Enumeration Comprehensive Examination

13:00 – 16:00, Thursday June 18, 2020

Kevin Purbhoo and David Wagner, examiners

There are 5 questions, worth a total of 100 points. Answer as many questions as possible. Solutions will be evaluated based on correctness, completeness, and quality of explanation. In case of an incomplete answer, a precise description of any gaps is preferred.

---

1. Prove the following identities.

   (a) $$\prod_{j=1}^{\infty}(1-x^j y)^{-1} = 1 + \sum_{k=1}^{\infty} \frac{x^{k^2} y^k}{\prod_{i=1}^{k}(1-x^i)(1-x^i y)}.$$ [8]

   (b) $$\prod_{j=1}^{\infty}(1+x^{2j-1} y) = 1 + \sum_{k=1}^{\infty} \frac{x^{k^2} y^k}{\prod_{i=1}^{k}(1-x^{2i})}.$$ [8]

2. (a) Let α and x be indeterminates. Find a formal power series $f(y)$ such that $f(xe^{-x}) = e^{\alpha x}$. [8]
   (Hint: $y = xe^{-x}$ implicitly determines x as a power series in y.)

   (b) Let β be another indeterminate. From part (a) or otherwise, prove that [9]

   $$(\alpha+\beta)(n+\alpha+\beta)^{n-1} = \alpha\beta \sum_{k=0}^{n} \binom{n}{k} (k+\alpha)^{k-1}(n-k+\beta)^{n-k-1}.$$

3. Let $\mathcal{F}_n$ denote the set of all endofunctions $\phi : [n] \to [n]$ of the set $[n] = \{1, 2, ..., n\}$.

   (a) For $\phi \in \mathcal{F}_n$, let $B(\phi) = \{1, \phi(1), \phi(\phi(1)), ...\} \subseteq [n]$, and let $b(\phi) = |B(\phi)|$. Prove that for integers $1 \le j \le n$, the number of endofunctions $\phi \in \mathcal{F}_n$ with $b(\phi) = j$ is [7]

   $$\binom{n-1}{j-1} \cdot j! \cdot n^{n-j}.$$

   (b) For $\phi \in \mathcal{F}_n$, let $\mathrm{fix}(\phi) = \{v \in [n] : \phi(v) = v\}$ and $p(\phi) = |\mathrm{fix}(\phi)|$. Consider the bivariate generating series [8]

   $$F(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{\phi \in \mathcal{F}_n} y^{p(\phi)}.$$

   Let $f(x) = \sum_{n\ge 1} n^{n-1}\frac{x^n}{n!}$. Find an expression for $F(x,y)$ in terms $f(x)$.

(c) For $n \geq 1$, the average value of $p(\phi)$ among all endofunctions $\phi \in \mathcal{F}_n$ is 1. (To see this, each endofunction ϕ is defined on n points, each of which is in $\text{fix}(\phi)$ with probability $1/n$.) Compute the average value of $p(\phi)^2$ among all endofunctions $\phi \in \mathcal{F}_n$. [10]

4. (a) Let $c_{n,k}$ denote the number of cycles of length k in the complete graph K_n. (Note that we have $c_{n,k} = 0$ if $k \leq 2$.) Obtain a closed formula for the generating series [8]

$$C(x, y) = \sum_{n,k \geq 0} c_{n,k} \frac{x^n y^k}{n!} .$$

   (b) Let $\mathcal{G}_n$ be the set of all graphs with vertex set $[n]$. Suppose a graph G is chosen uniformly at random from $\mathcal{G}_n$. Prove that the expected number of cycles in G is $\left[\frac{x^n}{n!}\right] C(x, \frac{1}{2})$. [8]

5. As usual, for $n \geq \mathbb{N}$, let $(2n-1)!! = \prod_{i=1}^{n}(2i-1)$. For a series of the form $A(x) = \sum_{n \geq 0} a_n x^n$, we let $\delta_x A(x) = a_0 + \sum_{n \geq 1} a_{2n}(2n-1)!!$, whenever this latter sum is formally defined.

Note that the coefficients a_n above may be either constants or power series in variables other than x. For example, $\delta_x(1 + xy + x^2y^2) = 1 + 3y^2$.

   (a) Let $\mathcal{M}_{2n}$ be the set of all perfect matchings in the complete graph K_{2n}. Prove that $|\mathcal{M}_{2n}| = \delta_x(x^{2n})$. [4]

   (b) Fix a matching $M_0 \in \mathcal{M}_{2n}$. Using the Principle of Inclusion-Exclusion, or otherwise, prove that the number of matchings $M \in \mathcal{M}_{2n}$ such that $M \cap M_0 = \emptyset$ is $\delta_x\big((x^2 - 1)^n\big)$. [10]

   (c) Prove that the number of ordered triples $(M_1, M_2, M_3) \in \mathcal{M}_{2n}^3$ such that $M_1 \cap M_2 = M_1 \cap M_3 = M_2 \cap M_3 = \emptyset$ is [12]

$$\left[\frac{t^{2n}}{2n!}\right] \delta_x \delta_y \delta_z \exp\left(txyz + \tfrac{1}{2}t^2(2 - x^2 - y^2 - z^2)\right) .$$