Discrete Optimization Comprehensive Exam — Spring 2015

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Instructions: Unless otherwise stated, do not use results without proofs. If you are asked to state or prove a result in a part of a question, you may use it without proof in subsequent parts of the question. The reference CCPS refers to the Cook-Cunningham-Pulleyblank-Schrijver book.

If you are pressed for time it is better to answer some questions completely than to give partial answers to all questions. Feel free to ask questions if any notation is unclear.

Problem 1: Network flows **(22 marks)**

Given an undirected graph $G = (V, E)$, say that G has a *balanced orientation* if one can direct the edges of E to obtain an arc-set A such that $\big||\delta_A^{\text{in}}(v)| - |\delta_A^{\text{out}}(v)|\big| \le 1$ for every node $v \in V$ (where $\delta_A^{\text{in}}(v)$ and $\delta_A^{\text{out}}(v)$ denote respectively the incoming and outgoing arcs of v in A).

(a) Formulate the problem of finding a balanced orientation of G as a network-flow problem.

(b) Use the construction in part (a) to show that G always has a balanced orientation. You may use standard results about flows.

Problem 2: Matroid theory **(22 marks)**

(a) Let $M = (U, \mathcal{I})$ be a matroid with rank function $r : 2^U \mapsto \mathbb{Z}_+$. Show that r is submodular, that is, it satisfies $r(A) + r(B) \ge r(A \cap B) + r(A \cup B)$ for all $A, B \subseteq U$.

(b) Let $D = (N, A)$ be a directed graph. We say that $T = (N, A')$, where $A' \subseteq A$, is a spanning tree of D if upon ignoring the directions of the arcs in A', we get an acyclic edge set that spans N. (Informally, T is a spanning tree of D if its undirected version is a spanning tree for the undirected version of D.) Let b_v be a nonnegative integer associated with each $v \in N$. Formulate the problem of determining if D has a spanning tree T with $|\delta_T^{\text{in}}(v)| \le b_v$ for all $v \in N$ as a matroid intersection problem.

(c) Apply the matroid-intersection theorem to the construction in part (b) to show that D has a spanning tree T with $|\delta_T^{\text{in}}(v)| \le 1$ for all $v \in N$ if and only if

$$\big|\{v \notin S : \delta_D^{\text{in}}(v) \ne \emptyset\}\big| \ge |N \setminus S| - 1 + \kappa(S) \qquad \text{for all } S \subseteq N \text{ s.t. } \delta_D^{\text{in}}(S) = \emptyset.$$

Here $\kappa(S)$ is the number of components of $\big(S, \{(u, v) \in A : u, v, \in S\}\big)$ when one ignores the directions of the arcs.

Problem 3: Matching theory **(22 marks)**

(a) Write down a linear programming formulation (P) for the minimum-weight perfect matching problem, and write down the dual (D) of (P). State the Perfect Matching Polytope Theorem.

(b) State the Tutte-Berge formula, and prove the formula using the termination conditions of the Blossom algorithm for maximum matching. Make sure to state the relevant conditions that hold at the termination of the algorithm.

(c) Recall that a vertex is *essential* if it is covered by all maximum matchings. Suppose that $A^* \subseteq V$ is a minimizer of the Tutte-Berge formula. Show that each vertex $v \in A^*$ is essential.

Problem 4: Polyhedral theory **(22 marks)**

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a non-empty polytope. You may use any result from the CCPS book (except the one you are asked to prove) to answer the following parts.

(a) Prove that F is a face of P if and only if $F = \{x \in P : A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$.

(b) Prove that if $\hat{x}$ is an extreme point of a face of P, then $\hat{x}$ is an extreme point of P.

Problem 5: Complexity **(12 marks)**

For each of the following problems, either indicate that the problem is *NP*-hard, or that it admits a polynomial-time algorithm. In the former case indicate a reduction from a well-known *NP*-hard problem; in the latter case, indicate briefly how the problem can be solved using results from CCPS. You may assume that the following problem is *NP*-hard:

MULTIWAY CUT: Given a graph $G = (V, E)$, a subset $T \subseteq V$ of terminals, and an integer K, determine if there is a subset $F \subseteq E$ of at most K edges whose removal disconnects every pair of terminals.

(a) Given a graph $G = (V, E)$, pairs of nodes $(s_1, t_1), (s_2, t_2), \ldots, (s_\ell, t_\ell)$, and an integer K, determine if there is a subset $F \subseteq E$ of at most K edges that intersects every s_i-t_i path for all $i = 1, \ldots, \ell$.

(b) Given a graph $G = (V, E)$ with integer edge-weights w_e for every edge e, find an optimal solution to the following LP-relaxation of (a weighted version of) the problem in part (a):

$$\min \sum_{e \in E} w_e x_e \qquad \text{s.t.} \qquad x(P) \geq 1 \qquad \forall\ s_i\text{-}t_i \text{ paths } P,\ \forall i = 1, \ldots, \ell$$

$$0 \leq x_e \leq 1 \qquad \forall\ e \in E.$$