Graph Theory Comprehensive

EXAMINERS:	Jim Geelen and Bruce Richter.
TIME/DATE:	1-4pm, July 3, 2007
INSTRUCTIONS:	Attempt all six questions.

Question 1. (Planarity.) Prove that: *every simple planar graph admits a straight-line embedding in the plane.*

HINT: Consider triangulations.

NOTE: This question is not intended as a test of planar point-set topology. You may freely assume, for instance, that each edge in a planar embedding lies in a disc that meets the graph in only the end-points of the edge.

Question 2. (Connectivity.) Let G be a graph and let A and B be subsets of $V(G)$. An *AB-path* is a path in G with one end in A and the other end in B. An *AB-separator* is a set W of vertices so that $G - W$ has no AB-path. Give a self-contained proof of the following form of Menger's Theorem:

Let G be a graph and let A and B be subsets of $V(G)$. If the size of a smallest AB-separator is k, then G contains k totally disjoint AB-paths.

Question 3. (Ramsey's Theorem.) A *tournament* is a directed graph D in which, for any two vertices u, v of D, precisely one of the two directed edges (u, v) and (v, u) occurs in D. Give a self-contained proof of the following form of Ramsey's Theorem:

For any positive integer k, there is an integer N so that if D is a tournament on n vertices and $n \geq N$, then D contains the tournament on k vertices having no directed circuits.

Question 4. (Transitive graphs.) Prove that: *if a connected graph is edge-transitive but not vertex-transitive, then it is bipartite.*

Question 5. (Matchings.) For a graph G, let $\nu(G)$ denote the size of a maximum matching in G and let $D(G)$ denote the set of all vertices v of G such that $\nu(G-v) = \nu(G)$. For distinct vertices $u, v \in V(G)$, we write $u \sim v$ if $\nu(G - \{u, v\}) < \nu(G)$.

(a) Prove that $\sim$ is transitive. (That is, if u, v, and w are distinct vertices in G such that $u \sim v$ and $v \sim w$, then $u \sim w$.)

(b) Prove that: *every connected vertex-transitive graph with an even number of vertices has a perfect matching.*

Question 6. (Flows.)

(a) Prove that: *Every* 3*-connected graph with at least* 4*-vertices has three spanning trees such that no edge is contained in all three.*

You may use, without proof, the fact that: *every* 3*-connected graph with at least* 5 *vertices has an edge whose contraction maintains* 3*-connectivity.*

(b) Prove that: *If T is a spanning tree of an oriented graph G, then there exists a* 2*-flow in G such that each edge in $E(G) - E(T)$ receives non-zero flow.*

(c) Prove that: *Every* 2*-edge-connected graph admits a nowhere-zero* 8*-flow.*