GRAPH THEORY COMPREHENSIVE

11 June 2014, 1-4pm

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Instructions: A complete examination consists of solutions to all six questions. All questions have equal value and, in multipart questions, all parts have equal value. All proofs should be derived from first principles unless stated otherwise.

Problem 1. Let u and v be distinct vertices in a graph G and let k be a positive integer. Prove Menger's Theorem that: either there are k pairwise edge-disjoint uv-paths in G or there is a set C of edges of G so that $|C| < k$ and u and v are in different components of $G - C$.

Problem 2. A *tournament* is a directed graph D so that, for any distinct vertices u and v, precisely one of uv and vu is an edge of D. A directed graph is *acyclic* if it has no directed cycles.

Prove Ramsey's Theorem that: for each positive integer k there is a positive integer $n(k)$ such that, if D is any tournament with at least $n(k)$ vertices, then D contains an acyclic subtournament with k vertices.

Problem 3. State and prove Tutte's perfect matching theorem.

Problem 4. (a) Let G be a graph with no isolated vertices. Prove that if G is edge-transitive but not vertex-transitive, then G is bipartite.

(b) Let G be a bipartite graph with adjacency matrix A and let λ be a real number. Prove that λ is an eigenvalue of A if and only if $-\lambda$ is an eigenvalue of A.

Problem 5. A cycle C in a graph G is *peripheral* if $G - V(C)$ is connected and C has no chords.

Prove that if G is a 3-connected plane graph, then a cycle C of G is a face boundary of G if and only if C is a peripheral cycle. (You may assume the Jordan Curve Theorem.)

Problem 6. Prove that every 3-edge-connected graph has a $(\mathbb{Z}_2 \times \mathbb{Z}_3)$-flow. (You may use the results of the other problems, above, even if you have not proved them.)