C&O Comprehensive
Graph Theory

Thursday 18 June, 1-4PM

Examiners: Jane Gao and Peter Nelson

Instructions: A complete examination consists of solutions to all six questions. All questions have equal value and, in multipart questions, all parts have equal value. All proofs should be derived from first principles unless indicated otherwise. Where assumptions are allowed, **clearly state any assumptions you make**. All graphs are simple.

1. State and prove Brooks' theorem.

2. Prove that if G is a 3-connected graph and $|V(G)| \geq 5$, then G has an edge e such that G/e is 3-connected.

3. Prove Tutte's theorem: a graph G has a perfect matching if and only if, for each set S of vertices of G, the graph $G - S$ has at most $|S|$ components that have an odd number of vertices.

4. For integers $a, b \geq 2$, let $R(a, b)$ denote the minimum integer n such that every colouring of the edges of K_n with the colours red and blue has either a red K_a-subgraph or a blue K_b-subgraph.
   (a) Prove that $R(a, b) \leq \binom{a+b-2}{a-1}$.
   (b) Prove that if $0 \leq p \leq 1$, then $R(a, b) \geq \min(\frac{a}{4}p^{-(a-1)/2}, \frac{b}{4}(1-p)^{-(b-1)/2})$.
   (Hint: Colour each edge of a complete graph independently with probability p. You may use the facts that $\mathrm{e}^2 < 8$ and that $\binom{m}{k} \leq (\mathrm{e}m/k)^k$ for all $1 \leq k \leq m$.)

5. Prove that every graph with a Hamilton cycle has a nowhere-zero $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-flow.

6. Let G be a connected k-regular graph that is not a complete graph. Prove that G has exactly three distinct eigenvalues if and only if G is strongly regular.

   You may use the following two assumptions without proof:

   - For each connected k-regular graph H, the integer k is an eigenvalue of H with multiplicity 1.
   - For every real symmetric $n \times n$ matrix A, there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A.