UW C&O Enumeration Comprehensive Examination

1:00pm – 4:00pm, June 4, 2025

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Each question is equally weighted. Answer as many questions as possible. Solutions will be evaluated based on correctness, completeness, and quality of explanation. In case of an incomplete answer, a precise description of any gaps is preferred.

1. Let $\mathcal{N}$ be the set of all finite sequence $\gamma = (n_1, \dots, n_k)$ of non-negative integers (also called "weak compositions") and define the following statistics:

   - the *size* $|\gamma| = n_1 + \cdots + n_k$,
   - the *length* $\ell(\gamma) = k$, and
   - the *degeneracy* $d(\gamma) = \big|\{i : c_i = 0\}\big|$, which tracks how many indices equal 0.

   For instance, if $\gamma = (0, 1, 2, 0, 3)$ then $|\gamma| = 6$ while $\ell(\gamma) = 5$ and $d(\gamma) = 2$.

   (a) Obtain a rational function expression for the generating function

   $$F(x, y, z) = \sum_{\gamma \in \mathcal{N}} x^{|\gamma|} y^{\ell(\gamma)} z^{d(\gamma)}.$$

   (b) Consider setting the possible subsets of $\{x, y, z\}$ equal to 1 to get eight specializations $F(x, y, z), F(1, y, z), F(x, 1, z), \dots, F(1, 1, 1)$ of this generating function. Some of these specifications make sense combinatorially and correspond to generating functions tracking some of the parameters, while some do not make sense combinatorially (i.e., are not valid as formal power series). Determine which of the specializations make sense combinatorially and explain what goes wrong for the ones that don't.

2. Prove that for all nonnegative integers n,

   $$\begin{bmatrix} 2n \\ n \end{bmatrix}_q = \sum_{k=0}^{n} q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q^2.$$

3. A *Motzkin path* is a lattice path on the step set $\nearrow$, $\searrow$, $\rightarrow$ starting at $(0, 0)$, ending on the x-axis, and never going strictly below the x-axis. Let m_n be the number of Motzkin paths of length n (often called the nth Motzkin number). It is known that

   $$m_n = \sum_{0 \le k \le \frac{n}{2}} \binom{n}{2k} \mathrm{Cat}_k \tag{1}$$

   for $n \ge 0$, where $\mathrm{Cat}_k = \frac{1}{k+1}\binom{2k}{k}$ is the kth Catalan number.

(a) Give a bijective proof of (1).

(b) Give an algebraic (or generating series based) proof of (1).

4. For a permutation $\sigma \in S_n$ and $i \in \{1, \ldots, n-1\}$ we say that i is a *descent* of σ if $\sigma(i) > \sigma(i+1)$. The set of all descents of σ is denoted $\mathsf{Des}(\sigma)$.

   (a) Let $w_{n,k}$ be the number of pairs (σ, α) where $\sigma \in S_n$ and α is a k-subset of $\mathsf{Des}(\sigma)$. Prove that
   $$\sum_{n,k \geq 0} w_{n,k} \frac{x^n t^k}{n!} = \left(1 - \frac{e^{xt} - 1}{t}\right)^{-1}.$$

   (b) Prove, using part (a) or otherwise, that the number of permutations in S_n with exactly k descents is
   $$n![x^n t^k] \left(1 - \frac{e^{x(t-1)} - 1}{t-1}\right)^{-1}.$$

5. Given a rooted labelled tree T, let $c_k(T)$ be the number of vertices with exactly $k \in \mathbb{N}$ children. Prove that the average value of $c_k(T)$ among the n^{n-1} rooted labelled trees with n vertices using the labels $\{1, \ldots, n\}$ is
   $$\frac{n^2}{(n-1)^{k+1}} \binom{n-1}{k} \left(1 - \frac{1}{n}\right)^n.$$

6. You live in a house that has a *devil's staircase* with m steps, which you attempt to climb. At each time unit, you either succeed in climbing up one step or fall back all the way down to the bottom (the bottom counts as the first step). On the top step, you always stumble and fall back down to the bottom.

   (a) Describe this scenario as a weighted walk on a graph, and give the $m \times m$ weighted adjacency matrix A.

   (b) When $m = 3$ give the rational generating function describing the number of walks of length n that start and end at the vertex corresponding to the bottom of the stairs (i.e., give the generating function for the number of ways to move n times and start and end at the bottom of the stairs).

   (c) Prove that the eigenvalues λ of A are given by the equation
   $$0 = \lambda^m - \lambda^{m-1} - \cdots - \lambda - 1. \tag{2}$$

   Hint: You might want to try computing with a general eigenvector.

   (d) It follows from the *Perron–Frobenius theorem* that A has a unique largest positive eigenvalue ρ and that all other eigenvalues of A have modulus strictly less than ρ (you do **not** need to prove this). Prove that there exists a constant C (that can depend on m) such that the number w_n of ways to move n times and start and end at the bottom of the stairs has asymptotic behaviour $w_n \sim C\rho^n$.