C&O — GRAPH THEORY

COMPREHENSIVE EXAM — Summer 2008

MC 5045, Monday, July 7, 2008, 1:00 – 4:00 (**3 hours**)

Examiners: Chris Godsil and Bruce Richter

1. Let V be the vector space of dimension d over the 2-element field with standard basis $e_1, \ldots, e_d$. Define

$$e_0 = e_1 + \cdots + e_d$$

and let $X(d)$ be the Cayley graph relative to the connection set

$$C := \{e_0, e_1, \ldots, e_d\}.$$

   (a) Show that $X(4)$ is strongly regular and determine its parameters (v, k, a, c) and its eigenvalues.

   (b) Determine if $X(d)$ is strongly regular when $d > 4$.

2. Let G be a simple graph with n vertices and m edges.

   (a) Prove that if G has no subgraph isomorphic to the complete graph K_3, then

$$m \leq \left\lfloor \frac{n^2}{4} \right\rfloor .$$

   (b) Furthermore, show that the only example having $\lfloor \frac{n^2}{4} \rfloor$ edges is the complete bipartite graph

$$K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil} .$$

3. König's Lemma states that: *for a bipartite graph G, the size of a smallest cover of G is the size of a largest matching in G.* Give a self-contained proof of this.

   Terminology: a *cover of G* is a set C of vertices of G so that every edge of G is incident with at least one vertex in C.

4. A graph G is *critical* if, for each $H \subsetneq G$, $\chi(H) < \chi(G)$.

   (a) Prove that every critical graph is 2-connected.

   (b) Suppose G is a critical graph and that G has edge disjoint subgraphs H and K so that $G = H \cup K$ and $H \cap K$ consists of precisely two vertices u and v. Show (i) u and v are not adjacent in G and (ii) either both H/uv and $K + uv$ are critical or both $H + uv$ and K/uv are critical. (The graph H/uv is the graph obtained from H by identifying the vertices u and v.)

   (c) Suppose G is 3-connected. Prove Brooks' Theorem: either G is a complete graph or $\chi(G) \leq \Delta(G)$.

5. Terminology: Let T be a triangulation of the plane. Then T is *simple* if T has no loops or parallel edges. An edge e of T is *contractible* if e is in precisely two triangles of T. If e is a contractible edge of a simple triangulation T, then T/e is the triangulation obtained from T by contracting e and deleting one edge from each of the two pairs of parallel edges.

   (a) Prove that each simple triangulation of the plane has a contractible edge.

   (b) Deduce that, for every simple triangulation T of G, there is a sequence $T = T_0, T_1, \ldots, T_k = K_4$ of simple triangulations T_i so that, for each $i = 1, 2, \ldots, k$, there is an edge e_i of T_{i-1} so that $T_i = T_{i-1}/e_i$.

6. Terminology: Let G be a graph and let C be a cover of G (see Question ?? for a definition of cover). Let x and y be any two vertices of G. A set P of paths in G is *C-disjoint* if $p, p' \in P$ and $v \in V(p \cap p')$ imply $v \notin C$. A subset D of C is an $\{x, y, C\}$- *cut* if there is no xy-path in $G-D$. (Note that if $x \in C$, say, then $\{x\}$ is an $\{x, y, C\}$-cut.)

   (a) Show that if P is a set of C-disjoint paths, then the paths in P are edge-disjoint.

   (b) Prove that if P is a set of C-disjoint xy-paths and D is an $\{x, y, C\}$-cut, then $|P| \leq |D|$.

   (c) Prove that there is a set P of C-disjoint xy-paths and an $\{x, y, C\}$-cut D so that $|P| = |D|$. Hint: Any proof of Menger's Theorem for vertex-disjoint paths should adapt.