Enumeration Comprehensive Examination

MC 5479, 1:00 – 4:00 pm, Thursday June 13, 2019

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There are six questions, worth a total of 75 points. Answer as many questions as possible. Solutions will be evaluated based on correctness, completeness, and quality of explanation. In case of an incomplete answer, a precise description of any gaps is preferred.

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1. [12 pts.] For a positive integer j and an integer partition $\lambda \vdash n$, let $m_j(\lambda)$ denote the multiplicity with which j occurs as a part of λ. Fix a positive integer p, and let

$$a_p(\lambda) = |\{j : \ m_j(\lambda) = p\}|$$

be the number of different sizes of parts which occur exactly p times in λ.

For $n \in \mathbb{N}$, let $M_p(n) = \sum_{\lambda \vdash n} m_p(\lambda)$ and $A_p(n) = \sum_{\lambda \vdash n} a_p(\lambda)$.

   (a) Show that

$$\sum_{n=0}^{\infty} M_p(n) x^n = \frac{x^p}{1-x^p} \prod_{j=1}^{\infty} \frac{1}{1-x^j}.$$

   (b) Show that for all $n \geq p$, $M_p(n) = \sum_{i=0}^{p} A_p(n-i)$.

2. [12 pts.] Recall that a plane planted tree (PPT) has a root node, and every node has a (possibly empty) sequence of child nodes, ordered from left to right. A PPT is *even* if every node has an even number of children.

   (a) Show that the number of even PPTs with $2n+1$ nodes is $\frac{1}{2n+1}\binom{3n}{n}$.

   (b) Show that the average degree of the root node, among all even PPTs with $2n+1$ nodes, is $4n/(n+1)$.

3. [11 pts.] Let $\mathcal{S}_{2n}$ be the set of permutations of $[2n] = \{1, 2, ..., 2n\}$. Let

$$\mathcal{J}_n = \{\sigma \in \mathcal{S}_{2n} \mid \sigma(i) \not\equiv \sigma(i+n) \bmod n, \ \text{for all } i \in [n]\}.$$

Prove that

$$|\mathcal{J}_n| = \sum_{k=0}^{n} \frac{(-2)^k (n!)^2 (2n-2k)!}{k!(n-k)!}.$$

4. [10 pts.] A permutation σ of $[n]$ is 231-*avoiding* if there do not exist three indices $1 \leq h < i < j \leq n$ with $\sigma(i) > \sigma(h) > \sigma(j)$. By any method you choose, determine the number of 231-avoiding permutations of $[n]$ for each $n \in \mathbb{N}$.

5. [10 pts.] Let $g_{p,q,r}$ denote the number of simple graphs with vertex set $[p]$ that have q edges and r components. Let $H(x,t) \in \mathbb{Q}[t][[x]]$ be the unique series such that $H(0,t) = 1$ and $\frac{\partial}{\partial x} H(x,t) = H(xt,t)$.

   Prove that
$$g_{p,q,r} = \left[\frac{x^p y^q z^r}{p!}\right] H(x, y+1)^z \,.$$

6. [20 pts.]

   (a) On a finite set X, a $\mathcal{Q}$-structure is a pair (γ, S) such that γ is a cyclic permutation of X and S is a (possibly empty) subset of X such that if $v \in S$, then $\gamma(v) \notin S$. Show that the bivariate generating series for $\mathcal{Q}$-structures is
$$Q(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{(\gamma,S)\in\mathcal{Q}_n} y^{|S|} = \log\left(\frac{1}{1-x-x^2y}\right).$$

   (b) Let $\mathcal{Z}$ be the class of rooted labelled trees (RLTs) in which

   - each vertex has at most two children, and
   - if vertex v has two children, then the children of v each have at most one child.

   Show that the generating series $Z(x) = \sum_{n=0}^{\infty} |\mathcal{Z}_n|\, x^n/n!$ of $\mathcal{Z}$ satisfies the equation
$$x^3 Z^2 + 2(x^3 + x - 1)Z + x(x^2+2) = 0\,.$$

   (c) Consider the class $\mathcal{M}$ of endofunctions $\phi : X \to X$ such that

   - for all $v \in X$, $|\phi^{-1}(v)| \leq 2$, and
   - if $|\phi^{-1}(v)| = 2$ and $w = \phi(v)$, then $\phi^{-1}(w) = \{v\}$.

   From parts (a) and (b), or otherwise, obtain a formula for the generating series $M(x) = \sum_{n=0}^{\infty} |\mathcal{M}_n|\, x^n/n!$.