graphs and $K = 3$.

Exact cover

INSTANCE: Collection $\mathcal{F}$ of subsets of a finite set X.

QUESTION: Is there a sub-collection of $\mathcal{F}$ that forms a partition of X.

Knapsack

INSTANCE: A finite set U, values $s_u, v_u \in \mathbb{Z}_+$ for all $u \in U$ and $B, K \in \mathbb{Z}_+$.

QUESTION: Is there a subset $U' \subseteq U$ such that

$$\sum_{u \in U'} s_u \leq B \qquad \sum_{u \in U'} v_u \geq K$$

Maximum clique

INSTANCE: Graph G and $K \in \mathbb{Z}_+$.

QUESTION: Does G contain a clique (induced complete subgraph) of size K?

Maximum cut

INSTANCE: A graph $G = (V, E)$ and $K \in \mathbb{Z}_+$.

QUESTION: Is there $S \subseteq V$ such that $|\delta(S)| \geq K$?

Minimum cover

INSTANCE: Collection $\mathcal{C}$ of subsets of a set $\mathcal{S}$, positive integer K.

QUESTION: Does $\mathcal{C}$ contain a *cover* for $\mathcal{S}$ of size K or less, that is, a subset $\mathcal{C}' \subseteq \mathcal{C}$ with $|\mathcal{C}'| \leq K$ and such that $\cup_{c \in \mathcal{C}'} C = S$?

k**-closure**

INSTANCE: Directed graph $G = (V, E)$ and integer $k \leq |V|$.

QUESTION: Does there exists a nonempty subset X of V, with $|X| = k$, such that there are no arcs uv of G with $u \in X$ and $v \notin X$?