GRAPH THEORY COMPREHENSIVE SPRING 2001

INSTRUCTIONS. *Attempt all questions. All questions have equal weight. A complete solution to a single question is worth more than partial solutions to several questions.*

QUESTION 1.

(a) Let G be a simple graph such that, if $u \in V(G)$, then $deg_G(u) \geq |V(G)|/2$. Give a self-contained proof that G has a Hamiltonian cycle.

(b) Let G be a graph in which every vertex has even degree. Prove that $E(G)$ partitions into the edge-sets of edge-disjoint cycles of G.

(c) Prove that every connected graph in which every vertex has even degree has an Euler tour.

QUESTION 2.
The chromatic number of a graph G is denoted $\chi(G)$. A graph G is *k-critical* if $\chi(G) = k$ but if H is a proper subgraph of G, then $\chi(H) < k$.

(a) Prove that if G is k-critical, then every vertex of G has degree at least $k - 1$.

(b) Prove that if $\chi(G) \geq k$, then G contains a k-critical subgraph.

The following parts lead to a proof of Brooks' Theorem. Either prove the parts below or give a self-contained proof of:

Brooks' Theorem. *Let G be a connected graph with maximum degree Δ. Then at least one of the following holds: $G = K_{\Delta+1}$; or G is an odd cycle; or $\chi(G) \leq \Delta$.*

(c) Prove that if G is k-critical, then G is 2-connected.

(d) Suppose G is k-critical and has subgraphs H and K, each containing a vertex not in the other, such that $G = H \cup K$ and $H \cap K$ consists just of the two vertices u and v. Prove that one, say X, of H and K is such that $\chi(X + uv) = k$ and the other, say Y, is such that $\chi((Y + uv)/uv) = k$. (Note that if e is an edge of a graph L, then L/e is L with e contracted.)

(e) Suppose G is 3-connected and not complete. Prove there is an ordering of its vertices as $v_1, v_2, \ldots, v_n$ such that $v_1 v_2 \notin E(G)$, $v_1 v_n, v_2 v_n \in E(G)$, and, for $j = 3, 4, \ldots, n-1$, v_j is adjacent to at least one vertex among $v_{j+1}, v_{j+2}, \ldots, v_n$.

(f) Prove Brooks' Theorem.

QUESTION 3.

(a) Let M be a matching in a graph G, and let C be a cycle in G of length $2k + 1$ for some integer $k \geq 1$. Suppose C contains exactly k edges of M, and has one vertex x that is not incident with an edge of M. Prove that M is maximum in G if and only if M' is maximum in G', where $M' = M \setminus E(C)$ and G' is the graph obtained from G by contracting the edges of C.

(b) Give an example to show that the existence of x is necessary for the previous statement.

(c) Prove Tutte's Theorem for matchings in graphs.

QUESTION 4.

(a) This question is about **Menger's Theorem:** *Let a and b be distinct non-adjacent vertices in a graph G, and let k denote the minimum size of a vertex-cut of G that*

separates a and b. Then the size of the largest set of vertex-disjoint paths joining a and b in G is k.

Complete the following steps (i)–(vi) leading to a proof of the above theorem OR state and give a self-contained proof of some version of Menger's Theorem.

(i) Prove the theorem for $k = 0$ and $k = 1$.

Suppose the theorem is not true, and let $k \geq 2$ be smallest such that there exists a counterexample. Assume that G is the smallest counterexample for this value of k.

(ii) Prove that if G contains a vertex x adjacent to both a and b, then G cannot be a counterexample.

(iii) Suppose W is a vertex cut of G separating a and b of size k, such that neither a nor b is adjacent to all vertices of W. Define G_a to be the graph obtained by replacing the component of $G - W$ containing a by a single vertex a_0, and joining a_0 to all of W. Prove that there are k vertex-disjoint paths from a_0 to b in G_a.

(iv) Prove that if W exists as in (iii), then G cannot be a counterexample.

(v) We may therefore assume that any vertex cut W of size k that separates a and b is such that either a or b is adjacent to all of W. Let $a x_1 x_2 \ldots b$ be a shortest path joining a and b in G. Prove that the graph $G - \{x_1 x_2\}$ obtained from G by removing the edge $x_1 x_2$ has a vertex cut of size $k - 1$ that separates a and b.

(vi) Prove Menger's Theorem as stated.

(b) Use Menger's Theorem to prove König's Theorem on matchings and covers in bipartite graphs.

(c) Use König's Theorem to prove Hall's Theorem on matchings in bipartite graphs.

QUESTION 5. Prove Turán's Theorem: *For each pair (n, r) of integers $n \geq r \geq 2$, there is a graph $T(n, r)$ on n vertices such that if G is a graph on n vertices not containing K_{r+1} as a subgraph, then $|E(G)| \leq |E(T(n, r))|$, with equality if and only if $G = T(n, r)$.*

QUESTION 6.
(a) Let G be a connected graph with maximum degree Δ. Prove that Δ is an eigenvalue of the adjacency matrix A of G if and only if G is Δ-regular.
(b) Prove that if Δ is an eigenvalue of A, then its multiplicity is 1.
(c) Prove that if $-\Delta$ is an eigenvalue of A, then G is bipartite.
(d) Suppose G is Δ-regular and has the property that, for some $a \geq 0$ and $b \geq 1$, every pair of adjacent vertices have exactly a common neighbours, and every pair of nonadjacent vertices have exactly b common neighbours. Prove that $A^2 + (b - a)A + (b - \Delta)I = cJ$, where I denotes the identity matrix and J the all-1's matrix. Use this to obtain a formula satisfied by all other eigenvalues of A (other than Δ).