Discrete Optimization Comprehensive Exam — Spring 2018

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Instructions: Unless otherwise stated, do not use results without proofs. If you are asked to state or prove a result in a part of a question, you may use it without proof in subsequent parts of the question. The reference CCPS refers to the Cook-Cunningham-Pulleyblank-Schrijver book.

Problem 1: Network Flows **(20 marks)**

a) State The Max-Flow Min-Cut (MFMC) Theorem.

b) Using the MFM Theorem, find a sufficient and necessary condition for the existence of a circulation in a directed graph $G = (V, E)$ with lower bounds ℓ and upper bounds u, i.e. a solution to,

$$\begin{aligned} f_x(u) &= 0 && (u \in V) \\ \ell_e \leq x_e &\leq u_e && (e \in E) \end{aligned}$$

HINT: Define $x'_e = x_e - \ell_e$ and $u'_e - \ell_e$ for all $e \in E$.

Problem 2: Matching polytope, blossom algorithm **(20 marks)**

Let $G = (V, E)$ be a graph with non-negative edge weights $c = (c_e : e \in E)$. Edmonds showed that the maximum weight of a matching of G is equal to the optimal value of the following linear-program (P).

$$\max \sum (c_e x_e : \ e \in E)$$

subject to

$$\begin{aligned} & x(\delta(v)) \leq 1 && (v \in V) && (1) \\ & x(\gamma(S)) \leq (|S| - 1)/2 && (S \subseteq V,\ |S| \geq 3, |S| \text{ odd}) && (2) \\ & x_e \geq 0, && (e \in E). && (3) \end{aligned}$$

a) Write the linear-programming dual (D) of (P).

b) Suppose that c_e is an integer for all $e \in E$. Then the blossom algorithm shows that (D) has an optimal solution such that the variables corresponding to constraints (1) are half-integer valued and the variables corresponding to constraints (2) are integer valued. Show how to modify such a half-integer optimal solution to (D) to obtain an integer optimal solution to (D).

   HINT: An optimal solution to (D) has an even number of variables that have non-integer values.

c) Using a result on Total Dual Integrality deduce that the polyhedron described by (1) – (3) is integral.

Problem 3: Matroid intersection **(20 marks)**

a) State the Matroid Intersection Theorem for a pair of matroids M_1 and M_2.

b) Prove that the $\max$ is at most equals to the $\min$.

c) Simplify the statement for the case where M_2 is the dual of M_1.

d) Given a connected planar graph G and a non-negative integer k, find a necessary and sufficient for the existence of k edges of G such that its removal keeps G and the planar dual of G connected.

e) Is the previous problem solvable in polynomial time? Justify your answer.

Problem 4: Complexity **(15 marks)**

a) Define the problem classes $\mathcal{NP}$, co-$\mathcal{NP}$, and $\mathcal{NP}$-complete.

b) Consider the following two problems.

- DIRECTED HAMILTONIAN CIRCUIT:
  Given a directed graph, does it have a directed Hamiltonian circuit?
- UNDIRECTED HAMILTONIAN CIRCUIT:
  Given an undirected graph, does it have a Hamiltonian circuit?

Use the fact that DIRECTED HAMILTONIAN CIRCUIT is $\mathcal{NP}$-complete to show that UNDIRECTED HAMILTONIAN CIRCUIT is also $\mathcal{NP}$-complete.