Graph Theory Comprehensive Examination
Summer 2009
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1. (a) Give a self-contained proof of Menger's theorem: for any subsets A and B of the vertex set of a graph G, the maximum cardinality of a set of disjoint paths joining A and B is equal to the minimum cardinality of a vertex cut separating A and B in G.

   (b) Let G be a k-connected graph ($k \geq 2$) that does not have a Hamilton cycle. Prove that G has an independent set of size at least $k+1$. (You may use Menger's theorem in the proof.)

2. Give a self-contained proof of Tutte's theorem: a graph G has a perfect matching if and only if $\text{odd}(G - X) \leq |X|$ for all $X \subseteq V(G)$, where $\text{odd}(H)$ denotes the number of odd components of the graph H.

3. (a) Let $\text{ex}(n, H)$ denote the Turán number for the graph H, that is, the maximum number of edges of a graph G with n vertices that does not contain H as a subgraph. Give a self-contained proof of Turán's Theorem: that $\text{ex}(n, K_r)$ is equal to the number of edges in the *Turán graph* $T_{r-1}(n)$, the $(r-1)$-partite graph with n vertices whose partition classes are as equal as possible in cardinality.

   (b) Let $R_k(H)$ denote the k-coloured Ramsey number for the graph H, that is, the minimum n such that in every k-colouring of the edges of K_n there exists a monochromatic copy of H. Prove that $\text{ex}(R_k(H)-1, H) \geq \frac{1}{k}\binom{R_k(H)-1}{2}$ for every graph H.

4. (a) A graph is *orientable* if its edges can be directed so that it is strongly connected. Let G be a connected graph. Prove that G is orientable if and only if each edge of G is contained in at least one cycle.

   (b) Prove that every 4-edge-connected graph G has a (nowhere-zero) $\mathbb{Z}_4$-flow. You may assume without proof that G has two edge-disjoint spanning trees, but the proof should otherwise be self-contained.

5. (a) Prove that if A is a minor of a graph G and A has maximum degree at most 3 then A is a topological minor of G.

   (b) Prove using (a) that every graph containing K_5 or $K_{3,3}$ as a minor contains either K_5 or $K_{3,3}$ as a topological minor. (Note that K_5 is denoted K^5 in Diestel's book.)

6. Give a self-contained proof of each of the following (except that any part may be used in the later parts).

(a) If H is a component of G then every eigenvalue of H is an eigenvalue of G.

(b) If matrices A and B satisfy $A = B^T B$, where A is real and symmetric and B is real, then A is positive semidefinite.

(c) If L is a line graph then its minimum eigenvalue is at least -2.

(d) If H is an induced subgraph of G then the maximum eigenvalue of H is at most that of G.