Discrete Optimization Comprehensive Exam — Spring 2012

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Instructions: Unless otherwise stated, do not use results without proofs. If you are asked to state or prove a result in a part of a question, you may use it without proof in subsequent parts of the question. The reference CCPS refers to the Cook-Cunningham-Pulleyblank-Schrijver book.

Problem 1: Network Flows **(25 marks)**

We are given an undirected graph $G = (V, E)$, and integer bounds $\{(\ell_v, \kappa_v)\}_{v \in V}$ on the nodes such that $\ell_v \leq \kappa_v$ for all nodes v. Say that G is (ℓ, κ)-*orientable* if one can direct the edges of E to obtain an arc-set A such that $\ell_v \leq |\delta_A^{\text{in}}(v)| \leq \kappa_v$ for every node $v \in V$ (where $\delta_A^{\text{in}}(v)$ denote the incoming edges of v in A).

(a) Formulate the problem of determining whether G is (ℓ, κ)-orientable as a network-flow problem.

Hint: Create a vertex for each node and edge of G, and have ∞-capacity arcs directed from each edge-vertex to the node-vertices corresponding to the end points of the edge.

(b) State Hoffman's circulation theorem, which gives necessary and sufficient conditions for the existence of a circulation satisfying lower bounds and capacities.

(c) Use the construction in part (a) to show that G is orientable if and only if

$$\sum_{v \in S} \kappa_v \geq |\gamma(S)|, \quad \sum_{v \in S} \ell_v \leq |\gamma(S)| + |\delta(S)| \qquad \text{for all } S \subseteq V.$$

Recall that $\gamma(S)$ is the set of edges with both ends in S, and $\delta(S)$ is the set of edges with exactly one end in S. You may use standard results about flows.

Problem 2: Matroid Theory **(25 marks)**

(a) Let U be a ground set. Let $T_1 \cup T_2 \cup \ldots T_k$ be a partition of U, and $t_1, \ldots, t_k$ be nonnegative integers. Show that $M = \big(U, \mathcal{I} := \{S \subseteq U : |S \cap T_i| \leq t_i\}\big)$ is a matroid.

(b) Let $G = (V = A \cup B, E)$ be a bipartite graph with bipartition (A, B). Let $\{b_v\}_{v \in V}$ be integers such that $1 \leq b_v \leq |\delta(v)|$ for every $v \in V$. A b-matching is a subset M of edges such that $|\delta(v) \cap M| \leq b_v$ for every node v. The size of a b-matching M is the number of edges in M. Formulate the problem of finding a maximum-size b-matching in G as a matroid-intersection problem.

(c) Define $b(S) = \sum_{v \in S} b_v$ for a subset $S \subseteq V$. Use the construction in part (b) and the matroid-intersection theorem to show that

$$\max\{|M| : M \text{ is a } b\text{-matching}\} = \min\{b(S) : S \text{ is a vertex cover of } G\}.$$

Problem 3: Complexity **(15 marks)**

Let $G = (V, E)$ be a graph and denote by $\mathcal{F}$ the set of all circuits of G and by $\mathcal{F}_{odd}$ the set of all circuits of G that have an odd number of edges (by a circuit we mean an set of edges that form a connected subgraph where every vertex has degree two). Consider the following polytopes in $\Re^E$,

$$P = \left\{ x : \sum_{e \in C} x_e \geq 1, C \in \mathcal{F}, \mathbf{0} \leq x \leq \mathbf{1} \right\} \qquad \text{and}$$
$$Q = \left\{ x : \sum_{e \in C} x_e \geq 1, C \in \mathcal{F}_{odd}, \mathbf{0} \leq x \leq \mathbf{1} \right\}.$$

Let w be a set of non-negative integer edge weights. For each of the following optimization problem, either prove that the problem is polynomial solvable by outlining an algorithm or prove that it is NP-complete by reducing it to one of the problems in CCPS.

(a) $\min\{w^T x : x \in P, x \text{ integer}\}$,

(b) $\min\{w^T x : x \in Q, x \text{ integer}\}$.

Problem 4: Matching **(20 marks)**

Let G be a connected graph with at least two vertices and with the property that for every vertex v there exists a maximum matching avoiding v. Given a pair of vertices u and v, we write $u \sim v$ if no maximum matching avoids both u and v.

(a) Observe that for every edge uv, $u \sim v$,

(b) Show that if $u \sim v$ and $v \sim w$ for distinct vertices u, v, w, then $u \sim w$,

(c) Prove that for every vertex v there exists a matching that covers every vertex of G except v.

A perfect 2-matching of a graph is an assignment of weights $0, 1$ or 2 to the edges, such that the weight of the edges incident with any vertex sum to 2.

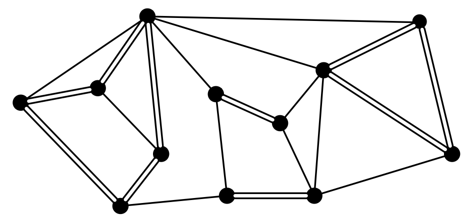
(d) Show that G has a perfect 2-matching.

Problem 5: Polyhedral Theory **(20 marks)**

A graph is *bad* if it contains as a spanning subgraph, a graph where the components consists of single edges, or circuits with an odd number of edges (by a circuit we mean an set of edges that form a connected subgraph where every vertex has degree two). Moreover, we require that we have at least one such odd circuit. As an example we indicate in the next figure a bad graph. The double edges indicate the forbidden subgraph. Consider the polytope,

$$P = \left\{ x \geq \mathbf{0} : \sum_{e \in \delta(v)} x_e = 1, \text{for all } v \in V \right\}.$$

Recall that a polytope is integral if every extreme point is integral.



(a) Show that if G is bad, then P is not integral.

Hint: Construct a fractional extreme point where all values are in $\{0, 1, \frac{1}{2}\}$.

(b) Show that if G is not bad then P is integral.

Hint: Express fractional points of P as the convex combination of two points of P.