Comprehensive Exam – Discrete Optimization

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Examiners: Jochen Könemann and Laura Sanità

Note:

The exam has five problems.

If you are pressed for time, then it is better to answer some questions completely than to give partial answers to all questions.

Do not use results without proof unless the question allows it.

If you are asked to state or prove a result in one part of a question, then you may use it to answer the other parts.

Problem 1: Network Flows (20 points)

(a) Let $G = (V, E)$ be a directed graph, with edge costs $c \in \mathbb{R}^E$ and $r \in V$. Define the term *feasible potential*. Prove that if G has a feasible potential, then there is no directed circuit C of G of negative-cost (i.e. such that $\sum_{e \in C} c_e < 0$).

(b) Given a directed graph $G = (V, E)$ with edge costs $c \in \mathbb{R}^E$, consider the linear programming formulation for the *minimum cost flow problem*:

$$\begin{array}{rll} \text{Minimize} & \sum_{e \in E} c_e x_e & \\ \text{subject to} & & \\ & f_x(v) = b_v & \text{for all } v \in V \\ & 0 \le x_e \le u_e & \text{for all } e \in E \end{array}$$

where here we assume $b \in \mathbb{R}^V$ with $\sum_v b_v = 0$, and $u \in (\mathbb{R}_{\geq 0} \cup \infty)^E$. Assuming that the above formulation has a feasible solution, show that it has an optimal solution if and only if there exists no negative-cost directed circuit of G, each of whose arcs has infinite capacity.

Hint: Use the dual of the above LP together with the following fact (which you may use without proof).

Fact 1. *A directed graph G with edge costs c has a feasible potential if and only if it has no negative-cost directed circuit.*

Problem 2: Matchings (22 points)

(a) Recall the Augmenting Path Theorem of Matchings:

A matching M in a graph $G = (V, E)$ is maximum if and only if there is no M-augmenting path.

  (i) Give a proof of the Augmenting Path Theorem.

  (ii) Suppose that $p > 0$ is the cardinality of a maximum matching of G, and let M be a matching of cardinality at most $p - \sqrt{p}$. Show that there is an M-augmenting path having less than $\sqrt{p}$ edges from M.

  Hint: Start by showing that there are at least $\sqrt{p}$ node-disjoint M-augmenting paths.

(b) Let $G = (V, E)$ be an undirected graph with non negative edge costs $c \in \mathbb{R}^E$ and with distinct vertices s and t. An odd s, t-path is an s, t-path that has an odd number of edges. Show that we can formulate the problem of finding an odd s, t-path of minimum cost as a minimum cost perfect matching problem.

Hint: Make a copy of the graph G and remove vertices s and t. Call the resulting graph G'. Construct a new graph H starting with the union of graphs G and G' and by joining every vertex $v \neq s, t$ in G with its copy v' in G'.

Problem 3: Matroid Theory (22 points)

Let $(\mathcal{S}, \mathcal{I}_1), \ldots, (\mathcal{S}, \mathcal{I}_k)$ be matroids with rank functions $r_1, \ldots, r_k$. Recall that a set $J \subseteq \mathcal{S}$ is called *partitionable* if $J = J_1 \cup \ldots \cup J_k$ and $J_i \in \mathcal{I}_i$ for all $1 \leq i \leq k$.

(a) Let $X \subseteq \mathcal{S}$, and show that the cardinality of the largest partitionable subset of X is at most $|X \setminus A| + \sum_{i=1}^{k} r_i(A)$, for any $A \subseteq X$.

(b) Once again let $X \subseteq \mathcal{S}$, and define $X' = X \times \{1, \ldots, k\}$. For $J \subseteq X'$, we let $J_i = \{e \in X : (e, i) \in J\}$, and define

$$\begin{aligned} \mathcal{I}^{(1)} &= \{J \subseteq X' : J_i \in \mathcal{I}_i \text{ for all } 1 \leq i \leq k\} \\ \mathcal{I}^{(2)} &= \{J \subseteq X' : J_i \cap J_l = \emptyset \text{ for all } i \neq l\}. \end{aligned}$$

Show that $(X', \mathcal{I}^{(1)})$ and $(X', \mathcal{I}^{(2)})$ are matroids, and state their rank functions. You may use the following fact without proof.

Fact 2. *If $(\mathcal{S}, \mathcal{I})$ is a matroid and $J, J' \in \mathcal{I}$, $|J| < |J'|$, then there is an element $e \in J' \setminus J$ such that $J \cup \{e\} \in \mathcal{I}$.*

(c) State (but do not prove) the matroid intersection theorem. Use it together with your findings in (a) and (b) to show that the largest partitionable subset of $X \subseteq \mathcal{S}$ has size exactly

$$\min_{A \subseteq X} |X \setminus A| + \sum_{i=1}^{k} r_i(A).$$

Hint: Argue first that there is a $1 - 1$ correspondence between partitionable subsets of X, and common independent sets of the matroids $(X', \mathcal{I}^{(1)})$ and $(X', \mathcal{I}^{(2)})$ defined in (b).

Problem 4: Polyhedral Theory (20 points)

(a) What is meant by the statement "matrix A is *totally unimodular*"?

(b) Let A be an $m \times n$ totally unimodular matrix and let $b \in \mathbb{Z}^m$. Prove that the polyhedron defined by $Ax \leq b$ is integral. You may use the following fact without proof.

Fact 3. *Let F be a minimal non-empty face of $P = \{x : Ax \leq b\}$. Then $F = \{x : A^0 x = b^0\}$ for some subsystem $A^0 x \leq b^0$ of $Ax \leq b$. Moreover, the rank of matrix A^0 is equal to that of A.*

(c) Let A be an $m \times n$ 0,1-matrix with the following extra property: for any two rows $1 \leq i_1 < i_2 \leq m$, the *support* of row i_2 is a subset of the support of row i_1; i.e.,

$$A_{i_1, j} \geq A_{i_2, j},$$

for all columns $1 \leq j \leq n$. Show that A is totally unimodular.

<u>Problem 5:</u> Complexity (16 points)

Are the following decision problems polynomial-time solvable or NP-complete? Give brief and convincing explanations; detailed proofs are not needed.

(a) Let $G = (V, E)$ be a bipartite graph, and w_v a non-negative weight for each vertex $v \in V$. Find a maximum-weight independent set in G; i.e., find a vertex set $S \subseteq V$ of largest total weight such that no two vertices in S are connected by an edge.

(b) Given an undirected graph $G = (V, E)$, edge weights $w_e \geq 0$ for all $e \in E$, and a parameter $k \in \mathbb{Z}_+$. Find a maximum-weight collection of at most k edge-disjoint forests in G.

Appendix: NP-Complete problems

Multiple choice branching

INSTANCE: Directed graph $G = (V, A)$, a weight $w_a \in \mathbb{Z}_+$ for all $a \in A$, a partition of A into disjoint sets $A_1, \ldots, A_m$ and a positive integer K.

QUESTION: Is there a subset A' of A with weight at least K such that no two arcs in A' enter the same vertex, A' is acyclic, and A' contains at most one arc from each set A_i?

Vertex cover

INSTANCE: Directed graph $G = (V, E)$, and a positive integer $K \leq |V|$.

QUESTION: Is there a vertex cover of size K or less in G; i.e., is there a set $V' \subseteq V$ of at most K vertices such that every edge $e \in E$ has at least one endpoint in V'?

Disjoint connecting paths

INSTANCE: Graph G, collection of disjoint vertex pairs $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$.

QUESTION: Does G contain k mutually vertex disjoint paths, connecting s_i and t_i for each $i = 1, \ldots, k$?

Edge colouring

INSTANCE: Graph $G = (V, E)$ and $K \in \mathbb{Z}_+$.

QUESTION: Can each edge be assigned one of K different colours so that edges incident to the same vertex get distinct colours? Remains hard for cubic graphs and $K = 3$.

Partition into forests

INSTANCE: Graph $G = (V, E)$, positive integer $K \leq |V|$.

QUESTION: Can the vertices of G be partitioned into $k \leq K$ disjoint sets $V_1, \ldots, V_k$ such that the subgraph induced by V_i contains no circuits, for all $1 \leq i \leq k$?

Knapsack

INSTANCE: A finite set U, values $s_u, v_u \in \mathbb{Z}_+$ for all $u \in U$ and $B, K \in \mathbb{Z}_+$.

QUESTION: Is there a subset $U' \subseteq U$ such that

$$\sum_{u \in U'} s_u \leq B \qquad \sum_{u \in U'} v_u \geq K$$

Maximum clique

INSTANCE: Graph G and $K \in \mathbb{Z}_+$.

QUESTION: Does G contain a clique (induced complete subgraph) of size K?

Maximum cut

INSTANCE: A graph $G = (V, E)$ and $K \in \mathbb{Z}_+$.

QUESTION: Is there $S \subseteq V$ such that $|\delta(S)| \geq K$?

Minimum cover

INSTANCE: Collection $\mathcal{C}$ of subsets of a set $\mathcal{S}$, positive integer K.

QUESTION: Does $\mathcal{C}$ contain a *cover* for $\mathcal{S}$ of size K or less, that is, a subset $\mathcal{C}' \subseteq \mathcal{C}$ with $|\mathcal{C}'| \leq K$ and such that $\cup_{c \in \mathcal{C}'} C = S$?