Graph Theory Comprehensive Examination
Summer 2013
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All graphs will be simple and undirected. If you wish to use the result of a well-known theorem, clearly state the theorem you are using. If you are in any doubt about what needs to be stated, please ask an examiner.

1. A graph is *outerplanar* if it can be drawn in the plane so that there are no crossings and all vertices lie in the unbounded face. Show that a graph is outerplanar if and only if it does not have $K_{2,3}$ or K_4 as a minor. [You may use Kuratowski's theorem without proving it.]

2. (a) Give a self-contained proof of Tutte's theorem: a graph G has a perfect matching if and only if $\text{odd}(G - X) \leq |X|$ for all $X \subseteq V(G)$, where $\text{odd}(H)$ denotes the number of odd components of the graph H.

   (b) Prove that for every $k \geq 1$, every k-regular k-edge-connected graph with an even number of vertices has a perfect matching.

3. Give a self-contained proof of Vizing's theorem: every graph G has an edge colouring with $\Delta(G) + 1$ colours.

4. Give a self-contained proof that every 3-connected graph G with at least five vertices has an edge e such that G/e is 3-connected.

5. (a) Prove that for $s, t \geq 3$

   $$R(s,t) \leq R(s-1,t) + R(s,t-1),$$

   where $R(s,t)$ denotes the usual Ramsey number.

   (b) Prove that $R(s,s) < 2^{2s-2}$ for each $s \geq 2$.

   (c) Prove that $R(s,s) > 2^{s/2}$ for each $s \geq 3$.

6. Let G be the additive group of the finite field $GF(q)$, where $q \equiv 1$ (modulo 4). Let S denote the set of non-zero squares in $GF(q)$. [You may assume without proof that $-1 \in S$.]

   (a) Show that the order of the automorphism group of the Cayley graph X of G with respect to S is at least $q(q-1)/2$, and that the stabilizer of a vertex has exactly three orbits.

   (b) Prove that X is isomorphic to its complement.

   (c) Prove that X is strongly regular.