C&O Comprehensive
Graph Theory

Wednesday 4 June, 1:00-4:00 PM

Examiners: Jim Geelen and Penny Haxell

Instructions: A complete examination consists of solutions to all six questions. All questions have equal value. All proofs should be derived from first principles unless indicated otherwise. Where assumptions are allowed, **clearly state any assumptions you make**. All graphs are simple. If loops and multiple edges are to be considered, the term *multigraph* is used.

1. (a) State and prove Tutte's theorem on perfect matchings in graphs.
   (b) Prove that in any cubic graph with no cut edge, each edge belongs to a perfect matching.

2. State and prove Brooks' Theorem on the chromatic number and the maximum degree in graphs.

3. Recall that a k-regular graph with n vertices is called (n, k, a, c)-strongly regular if every pair of adjacent vertices have exactly a common neighbours and every pair of nonadjacent vertices have exactly c common neighbours.

   Prove that if $k \geq 2$ and G is a $(k^2 + 1, k, 0, 1)$-strongly regular graph then k is one of a set of four possible values, and find these values. You may assume without proof that k is an eigenvalue of the adjacency matrix of G of multiplicity 1, as well as basic results from linear algebra.

4. For any graph G and integer $k \geq 1$, prove that, if G has a nowhere-zero flow over the group $\mathbb{Z}_k$, then G has a nowhere-zero k-flow.

5. For a positive integer k, prove that, if G is a k-connected graph with at least three vertices and with no independent set of size greater than k, then G has a Hamilton cycle.

6. Let C be a cycle in a 3-connected planar embedding G. Prove that C bounds a face in G if and only if C is induced and $G - V(C)$ is connected.