Graph Theory Comprehensive Examination

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1. We construct a circle graph on $2n$ vertices as follows. Draw the cycle C_{2n} in the plane in the usual way. Now divide the $2n$ vertices into n pairwise disjoint pairs, and join the vertices in each pair by an edge. We call this set of n edges the *chords* of the circle graph. The *intersection graph* of the circle graph is the graph with the n chords as vertices, and two chords are adjacent if they are overlapping bridges of the original cycle. (In other words, if we draw them as straight lines, then they cross.)

   (a) Prove that if the intersection graph of a circle graph is bipartite, then the circle graph is planar.

   (b) Prove that if the intersection graph of a circle graph is not bipartite, the circle graph is not planar.

2. Let G be a connected regular graph and let $L(G)$ denote its line graph.

   (a) Let B be the incidence matrix of G. Express the adjacency matrices of G and $L(G)$ in terms of the matrices BB^T and B^TB and the identity matrix.

   (b) If x is an eigenvector of BB^T with eigenvalue λ, prove that B^Tx is an eigenvector of B^TB with eigenvalue λ. If y is an eigenvector of B^TB with eigenvalue λ, prove that By is an eigenvector of BB^T with eigenvalue λ.

   (c) Prove that λ is a non-zero eigenvalue of BB^T with multiplicity m if and only if it is a non-zero eigenvalue of B^TB with multiplicity m.

   (d) Use the result of (c) (whether or not you prove it) to express the eigenvalues of $L(G)$ in terms of the eigenvalues of G.

   (e) Given that K_5 has -1 as an eigenvalue with multiplicity four, determine the eigenvalues of $L(K_5)$.

3. (a) Let D be a directed graph with n vertices, with no loops or multiple arcs. Suppose that for each vertex x, $d^-(x) \geq \frac{n}{2}$ and $d^+(x) \geq \frac{n}{2}$. Prove that D has a directed Hamilton cycle.

   (b) Let D be a strongly connected (diconnected) directed graph whose underlying graph contains an odd cycle. Prove that D contains a directed odd cycle.

4. A graph G is vertex critical if $\chi(H) < \chi(G)$ for each proper subgraph of G.

   (a) Prove that a vertex-critical graph is a block.

   (b) Prove that if G is vertex critical and $\{u, v\}$ is a vertex cut in G, then u is not adjacent to v.

   (c) Suppose G is vertex critical with chromatic number k and $\{u, v\}$ is a vertex cut. Show that G is the *2-sum* of graphs G_1 and G_2, where one of G_1 and G_2 is vertex critical with chromatic number k.

   We say G is the 2-sum of graphs H_1 and H_2 if

   (i) $V(H_1) \cup V(H_2) = V(G)$.

   (ii) $V(H_1) \cap V(H_2) = \{u, v\}$.

   (iii) The vertices u and v are adjacent in H_1 and H_2.

   (iv) $E(G) = (E(H_1) \cup E(H_2)) \setminus \{uv\}$.

5. (a) State the vertex version of Menger's theorem, and give a self-contained proof.

   (b) Let $a, b, x_1 \cdots x_k$ be distinct vertices in a $(k+1)$-connected graph. Prove that there is a path from a to b that contains all the points $x_1 \cdots x_k$. (You may assume Menger's Theorem.)

6. (a) State and prove Hall's theorem for matchings in bipartite graphs.

   (b) Let G be a connected graph such that for each $x \in V(G)$, $\nu(G-x) = \nu(G)$ (here $\nu(H)$ denotes the size of the largest matching in H). Prove that G is factor-critical, that is, $G-x$ has a perfect matching for every vertex x of G. [Hint: first show that $\nu(G-x-y) < \nu(G)$ for every pair of vertices $x \neq y$]