First Stage Comprehensive Examination in Enumeration
Department of C&O, U. Waterloo
Monday, June 19, 2017, 1:00 - 4:00 pm, MC 4044

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Answer as many questions as time permits. Each question part is worth 10 points (total: 90 points).

1. An *R-sequence* is a sequence $(c_1, ..., c_n)$ of integers $c_i \geq -1$ such that $c_1 + \cdots + c_n = 0$ and for all $1 \leq i < n$, $c_1 + \cdots + c_i \geq 0$. For each $n \in \mathbb{N}$, determine the number of R-sequences of length n.

2. Let $\phi : X \to X$ be an endofunction. A vertex $v \in X$ is *recurrent* if there is some positive integer $k \geq 1$ such that $\phi^k(v) = v$. Let $\mathcal{Q}$ be the species (class) of endofunctions $\phi : X \to X$ such that if $v \in X$ has $|\phi^{-1}(v)| \geq 2$, then v is recurrent. Obtain a formula for the exponential generating function

$$Q(x) = \sum_{n=0}^{\infty} |\mathcal{Q}_n| \frac{x^n}{n!}.$$

(Note: $\mathcal{Q}_n$ denotes the set of all such endofunctions when $X = \{1, 2, \ldots, n\}$.)

3. Let $\mathcal{R}$ be the species (class) of rooted labelled trees (RLTs). Let $c_k(T, v)$ denote the number of nodes with exactly $k \in \mathbb{N}$ children in a RLT (T, v). Let $\overline{c}_k(n)$ denote the average value of $c_k(T, v)$ among all n^{n-1} RLTs on the set $\{1, 2, \ldots, n\}$.

(a) Obtain an expression which determines the exponential generating function

$$R(x, y) = \sum_{n=0}^{\infty} \left(\sum_{(T,v) \in \mathcal{R}_n} y^{c_k(T,v)} \right) \frac{x^n}{n!}.$$

(b) Show that for $0 \leq k \leq n$,

$$\overline{c}_k(n) = \frac{n^2}{(n-1)^{k+1}} \binom{n-1}{k} \left(1 - \frac{1}{n}\right)^n.$$

4. For $n, k \in \mathbb{N}$, let $a_{n,k}$ denote the number of lattice paths starting at $(0,0)$ and ending at (n,n), such that all points (x,y) on the path satisfy $x \leq y \leq x+k$.

Let $A_k(t) = \sum_{n \geq 0} a_{n,k} t^n$. Find a recurrence relation for $A_k(t)$, $k \geq 1$. Hence, or otherwise, prove that

$$A_k(t) = \frac{\sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^m \binom{k-m}{m} t^m}{\sum_{m=0}^{\lfloor \frac{k+1}{2} \rfloor} (-1)^m \binom{k+1-m}{m} t^m}.$$

5. For $m, n, k \in \mathbb{N}$, let $a(m,n,k)$ denote the number of m-by-n matrices with entries in $\{0,1\}$, with no rows or columns consisting entirely of 0s, and with exactly k 1s. Consider the generating series

$$A(x,y,z) = \sum_{m,n,k \in \mathbb{N}} a(m,n,k) \frac{x^m}{m!} \frac{y^n}{n!} z^k.$$

(a) Explain why

$$A(x,y,z) = \exp(-x-y) \sum_{m,n \in \mathbb{N}} (1+z)^{mn} \frac{x^m}{m!} \frac{y^n}{n!}.$$

(b) From part (a) or otherwise, derive a partial differential equation satisfied by $A(x,y,z)$, and show that $\{a(m,n,k) : \; m,n,k \in \mathbb{N}\}$ satisfies a linear recurrence relation.

6. Let $\mathcal{S}_n$ denote the set of permutations of $\{1,2,\ldots,n\}$. A permutation $\sigma \in \mathcal{S}_n$ has a *descent* at position i if $\sigma(i) > \sigma(i+1)$.

(a) Let $1 \leq a_1 < a_2 < \cdots < a_m \leq n-1$ be integers. Use Inclusion/Exclusion to show that the number of permutations in $\mathcal{S}_n$ with descents at the positions $a_1, a_2, ..., a_m$ and nowhere else is

$$n! \sum_{j=0}^{m} (-1)^{m-j} \sum_{1 \leq i_1 < i_2 < \cdots < i_j = m} \prod_{\ell=1}^{j+1} \frac{1}{(a_{i_\ell} - a_{i_{\ell-1}})!}.$$

In the product, we make the convention that $a_{i_0} = 0$ and $a_{i_{j+1}} = n$.

(b) Let $A = (A_{ij})_{i,j=0,\ldots,m}$ be the matrix

$$A_{ij} = \begin{cases} \frac{1}{(a_{j+1}-a_i)!} & \text{if } j+1 \geq i \\ 0 & \text{otherwise,} \end{cases}$$

where $a_0 = 0$ and $a_{m+1} = n$. Prove that the summation in part (a) is equal to $n! \det(A)$.