GRAPH THEORY COMPREHENSIVE

Friday 10 June 2011, 1-4pm

This examination has two parts. A complete paper consists of all four questions from Part A and any two of the three from Part B.

PART A: attempt all four questions

Question 1. Let G be a simple, connected graph with maximum degree $\Delta(G)$. Prove Brooks' Theorem: if $\Delta(G) \geq 3$, then either G is isomorphic to $K_{\Delta(G)+1}$ or $\chi(G) \leq \Delta(G)$.

Question 2. State and prove Turán's Theorem on the number of edges in a simple graph not having the complete graph K_t as a subgraph.

Question 3. (i) State and prove Hall's Theorem on matchings in a bipartite graph.

(ii) Using Hall's Theorem, prove the following version of Petersen's 2-factor Theorem. Let r be a positive integer. If G is a $2r$-regular simple graph, then G has a 2-regular subgraph.
Hint. You may use, without proof, Euler's Theorem that there is a closed Euler tour if and only if every vertex has even degree.)

Question 4. Let G be a connected simple graph with maximum degree Δ and adjacency matrix A. Prove each of the following.

(i) Δ is an eigenvalue of A if and only if G is Δ-regular;

(ii) If Δ is an eigenvalue of A, then its multiplicity is 1.

(iii) Suppose G is Δ-regular and there exist integers $a \geq 0$ and $b \geq 1$ so that

- every pair of adjacent vertices have exactly a common neighbours, and
- every pair of non-adjacent vertices have exactly b common neighbours.

(a) Prove that $A^2 + (b-a)A + (b-\Delta)I = cJ$, where, respectively, I and J denote the identity and all-1's matrices of the appropriate sizes.

(b) Using (iiia) or otherwise, obtain a formula satisfied by all eigenvalues of A other than Δ.

... over for Part B

PART B: attempt any two of three

Question 5. Let G be a graph that has either K_5 or $K_{3,3}$ as a (contraction and deletion) minor. Prove that G has a subgraph that is a topological minor of either K_5 or $K_{3,3}$. (In a topological minor, an edge may be replaced by a path, and all such paths must be internally disjoint.)

Question 6. Let G be a 3-connected simple graph having at least 5 vertices. Prove that G has an edge e so that G/e (the graph obtained from G by contracting e) is 3-connected.

Remark. We remove parallel edges in obtaining G/e.

Question 7. Let G be a simple graph, with chromatic number $\chi(G)$, and let k be a positive integer. Prove:

$\chi(G) \leq k$ if and only if there is an orientation D of G so that every directed path in D has at most k vertices.

(Hint. To show the existence of D implies $\chi(G) \leq k$, let E_0 be a minimal set of edges so that $D - E_0$ has no directed cycles. Define $c(v)$ to be the number of vertices in a longest directed path in $D - E_0$ beginning at v and show that c defines a colouring of G with at most k colours.)