Graph Theory First Stage Comprehensive June 2006

There are six questions. All questions have equal value.

Question 1. Let T be a strongly-connected tournament on n vertices. Prove that, for every integer k with $3 \leq k \leq n$, there is a directed circuit in T of length k.

Question 2. Let A be the adjacency matrix of a graph G and let ρ be the spectral radius of A. Prove the following are equivalent:

(1) G is bipartite;
(2) $-\rho$ is an eigenvalue of A;
(3) the spectrum of A is symmetric about the origin, i.e., λ is an eigenvalue of A if and only if $-\lambda$ is an eigenvalue of A.

Question 3. State and give a self-contained proof of Hall's Bipartite Matching Theorem.

Question 4. State and give a self-contained proof of Brooks' Vertex Colouring Theorem.

Question 5. Prove Ramsey's Theorem: for any positive integers r, s, there is a positive integer $N = N(r,s)$, for every $n \geq N$, any r-colouring of the edges of K_n contains a monochromatic K_s.

Question 6. Let H be a subgraph of a graph G. A walk in G is *H-avoiding* if at most its endvertices are in H (so no other vertex and no edge – even if the walk has length 1 – can be in H). Define the relation $\sim$ on $E(G) \setminus E(H)$ by: $e \sim f$ if there is an H-avoiding walk in G containing both e and f.

[TERMINOLOGY: *A* walk $(v_0, e_1, v_1, \ldots, e_k, v_k)$ *is an alternating sequence of vertices* v_i *and edges* e_i *such that, for* $i = 1, 2, \ldots, k$, e_i *has ends* v_{i-1} *and* v_i. *Its* ends *are* v_0 *and* v_k. *The walk is a* path *if* $v_0, v_1, \ldots, v_k$ *are distinct, and it is a* cycle *if* $v_1, \ldots, v_k$ *are distinct and* $e_1, e_2, \ldots, e_k$ *are distinct.*]

(1) Prove that $\sim$ is an equivalence relation.
(2) Prove that if $e \sim f$, then either there is an H-avoiding path containing both e and f, or e and f are incident with the same vertex of H and there is an H-avoiding cycle containing both e and f.
(3) Prove that $e \sim f$ if and only if either $e = f$ or e and f each have at least one end not in H and such ends are in the same component of $G - V(H)$.