# Discrete Optimization Comprehensive Exam - Spring 2017

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## Instructions:

- All five questions have the same number of marks.
- If you are pressed for time it is better to answer some questions completely than to give partial answers to all questions.
- Unless otherwise stated, do not use results without proofs.
- If you are asked to state or prove a result in a part of a question, you may use it without proof in subsequent parts of the question.
- Feel free to ask the proctors if any notation is unclear.

**Question 1 [Network Flows] - 20 points**

Let $D = (V, A)$ be a directed graph, $u \in \mathbb{R}_+^A$ and $b \in \mathbb{R}^V$. For any $W \subseteq V$, let $\delta^{out}(W)$ be the set of arcs leaving $W$ and $\delta^{in}(W)$ be the set of arcs entering $W$. Moreover, let $f_x(v) = \sum_{wv \in A} x_{wv} - \sum_{vw \in A} x_{vw}$.

(In both questions, you can use the max-flow/min-cut theorem without proof)

(a) (10 points) Prove that there exists $x$ satisfying $0 \leq x \leq u$ and $f_x(v) = b_v$ for all $v \in V$ if and only if $b(V) = 0$ and, for every $W \subseteq V$ we have $b(W) \leq u(\delta^{in}(W))$.

(b) (10 points) Let $b \in \mathbb{R}^V$ be such that $b(V) \geq 0$. Prove that there exists $x$ satisfying $0 \leq x \leq u$ and $f_x(v) \leq b_v$ for all $v \in V$ if and only if, for every $W \subseteq V$ we have $b(W) + u(\delta^{out}(W)) \geq 0$.

**Question 2 [Matroid Theory] - 20 points**

(a) (4 points) State the definition of rank function of a matroid. Prove that if $M_a = (S, \mathcal{I}_a)$ and $M_b = (S, \mathcal{I}_b)$ are two matroids with rank functions $r_a$ and $r_b$, respectively, such that $r_a(A) = r_b(A)$ for all $A \subseteq S$, then $M_a = M_b$.

(b) (4 points) Recall that the *dual* matroid $M^*$ of a matroid $M = (S, \mathcal{I})$ is given by $M^* = (S, \mathcal{I}^*)$ with

$$\mathcal{I}^* = \{J \subseteq S : r(S \setminus J) = r(S)\}$$

where $r$ is the rank function of $M$.

Prove that the rank function $r^*$ of $M^*$ satisfies

$$r^*(A) = |A| + r(S \setminus A) - r(S) \qquad \forall A \subseteq S$$

(c) (4 points) Use (a) and (b) to prove that $(M^*)^* = M$.

(d) (8 points) Use duality and matroid intersection to show that a connected graph $G = (V, E)$ has two edge-disjoint spanning trees if and only if for all partitions $V_1, V_2, \ldots, V_p$ of $V$, one has $|\delta(V_1, V_2, \ldots, V_p)| \geq 2(p-1)$, where $\delta(V_1, V_2, \ldots, V_p)$ is the set of edges with endpoints in different partitions.

(Note: for this question you can use the matroid intersection theorem without proving it)

**Question 3 [Matching Theory] - 20 points**

Let $G = (V, E)$ be a connected undirected graph. Recall the Tutte-Berge formula:

$$\max\{|M| : M \text{ is a matching of } G\} = \min\{\frac{1}{2}(|V| - oc(G \setminus A) + |A|) : A \subseteq V\}$$

where $oc(G \setminus A)$ denotes the number of odd components in the graph obtained by removing $A$ from $G$.

(a) (10 points) A vertex $v$ is called *essential* if every maximum matching of $G$ covers it. Let $A \subseteq V$ be a minimizer in the Tutte-Berge formula. Show that every vertex $v$ that is *not* a vertex of an odd component in $G \setminus A$, is essential.

(b) (10 points) Assume that $G$ is a $k$-vertex-connected graph, let $\nu(G)$ denotes the maximum cardinality of a matching of $G$, and $\tau(G)$ denote the minimum cardinality of a *vertex-cover* of $G$ (i.e. a subset $C \subseteq V$ such that each edge is incident into at least one vertex in $C$).
Show that if $\nu(G) < (|V| - 1)/2$, then $\nu(G) \geq k$, and $\tau(G) < |V| - k - 1$.

**Question 4 [Polyhedral Theory] - 20 points**

Let $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polyhedron with $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$. Inequality $\alpha^T x \leq \alpha_o$ is said to be a Chvátal-Gomory cut for $P$ if $\alpha = A^T y \in \mathbb{Z}^n$ and $\alpha_o = \lfloor y^T b \rfloor$ for some $y \in \mathbb{R}^m_+$.

(a) (5 points) Let $P' := P \cap \{x \text{ satisfying all Chvátal-Gomory cuts for } P\}$ be the Chvátal-Gomory closure of $P$. Prove that $P'$ is a rational polyhedron.

(b) (5 points) Assuming $P$ is nonempty, show that $P'$ can be generated by $Ax \leq b$ and Chvátal-Gomory cuts for which $y^T b = \max\left\{\alpha^T x : x \in P\right\}$.

(c) (4 points) Assuming $P$ is pointed and nonempty, show that $P'$ can be generated by $Ax \leq b$ and Chvátal-Gomory cuts such that $y^T A\bar{x} = y^T b$ for $\bar{x}$ a nonintegral vertex of $P$.

(d) (6 points) Let $k \in \mathbb{Z}$ and $k \geq 1$.
Show that the Chvátal rank of $P(k) = \left\{x \in \mathbb{R}^2 : x_2 \geq 0; x_2 \leq 2kx_1; x_1 + \frac{1}{2k}x_2 \leq 1\right\}$ is at least $k$.
(you may use, without proof, the fact that if $Q$ and $P$ are rational polyhedra with $Q \subseteq P$, then $Q' \subseteq P'$)

**Question 5 [Complexity] - 20 points**

Are the following problems polynomial-time solvable or NP-complete? Give a brief and convincing explanation. Detailed proofs are not needed. You may use the list of NP-complete problems given in the Appendix.

(a) (10 points) Given a directed graph $D = (V, A)$, does there exist a collection of vertex-disjoint directed cycles in $D$ partitioning $V$? (i.e., such that each vertex $v \in V$ belongs to exactly one cycle of the collection).

(b) (10 points) Given a directed graph $D = (V, A)$, does there exist a collection of vertex-disjoint triangles in $D$ partitioning $V$? (i.e., such that each vertex $v \in V$ belongs to exactly one triangle of the collection, where a triangle is a directed cycle with 3 edges).

**Appendix: NP-Complete problems**

**Multiple choice branching**

INSTANCE: Directed graph $D = (V, A)$, a weight $w_a \in \mathbb{Z}_+$ for all $a \in A$, a partition of $A$ into disjoint sets $A_1, \dots, A_m$ and a positive integer $K$.

QUESTION: Is there a subset $A'$ of $A$ with weight at least $K$ such that no two arcs in $A'$ enter the same vertex, $A'$ is acyclic, and $A'$ contains at most one arc from each set $A_i$?

**Vertex cover**

INSTANCE: Graph $G = (V, E)$, and a positive integer $K \leq |V|$.

QUESTION: Is there a vertex cover of size $K$ or less in $G$; i.e., is there a set $V' \subseteq V$ of at most $K$ vertices such that every edge $e \in E$ has at least one endpoint in $V'$?

**Disjoint connecting paths**

INSTANCE: Graph $G$, collection of disjoint vertex pairs $(s_1, t_1), (s_2, t_2), \dots, (s_k, t_k)$.

QUESTION: Does $G$ contain $k$ mutually vertex disjoint paths, connecting $s_i$ and $t_i$ for each $i = 1, \dots, k$?

**3-dimensional matching**

INSTANCE: Three disjoint sets of elements $X, Y, Z$, with $|X| = |Y| = |Z| = q$, collection $\mathcal{S} \subseteq X \times Y \times Z$.

QUESTION: Does $\mathcal{S}$ contain a sub-collection $\mathcal{S}' \subseteq \mathcal{S}$ of size $|\mathcal{S}'| = q$, such that each element of $X \cup Y \cup Z$ is contained in exactly one set of the sub-collection $\mathcal{S}'$?

**Edge colouring**

INSTANCE: Graph $G = (V, E)$ and $K \in \mathbb{Z}_+$.

QUESTION: Can each edge be assigned one of $K$ different colours so that edges incident to the same vertex get distinct colours? Remains hard for cubic graphs and $K = 3$.

**Partition into forests**

INSTANCE: Graph $G = (V, E)$, positive integer $K \leq |V|$.

QUESTION: Can the vertices of $G$ be partitioned into $k \leq K$ disjoint sets $V_1, \dots, V_k$ such that the subgraph induced by $V_i$ contains no circuits, for all $1 \leq i \leq k$?

**Knapsack**

INSTANCE: A finite set $U$, values $s_u, v_u \in \mathbb{Z}_+$ for all $u \in U$ and $B, K \in \mathbb{Z}_+$.

QUESTION: Is there a subset $U' \subseteq U$ such that

$$\sum_{u \in U'} s_u \leq B \qquad \sum_{u \in U'} v_u \geq K$$

**Maximum clique**

INSTANCE: Graph $G = (V, E)$ and $K \in \mathbb{Z}_+$.

QUESTION: Does $G$ contain a clique (induced complete subgraph) of size $K$?

**Maximum cut**

INSTANCE: A graph $G = (V, E)$ and $K \in \mathbb{Z}_+$.

QUESTION: Is there $S \subseteq V$ such that $|\delta(S)| \geq K$?

**Minimum cover**

INSTANCE: Collection $\mathcal{C}$ of subsets of a set $\mathcal{S}$, positive integer $K$.

QUESTION: Does $\mathcal{C}$ contain a *cover* for $\mathcal{S}$ of size $K$ or less, that is, a subset $\mathcal{C}' \subseteq \mathcal{C}$ with $|\mathcal{C}'| \leq K$ and such that $\cup_{c \in \mathcal{C}'} C = S$?