Finding 3-connected 2-crossing-critical graphs with V_8 minors and no V_{10} minors.

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Contents

Chapter 1

Introduction

Kuratowski's Theorem states that every non-planar graph contains a subdivision of either $K_{3,3}$ or K_5. In the language of crossing numbers, this means that every graph with positive crossing number contains a subdivision of either $K_{3,3}$ or K_5.

Definition 1.1. The *crossing number* of a graph G is the smallest number of pairwise crossings of edges in a drawing of G on the plane. A graph G is *k-crossing-critical* if it has crossing number at least k and every proper subgraph of G has crossing number strictly less than k.

For example, by Kuratowski's Theorem, there are exactly two graphs that are 1-crossing critical: K_5 and $K_{3,3}$.

Currently the classification of all 2-crossing critical graphs is not complete, but significant advances have been made.

The first time these graphs where studied was in 1981 by Bloom, Kennedy, and Quintas in [2], where they found twenty-one 2-crossing critical graphs. Later Kochol [3] found an infinite family of 3-connected, simple 2-crossing-critical graphs, and Siráň [5] proved that there exists an infinite family of 3-connected n-crossing critical graphs for all $n \geq 3$. In [4] Richter proved that there are eight cubic 2-crossing critical graphs.

This project continues with the work of describing all 2-crossing critical graphs. Previous work by Bokal, Oporowski, Richter, and Salazar has dealt with most kinds of 2-crossing-critical graphs [1].

Definition 1.2. Let $e = (u, v)$ be an edge of graph G with edge set $E(G)$ and vertex set $V(G)$. Then then the *contraction* of e denoted G/e is the graph with edge set $\{(x, y) \mid (x, y) \in E(G) \textit{ and } \{x, y\} \cap \{u, v\} = \emptyset\} \cup \{(e, x) \mid (u, x) \in E(G)\} \cup \{(e, x) \mid (v, x) \in E(G)\}$ and vertex set $V(G) \cup \{e\} - \{u, v\}$.

Definition 1.3. A graph G has a F *minor* if F can be obtained by deleting vertices and by deleting and contracting edges of G.

Definition 1.4. A graph G is *topologically isomorphic* to F if F can be obtained replacing induced paths of G by induced paths of any length, and relabeling vertices. We denote this as $G \cong F$.

The work so for on the characterization of 2-crossing critical graphs is done in terms of the values of n the graphs have a V_{2n} minor. The graph V_{2n} is obtained from the $2n$-cycle $R = (1, 2, \ldots, 2n-1, 2n, 1)$ by adding all the diagonals $(i, i+n)$ to it.

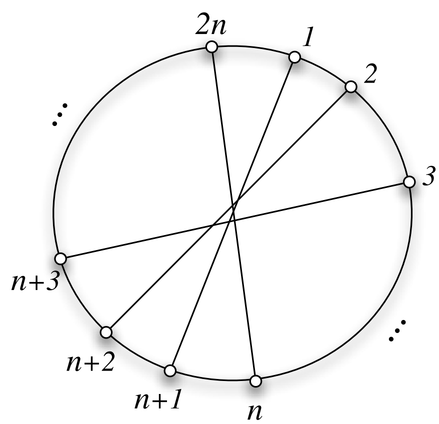

Figure 1.1: Graph V_{2n}.

We can divide 2-crossing critical graphs into the following categories:

1. *2-crossing-critical graphs that are not 3-connected*: In [1] it is shown that there are 49 examples, plus those which arise from 3-connected examples from a certain expansion of pairs of parallel edges. This material will not be discussed in this essay. See [1] for discussion.

2. *3-connected 2-crossing-critical graphs*: These have been broken up into the following cases:

   (a) *Graphs with a V_{10} minor*: These graphs were studied in [1]. It was shown that if G is a 3-connected 2-crossing-critical graph with a V_{10} minor, then G is one of a completely described infinite family of 3-connected 2-crossing-critical graphs. It was also shown that there are a finite number of 3-connected 2-crossing-critical graphs with no V_{10} minor.

   (b) *Graphs with a V_8 minor and with no V_{10} minor*: It was proved it [1] that these graphs can have at most 7 million vertices, thus proving that there is a finite number of them.

(c) *Graphs with no V_8-minor*: In [1] it was shown how to find all 2-crossing-critical graphs that do not have a V_8-minor.

This project begins the work of explicitly finding all 3-connected, 2-crossing-critical graphs that have a V_8-minor but no V_{10}-minor. Using our method we were able to find 326 such graphs. We were provided with a list of 531 3-connected, 2-crossing-critical graphs by Oporowski, of which 201 have a V_8-minor, but no V_{10}-minor.

We describe how adding certain edges to a V_8 can limit where the crossings in a drawing with only one crossing can occur. Adding enough of these edges will therefore result in a graph with crossing number 2.

In this essay, we:

- describe a particular set of edges to add;
- examine how these edges affect the crossing number of our graph, and in particular how they affect particular edges of the graph;
- use the computer to show that 326 graph are found which are 2-crossing-critical.

Definition 1.5. A *branch* of G is a maximal path in G, all of whose internal vertices have degree two.

Definition 1.6. Let H be a subdivision of V_{2n}. The *rim* R of V_{2n} is the eight branches that form the $2n$-cycle rim of the V_{2n} and the *spokes* of H are the n diagonals of the V_{2n}. See Figure 1.1.

The structure of this work is the following. Later in this section we introduce notation that we will be working with throughout the manuscript. The basis of our construction consists in adding edges to $H \cong V_8$ (which has crossing number one) trying to minimally force a second crossing. We study the ways of drawing V_8 with one crossing in Section 2. In Section 3 we introduce the concept of *covered* edges. We observe that adding certain edges to the $H \cong V_8$ forces some rim branches not to be crossed in a 1-drawing of G; we call this *covering a rim branch*. In Section 4 we define green cycles and green edges, and prove that in a graph G such that $V_8 \cong H \subseteq G$, if G has a green cycle C, then C is not crossed in a 1-drawing of G (Theorem 4.7). In Section 5 we specify which green edges we consider for our constructions. We call these green edges *rim configurations*. In Section 6 we talk about combinations of rim configurations that we do not consider, because they do not result in graphs that are are critical for the property of having crossing number two. In Section 7 we describe the computer program that was used to count the graphs we obtained with our construction.

Setup and some notation

We consider different ways of adding edges between vertices, spokes and rim edges of V_8 that will affect a 1-drawing of V_8. If we are attaching one end of the edge to a spoke or rim, then we subdivide the spoke or rim and create a new node where the edge is incident to it.

However, we will not consider all such edges, as not all of them can affect the crossing number of the graph in the case we are considering. The edges we examine are the ones shown in Figure 5.1.

Notation. We will be considering graphs that have a subgraph $H \cong V_8$. The vertices of H corresponding to the vertices of V_8 are, in cyclic order on the rim of H, $(i, i+1, \ldots, i+7, i)$ (always taken modulo 8). See Figure 1.2.

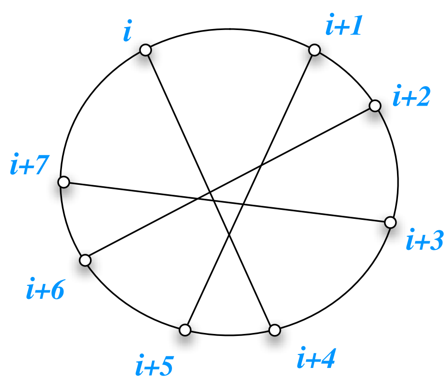

Figure 1.2: A V_8 with rim vertices i, $i+1$, $i+2$, $i+3$, $i+4$, $i+5$, $i+6$, $i+7$, and rim branches $(i, i+1)$, $(i+1, i+2)$, $(i+2, i+3)$, $d = (i+3, i+4)$, $(i+4, i+5)$, $(i+5, i+6)$, $(i+6, i+7)$, $(i+7, i)$.

Chapter 2

1-drawings

In this chapter, we study the 1-drawings of V_8. We show that there are only two 1-drawings of V_8 (Lemma 2.5), and make some additional remarks about which edges can cross each other in 1-drawings.

Auxiliary lemmas

In this section, we prove two general lemmas that help determine the 1-drawings of V_8. They will also prove useful in analyzing which edges can or cannot be crossed in a 1-drawing of a graph obtained by adding edges to a V_8.

Definition 2.1. Let D be a drawing of G, and let S be a subgraph of G. Then we say that $D[S]$ is the drawing of S in D.

Lemma 2.2. *Let G be a graph with vertex-disjoint cycles C_1 and C_2, and let D be a 1-drawing of G. Then $D[C_1] \cap D[C_2] = \emptyset$.*

Proof. Let D be a 1-drawing of G. Suppose that one of $D[C_1]$ or $D[C_2]$ is self-intersecting in D. Then clearly $D[C_1] \cap D[C_2] = \emptyset$, as an intersection would result in a second crossing in D.

Now suppose that $D[C_1]$ and $D[C_2]$ are both non-self-intersecting. By the Jordan Curve Theorem [6], $D[C_1]$ divides the plane into two regions; let f be one of these. Then the directed closed curve defined by traversing $D[C_2]$ once must leave f the same number of times it enters f. Therefore $D[C_1] \cap D[C_2]$ is even. Since D has at most one crossiing, C_1 and C_2 intersect at most once; thus must not intersect at all. □

Observe that using this lemma, in order to prove that the rim branch $(i, i+1)$ is not crossed by another rim branch, $(i+3, i+4)$, $(i+4, i+5)$ or $(i+4, i+5)$, in a 1-drawing D of G it suffices to find a pair of disjoint cycles C_1 and C_2 such that $(i, i+1)$ is in C_1 and the other rim branch is in C_2. By the lemma, $D[C_1] \cap D[C_2] = \emptyset$, therefore the rim branches cannot cross $(i, i+1)$.

Lemma 2.3. *Let G be a graph with a 1-drawing D. If e is an edge of G such that $G - \{e\}$ is not planar then e is not crossed in D.*

Proof. Since $G - \{e\}$ is not planar, there is a crossing in $D[G - \{e\}]$. This must be the only crossing of D, and therefore $D[e]$ is not crossed. □

Corollary 2.4. *Let G be a graph with a 1-drawing D. If e is an edge of G and $G - \{e\}$ has a $K_{3,3}$ minor, then e cannot be crossed in D.*

Proof. $G - \{e\}$ in not planar, so we apply Lemma 2.3. □

1-drawings of V_8

Observe that in drawing D_1 (as shown in Figure 2.1) the rim branch crossing $(i, i+1)$ is $(i+4, i+5)$, and in drawing D_2 (as shown in Figure 2.2) the rim branches crossing $(i, i+1)$ are either $(i+3, i+4)$ or $(i+5, i+6)$. In this section we prove that these are the only rim branches that can be crossed in a 1-drawing of V_8.

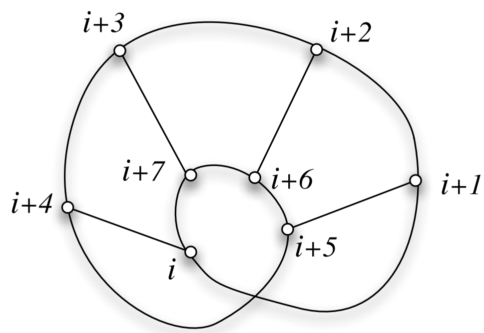

Figure 2.1: D_1

Lemma 2.5. *The only 1-drawings of V_8 are D_1 and D_2, shown in Figure 2.1 and in Figure 2.2 respectively.*

Proof. Clearly any drawing of V_8 has at least one crossing, as it has a $V_6 = K_{3,3}$ minor.

We now examine which edges of the V_8 can be crossed in a 1-drawing: Observe that no spoke S can be crossed in a 1-drawing of V_8, by Corollary 2.4. Therefore the crossing in the 1-drawing must be between two rim branches. Observe that two consecutive rim branches cannot cross as they are incident to a common vertex.

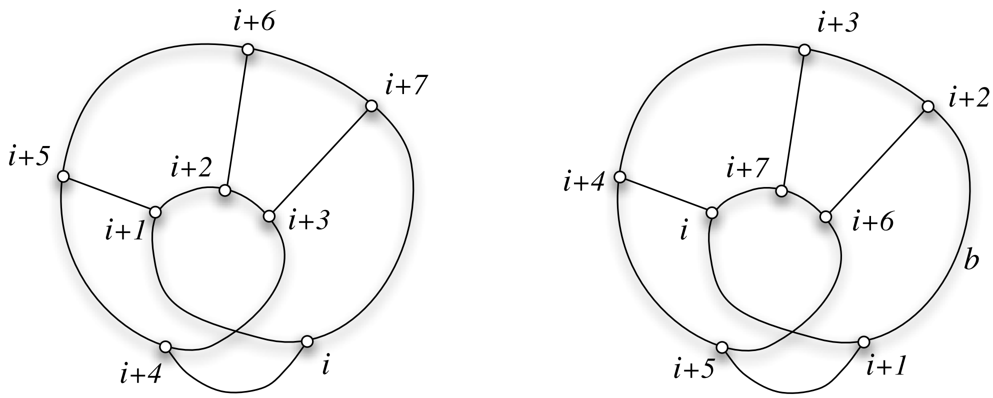

Figure 2.2: D_2

Applying Lemma 2.2 to the disjoint cycles $(i, i+1, i+5, i+4, i)$ and $(i+2, i+3, i+7, i+6, i+2)$, we see that rim branch $(i, i+1)$ cannot cross rim branches $(i+2, i+3)$ and $(i+6, i+7)$.

Therefore the only possible crossing in a 1-drawing must be of the rim branch $(i, i+1)$ with one of $(i+3, i+4)$, $(i+4, i+5)$ or $(i+5, i+6)$.

Observe that the only way to draw the rim cycle of the V_8 on a sphere with one crossing is by drawing it as an 8 shape (up to isomorphism), and the rim cycle with one crossing between $(i, i+1)$ and either $(i+3, i+4)$, $(i+4, i+5)$ or $(i+5, i+6)$ can only be completed to a 1-drawing of a V_8 in the two ways shown in Figures 2.1 and 2.2. All pictures on a plane can be obtained by projecting these drawings to the plane by putting our projective point (corresponding to infinity) in different faces of the graph of the drawing on a sphere. □

Notice that in the previous proof we also determined which edges must be the ones crossed in a 1-drawing of a V_8.

Corollary 2.6. *In a 1-drawing of* $V_8 \cong H$ *with rim cycle* $\{i, i+1, i+2, i+3, i+4, i+5, i+6, i+7, i\}$*, and spokes* $(i, i+4)$*,* $(i+1, i+5)$*,*$(i+2, i+6)$ *and* $(i+3, i+7)$*, the crossing in the* 1*-drawing must be between a rim branch* $(i, i+1)$ *and one of* $(i+3, i+4)$*,* $(i+4, i+5)$ *and* $(i+5, i+6)$*,* $\pmod 8$. *(See Figures 2.1, 2.2.)*

Chapter 3

Covered edges

In this section, we introduce the concept of covered edges, which will help us by giving some information about the structure of the drawings.

Definition 3.1. We say that a rim branch r (or an edge in a rim branch r) of a V_8 is *covered* by an edge s if it cannot be crossed in a 1-drawing of $V_8 \cup s$.

For example, let G consist of V_8 plus an edge s parallel to the rim edge $r = (i, i+1)$. Then $G - r \cong V_8$. Notice that V_8 is not planar, so by Lemma 2.3 r is not crossed in any 1-drawing of G. Therefore edge r is covered by edge s.

The main results in this section are: *(i)* if five consecutive rim branches are covered, then the resulting graph has crossing number two; and *(ii)* if no five consecutive rim branches are covered, then some pair of uncovered edges can be crossed in a 1-drawing of the V_8.

Using Corollary 2.6, we can observe that the least number of rim branches we need to cover is five, and the best way to cover five rim branches is by covering five rim branches in a row on the rim cycle. We formalize this idea in the next lemma.

Lemma 3.2. *Let H be a graph such that $V_8 \cong H$, and let G be a graph obtained by adding edges such that cover rim branches. If there are 5 consecutive rim branches of H that are covered, then the crossing number of G is at least 2. If no 5 consecutive rim branches of H are covered, then there is some pair of rim branches that have uncovered rim branches and can be crossed in a 1-drawing of H.*

Proof. From Corollary 2.6 we observe that one of every pair of opposite rim branches, $(i, i+1)$ and $(i+4, i+5)$, must be covered, since if they are both uncovered, they could cross in a 1-drawing of $V_8 \cong H$. Therefore at least four rim branches must be covered.

Suppose that G is obtained by adding edges that cover exactly four rim branches.

Let $n =$ *max number of consecutive covered rim edges on the rim cycle of H*. Then one of the following cases must occur:

1. $n = 4$. Without loss of generality, suppose that branches $(i, i+1)$, $(i+1, i+2)$, $(i+2, i+3)$, $(i+3, i+4)$ are covered. Then rim branches $(i+4, i+5)$ and $(i+7, i)$ are both uncovered, so they can cross in a 1-drawing of H.

2. $n = 3$. Suppose that branches $(i, i+1)$, $(i+1, i+2)$ are $(i+2, i+3)$ are covered. Then rim branches $(i+3, i+4)$ and $(i+7, i)$ are both uncovered, so they can cross in a 1-drawing.

3. $n = 2$. Without loss of generality, suppose that branches $(i, i+1)$ and $(i+1, i+2)$ are covered. Then rim branches $(i+2, i+3)$ and $(i+7, i)$ are both uncovered. Then they can cross in a 1-drawing of H.

4. $n = 1$. Observe that if there are no two consecutive rim branches that are covered, then the covered branches must alternate with uncovered rim branches on the rim cycle of H. Without loss of generality, suppose that branches $(i, i+1)$, $(i+2, i+3)$, $(i+4, i+5)$, $(i+6, i+7)$ are covered, and rim branches $(i+1, i+2)$, $(i+3, i+4)$, $(i+5, i+6)$, $(i+7, i)$ are not covered. Then observe that either rim branches $(i+1, i+2)$ and $(i+5, i+6)$, or $(i+3, i+4)$ and $(i+7, i)$ can cross in a 1-drawing of H.

Therefore at least five consecutive rim branches must be covered in G (as G has crossing number at least two).

Observe that if G has five consecutive covered rim branches, then it follows directly from Corollary 2.6 that all the rim branches of G are covered, and none can cross in a 1-drawing of G. Therefore G must have crossing number at least two.

□

We see from our casework from the proof of Lemma 3.2 that if we had fewer than five consecutive covered rim branches, then G could have a 1-drawing.

Definition 3.3. A 5-*covered graph* is a V_8 graph with five consecutive rim branches $(i, i+1)$, $(i+1, i+2)$, $(i+2, i+3)$, $(i+3, i+4)$, $(i+4, i+5)$ that are covered.

Chapter 4

Green cycles

The edges we add to $H \cong V_8$ to cover rim branches will all be green edges. In this section we give the definition of a green edge, and green cycle. We also prove that green edges will cover the rim branches in the green cycles they are contained in (Theorem 4.7).

We start with a series of definitions.

Definition 4.1. Let F be a subgraph of G. We say a u, v-path P of G is F*-avoiding* if $F \cap P \subseteq \{u, v\}$.

Definition 4.2. A *node* is a vertex whose degree is not two, and an H-node is a vertex of H that is a node.

Definition 4.3. A G*-branch* is a path in G whose ends are G nodes, but whose internal vertices are not, or a cycle in G that contains exactly one node.

Definition 4.4. We write $F \subseteq G$ if F is a subgraph of G.

Using these definitions, we can now define green cycles. (The term 'green' was used in [1] to distinguish how an edge in the rim can be covered; other edges were called 'red' or 'yellow.' We do not discuss red or yellow edges in this project.)

Definition 4.5. Suppose G is a graph and $V_8 \cong H \subseteq G$ with rim cycle R.

1. A cycle C in G is H*-green* if C is the composition $P_1P_2P_3P_4$ of four paths such that:
   (a) $P_2P_3P_4$ is R-avoiding;
   (b) $P_2 \cup P_4 \subseteq H$;
   (c) P_3 is H-avoiding; and
   (d) $P_1 \subseteq R$ and either:
      i. P_1 contains at most 2 H-nodes or

ii. for some $i \in \{0, 1, 2, \ldots, 7\}$ and indices read modulo 8, P_1 consists of precisely two H-rim branches.

2. An edge of P_3 is *H-green* if it is in an H-green cycle.

See Figure 4.1 for some examples of green cycles.

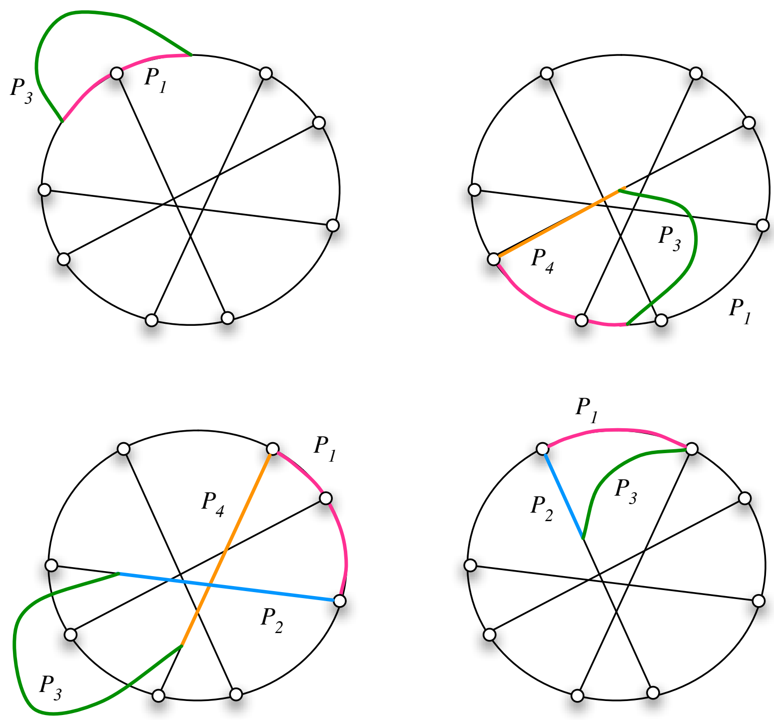

Figure 4.1: Some examples of green cycles, graph $H \cong V_8$ is everything but P_3.

We begin begin our analysis of green cycles with two observations.

1. Observe that if C is an H-green cycle in G and $V_8 \cong H \subseteq G$, then P_2 and P_4 are trivial or they are subpaths of spokes of H, as they must be subgraphs of H, and $P_2 \cup P_3 \cup P_4$ has at most two H-nodes.

2. Green cycles are defined specifically so that we can prove that all of C is not crossed in a 1-drawing of G (as we prove in Theorem 4.7). Notice in particular that if we relax condition *(d)* of Definition 4.5, then Theorem 4.7 is no longer true.

Lemma 4.6. *Let G be obtained from V_8 by adding edge (j, k), such that j is on spoke branch $(i, i+4)$ and k is on rim branch $(i+2, i+3)$. Then (j, k) covers rim branches $(i+1, i+2)$ and $(i+2, i+3)$, but does not cover the portion $(i+2, j)$ of $(i+2, i+3)$.*

Proof. Observe that $G - \{(i+1, i+2), (i+2, i+3)\}$ contains a V_6 minor with rim cycle $\{i, j, k, i+7, i+6, i+5, i+4, i\}$ and spokes $(i, i+7)$, $(j, i+1, i+5)$ and $(k, i+3, i+4)$. Therefore by Corollary 2.4, we know that neither $(i+1, i+2)$ nor $(i+2, i+3)$ can be crossed in a 1-drawing of G, so $(i+1, i+2)$ and $(i+2, i+3)$ are not covered by (j, k). To see that rim branch $(i+2, j)$ is not covered by (j, k), see Figure 4.2, which is a 1-drawing of G such that $(i+2, j)$ is crossed. □

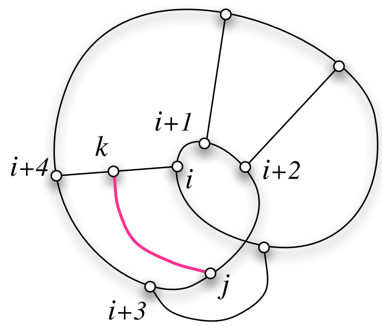

Figure 4.2: Illustration for Lemma 4.6.

Theorem 4.7. *Let G be a graph, $V_8 \cong H \subseteq G$, and C an H-green cycle in G. If D is a 1-drawing of $H \cup C$, then C is not crossed in D.*

Proof. First observe that by Corollary 2.4, P_3 can never be crossed in a 1-drawing of G, as H has several subgraphs $J \cong V_6$. In particular, for any spoke S of H, $(H - \{S\}) \cong V_6$.

Also, observe that P_2 is contained in a spoke S of $H \cong V_8$, so $(H - \{S\}) \cong V_6$. Therefore Corollary 2.4 implies that neither that neither P_2 nor P_4 can be crossed in a 1-drawing of G.

It remains to be shown that P_1 is not crossed in any 1-drawing of G. In what follows we consider all possible H-green cycles.

1. P_3 has ends s and t in the interiors of distinct spokes S and T, respectively. ($P_2 \neq \emptyset \neq P_4$, $P_2 \subset S$).

   (a) S and T are consecutive ($P_4 \subset T$). See Figure 4.3.

   Then $G - P_1$ contains a V_6 minor with rim cycle $\{i, s, t, i+1, i+2, i+3, i+4, i+5, i+6, i+7, i\}$ and spokes $(s, i+4)$, $(i+2, i+6)$ and $(i+3, i+7)$. Therefore by Corollary 2.4, we know that P_1 cannot be crossed in a 1-drawing of G.

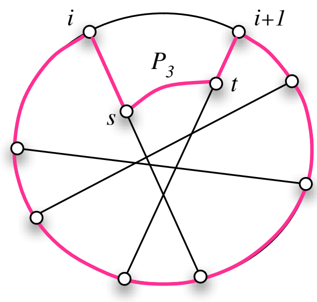

Figure 4.3: Proof of Theorem 4.7.

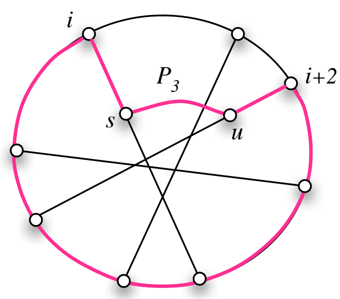

Figure 4.4: Proof of Theorem 4.7.

(b) S and T are not consecutive ($P_2 \subset S$ and $P_4 \subset U$). See Figure 4.4.
Then subgraph $G - P_1$ contains a V_6 minor with rim cycle $\{i, s, u, i+2, i+3, i+4, i+5, i+6, i+7, i\}$ and spokes $(s, i+4)$, $(u, i+6)$ and $(i+3, i+7)$. Hence by Corollary 2.4, we know that P_1 cannot be crossed in a 1-drawing of G.

2. P_3 has one end s in the interior of a spoke S and the other end j on the rim ($P_2 \neq \emptyset$ and $P_4 = \emptyset$, $P_2 \subset S$).

Observe that if $P_2 \neq \emptyset$, then P_1 must contain at least one H-node v. In fact, we must have that $i = P_1 \cap P_2$ is an H-node. Then we have one of the following cases:

(a) P_1 contains exactly one H-node, which is i. See Figure 4.5.
Then subgraph $G - P_1$ contains a V_6 minor with rim cycle $\{i, s, j, i+1, i+2, i+3, i+4, i+5, i+6, i+7, i\}$ and spokes $(i+1, i+4)$, $(i+2, i+6)$ and $(i+3, i+7)$. Therefore by Corollary 2.4, we know that P_1 cannot be crossed in a 1-drawing of G.

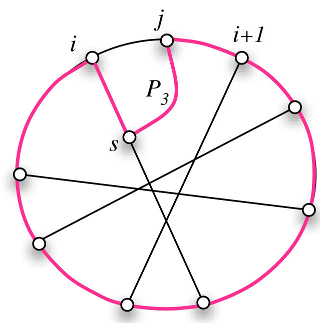

Figure 4.5: Proof of Theorem 4.7.

(b) P_1 contains two H-nodes. See Figure 4.6.

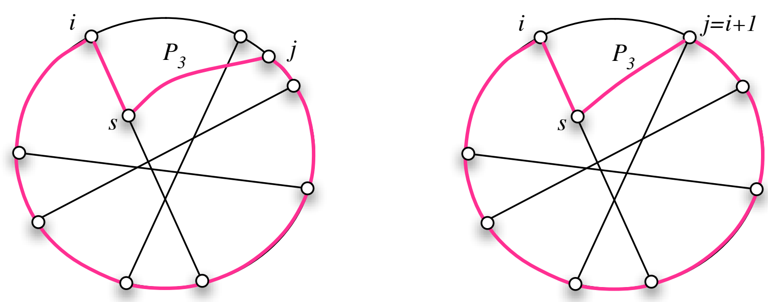

Figure 4.6: Proof of Theorem 4.7.

Then subgraph $G - P_1$ contains a V_6 minor with rim cycle $\{i, s, j, i+2, i+3, i+4, i+5, i+6, i+7, i\}$ and spokes $(s, i+4)$, $(i+2, i+6)$ and $(i+3, i+7)$. So by Corollary 2.4, we know that P_1 cannot be crossed in a 1-drawing of G.

(c) P_1 contains three H-nodes, i.e. P_1 consists of precisely two H-rim branches. See Figure 4.7.

Then subgraph $G - P_1$ contains a V_6 minor with rim cycle $\{i, s, i+2, i+3, i+4, i+5, i+6, i+7, i\}$ and spokes $(s, i+4)$, $(i+2, i+6)$ and $(i+3, i+7)$. So by Corollary 2.4, we know that P_1 cannot be crossed in a 1-drawing of G.

3. Both ends of P_3 are in R ($P_2 = \emptyset$ and $P_4 = \emptyset$).

(a) One end of P_3 is the H-node i. Let p be the other end of P_3. Then it is straight-

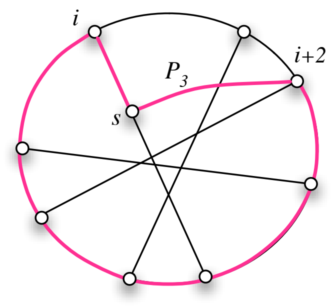

Figure 4.7: Proof of Theorem 4.7.

forward to see that that P_1 is not crossed in a 1-drawing of G by contracting edge (i, s) of the V_6 minors found for Case 2.

(b) Suppose without loss of generality that path $P_1 = (j, p)$, where j is a vertex on the interior of rim branch $(i, i+1)$ (so $i \neq j \neq i+1$). Observe that if p is a vertex of a spoke branch, then this case is equivalent to Case 3.a, so suppose that p is a vertex on a rim branch, and p is not an H node.

Then we must have one of the following cases:

i. p is in the interior of rim branch $(i, i+1)$. Then $P_1 = (j, p)$ is a parallel edge, $G - P_1 \cong V_8$, so G contains a V_6 minor, and by Corollary 2.4, we know that P_1 cannot be crossed in a 1-drawing of G.

ii. p is in the interior of rim branch $(i+1, i+2)$. See Figure 4.8. Then subgraph $G - P_1$ contains a V_6 minor with rim cycle $\{i, j, p, i+2, i+3, i+4, i+5, i+6, i+7, i\}$ and spokes $(i, i+4)$, $(i+2, i+6)$ and $(i+3, i+7)$. So by Corollary 2.4, we know that P_1 cannot be crossed in a 1-drawing of G.

iii. p is in the interior of rim branch $(i+2, i+3)$. See Figure 4.9.
Consider subpaths $P_{1.2} = (j, i+2)$ and $P_{1.1} = (i+1, p)$ of P_1, the subpaths with end vertices j and $i+2$, and $i+1$ and p respectively.
Observe that $G - P_{1.1}$ contains a V_6 minor with rim cycle $\{i, j, p, i+3, i+4, i+5, i+6, i+7, i\}$ and spokes $(i, i+4)$, $(p, i+2, i+6)$ and $(i+3, i+7)$. Then by Corollary 2.4, we know that $P_{1.1}$ cannot be crossed in a 1-drawing of G.
Similarly, observe that that $G - P_{1.2}$ contains a V_6 minor with rim cycle $\{i, j, p, i+3, i+4, i+5, i+6, i+7, i\}$ and spokes $(i, i+4)$, $(j, i+1, i+5)$ and $(i+3, i+7)$. Therefore by Corollary 2.4, we know that $P_{1.1}$ cannot be crossed in a 1-drawing of G.

From the above case work, we have shown that P_1 can never be crossed in a 1-drawing

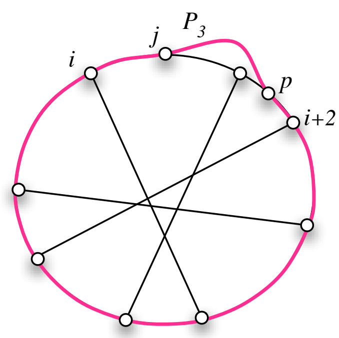

Figure 4.8: Proof of Theorem 4.7.

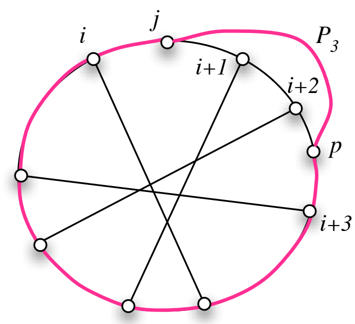

Figure 4.9: Proof of Theorem 4.7.

of G. Therefore C cannot be crossed in D.

□

Corollary 4.8. *Let G be a graph, $V_8 \cong H \subseteq G$, and C an H-green cycle in G such that $P_3 = e \in E(G)$. Then H-green edge e covers all rim branches in P_1.*

Corollary 4.8 just is a restatement of 4.7 in terms of covering edges.

A 2-$\frac{1}{2}$-jump can be viewed as a 1-$\frac{1}{2}$-jump, but it is only as the latter that it is in an H-green cycle.

Chapter 5

What covering edges we consider

We examine ways of adding green edges between vertices, spokes and rim branches of H that will affect a 1-drawing of H. If we are attaching the edge to a spoke or rim then we subdivide the spoke or rim and create a new H-node where the edge is incident to it.

Definition 5.1. When an edge is added between two vertices on the rim of the V_8 we say that the edge is a *bump*. When the edge is between a vertex on the rim of the V_8 and a vertex on a spoke we say that the edge is a *jump*. When an edge is added between two spokes of the V_8 we say that it is an *across edge*.

The covering edges we consider are $\frac{1}{2}$-$\frac{1}{2}$-bumps, 1-$\frac{1}{2}$-bumps, 2-bumps, across edges, bumps, jumps, $\frac{1}{2}$-bumps and $\frac{1}{2}$-jumps. The notation here is natural. For example, a $\frac{1}{2}$-$\frac{1}{2}$-bump is an edge joining the interior of consecutive rim branches, while a 1-$\frac{1}{2}$-bump joins a rim node i to the interior of one of the rim branches $(i-2, i-1)$ and $(i+1, i+2)$. Figure 5.1 illustrates all of these *rim configurations*. Note that these are all the covering edges that create H-green cycles (see the proof of Theorem 4.7), except we omit the 1-$\frac{1}{2}$-jump.

To build our graphs, we consider ways of adding covering edges to $H \cong V_8$, with the following restrictions:

1. We subdivide each rim branch at most once, so if we have two rim configurations that both end in the interior of the same branch, we suppose that they meet at a common vertex.

2. We try not to cover edges unnecessarily. In our construction of 5-covered graphs, if a and b are two rim configurations such a is a path $(P_3)_a$ of a green cycle C_a, and b is a path $(P_3)_b$ of a green cycle C_b, and $(P_1)_b$ is a subset of $(P_1)_a$, then we do not use both rim configurations a and b to construct the same graph.

   Observe that if a and b are two rim configurations such that the rim branches that are covered by b are a subset of the rim branches covered by a, then a 5-covered graph G that has both rim configurations a and b cannot be critical, because removing rim configuration b will not leave any of the five consecutive rim branches uncovered.

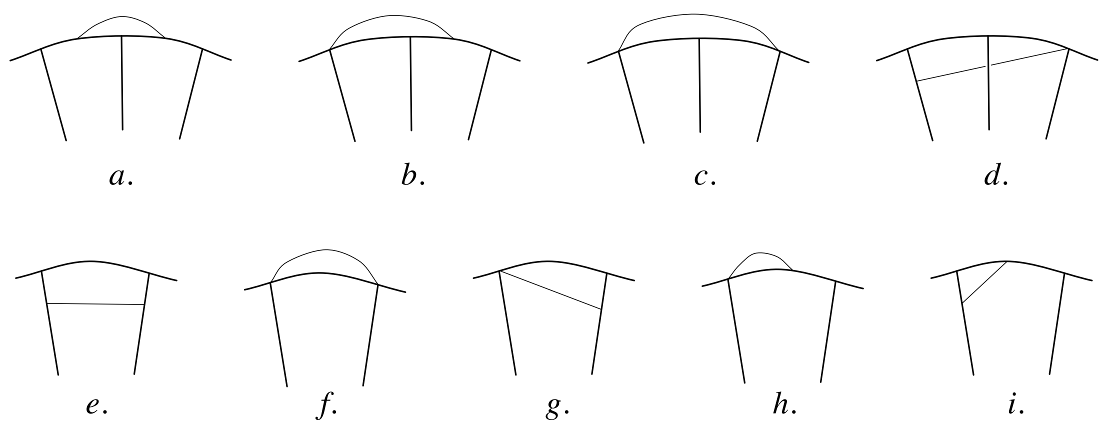

Figure 5.1: Rim configurations: *a.* $\frac{1}{2}$-$\frac{1}{2}$-bump, *b.* 1-$\frac{1}{2}$-bump, *c.* 2-bump, *d.* 2-jump, *e.* across edge, *f.* bump, *g.* jump, *h.* $\frac{1}{2}$-bump, *i.* $\frac{1}{2}$-jump.

We define the orientation of a jump as follows:

Left jump or left $\frac{1}{2}$-jump first vertex is the rim, second one is on the spoke.

Right jump or right $\frac{1}{2}$-jump first vertex is on the spoke, second one is on the rim.

Some vocabulary for rim configurations

Definition 5.2. We define the *length* of a rim configuration. Let edge $e = (u, v)$ be a rim configuration.

1. If e is a $\frac{1}{2}$-bump or a $\frac{1}{2}$-jump we say that it is of length one half.
2. If e is a bump, jump, or an across edge we say that it is of length one.
3. If e is a 2-bump or a 2-jump we say that it is of length two.

For example, we say that a 2-bump and a 2-jump are rim configurations of length two, that a bump, jump and across edge are rim configuration of length one, and a $\frac{1}{2}$-bump and $\frac{1}{2}$-jump are rim configuration of length one half.

Which edges are not considered

Lemma 5.3. *Let G be obtained from a V_8 by adding a 2-across edge. Then G is the Peterson graph, all 8 rim branches are covered, and G is 2-crossing-critical. See Figure 5.2.*

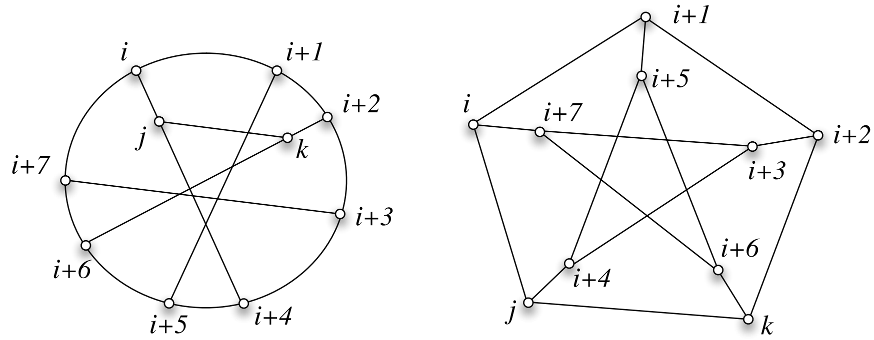

Figure 5.2: Illustration for Lemma 5.3

We will suppose, without loss of generality, that the covered rim edges of a 5-covered graph are $(i, i+1)$, $(i+1, i+2)$, $(i+2, i+3)$, $(i+3, i+4)$, $(i+4, i+5)$.

Note that across edges cover two rim branches at once; if there is an across edge between spokes $(i, i{+}4)$ and $(i{+}1, i{+}5)$, then both rim branches $(i, i{+}1)$ and $(i{+}4, i{+}5)$ are covered, and all other rim configurations that we are considering cover as many rim branches as the length of the rim configurations. (We use two rim configurations of length one half to cover a rim branch.)

Why we don't consider 1-$\frac{1}{2}$-bumps or $\frac{1}{2}$-$\frac{1}{2}$-bumps

Theorem 5.4. *Any 2-crossing-critical 5-covered graph using the nine configurations in Figure 5.1 can be isomorphically covered without using the first two configurations (i.e. without* 1-$\frac{1}{2}$*-bumps or* $\frac{1}{2}$-$\frac{1}{2}$*-bumps.)*

Proof. We define two straightening operations on 5-covered graphs; straighten-($\frac{1}{2}$-$\frac{1}{2}$) and straighten-(1-$\frac{1}{2}$).

For the straighten-($\frac{1}{2}$-$\frac{1}{2}$) operation (see Figure 5.10): Suppose we have a $\frac{1}{2}$-$\frac{1}{2}$-bump (u, v) where u and v are midpoints of two consecutive rim branches, and w is the H-node of the spoke branch S that is incident to these two rim branches. We construct $H' \cong V_8$ by first replacing the (u, w, v) portion of the rim with the $\frac{1}{2}$-$\frac{1}{2}$-bump (u, v). The spoke incident with w is extended to include the portion (w, v) of the original rim. Therefore this set of rim configurations is isomorphic to the set of rim configurations with a $\frac{1}{2}$-jump from u to w. We call this a *straighten-*($\frac{1}{2}$-$\frac{1}{2}$) *operation.*

Observe that if there are any covered rim branches immediately to the left of u, they remain unchanged after applying the straighten-($\frac{1}{2}$-$\frac{1}{2}$) operation.

We examine the rim configurations that could be immediately to the right of (u, v) by cases:

1. Suppose that there is a bump immediately to the left of (u, v), on rim branch (w, x) where x is the next vertex on the outer cycle of the V_8 following u and v as shown in Figure 5.3. Then when we perform a straighten-($\frac{1}{2}$-$\frac{1}{2}$) operation we obtain a rim configuration that has a right $\frac{1}{2}$-jump immediately to the right of a left jump. By Lemma 6.1 this configuration is not 2-crossing-critical, so we don't need to consider it.

   See Figure 5.3.

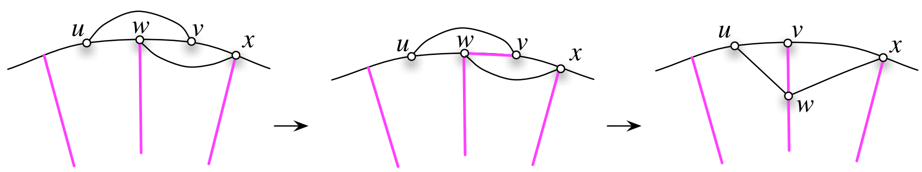

Figure 5.3: Straighten-$\frac{1}{2}$-$\frac{1}{2}$ operation, Case 1.

2. Suppose there is an across edge immediately to the left of (u, v), between vertices y and z where y lies on spoke S and z lies on the spoke left of S, as shown in Figure 5.4. Then performing a straighten-($\frac{1}{2}$-$\frac{1}{2}$) operation will result in a graph that also has an across edge immediately to the left of (u, v). See Figure 5.4.

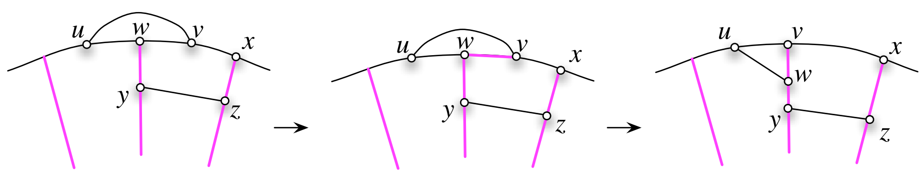

Figure 5.4: Straighten-$\frac{1}{2}$-$\frac{1}{2}$ operation, Case 2.

3. Suppose that there is a 2-bump immediately to the left of (u, v), between vertices w and y where the x and y appear on the 8-outer cycle of the V_8 subgraph of G in order u, v, x, y, as shown in Figure 5.5. Then a straighten-($\frac{1}{2}$-$\frac{1}{2}$) operation will result in a graph that has a right $\frac{1}{2}$-jump immediately to the right of a left 2-jump. See Figure 5.5.

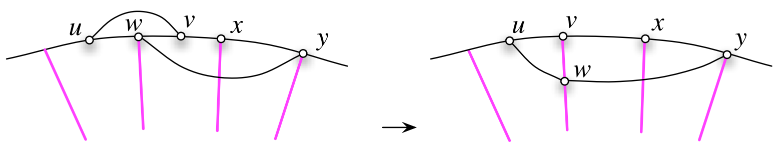

Figure 5.5: Straighten-$\frac{1}{2}$-$\frac{1}{2}$ operation, Case 3.

4. Suppose there is a left 2-jump immediately to the left of rim branch (u, v), between vertices w and z, where z is a vertex on a spoke such that (w, z) forms a right 2-jump, as shown in Figure 5.6. Then performing a straighten-($\frac{1}{2}$-$\frac{1}{2}$) operation will result in a graph with a 2-across jump. This graph cannot be critical with crossing number two because it has a subgraph that is 2-crossing-critical by Lemma 5.3, so we do not need to consider it in our case analysis. See Figure 5.6.

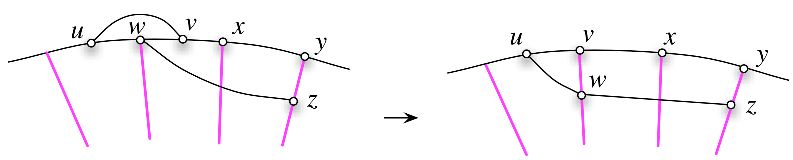

Figure 5.6: Straighten-$\frac{1}{2}$-$\frac{1}{2}$ operation, Case 4.

5. Suppose that there is a right 2-jump immediately to the left of rim branch (u, v), between vertices z and y, such that z in a vertex on spoke S and y is a vertex on the 8-outer cycle of the V_8 subgraph as shown in Figure 5.7. Then a straighten-($\frac{1}{2}$-$\frac{1}{2}$) operation will result in a graph that also has a right 2-jump immediately to the left of rim branch (u, v). See Figure 5.7.

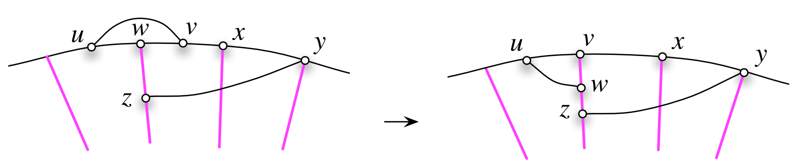

Figure 5.7: Straighten-$\frac{1}{2}$-$\frac{1}{2}$ operation, Case 5.

6. Suppose that there is a right jump immediately to the left of rim branch (u, v), between vertices w and y, such that y in a vertex on spoke S and w is a vertex on the 8-outer cycle of the V_8 subgraph as shown in Figure 5.8. Then a straighten-($\frac{1}{2}$-$\frac{1}{2}$) operation will result in a graph that also has an across edge immediately to the left of rim branch (u, v). See Figure 5.8.

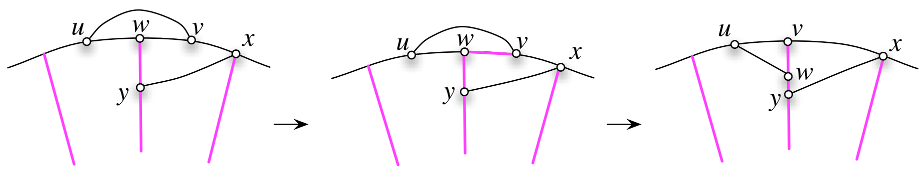

Figure 5.8: Straighten-$\frac{1}{2}$-$\frac{1}{2}$ operation, Case 6.

7. Suppose that there is a left jump immediately to the left of rim branch (u, v), between vertices w and y, such that y in a vertex on spoke immediately to the left of spoke S and w is a vertex on the 8-outer cycle of the V_8 subgraph as shown in Figure 5.9. Then a straighten-($\frac{1}{2}$-$\frac{1}{2}$) operation will result in a graph that also has an across edge immediately to the left of rim branch (u, v). See Figure 5.9.

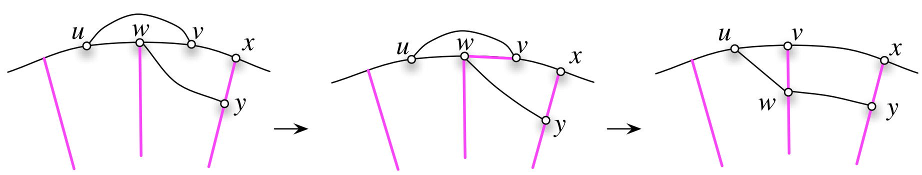

Figure 5.9: Straighten-$\frac{1}{2}$-$\frac{1}{2}$ operation, Case 7.

8. The only other rim configurations that could be immediately to the right of (u, v), are those that start on a vertex that is the midpoint of a rim branch. That is, the rim configuration must be a $\frac{1}{2}$-bump, $\frac{1}{2}$-jump, a 1-$\frac{1}{2}$-bump or a $\frac{1}{2}$-$\frac{1}{2}$-bump. Observe that performing a straighten-($\frac{1}{2}$-$\frac{1}{2}$) operation on (u, v) will transform the rim configurations immediately to the right of v as follows: a $\frac{1}{2}$-bump into a bump, a $\frac{1}{2}$-jump into a jump, a $\frac{1}{2}$-$\frac{1}{2}$-bump into a 1-$\frac{1}{2}$-bump, and a 1-$\frac{1}{2}$-bump into a 2-bump.

   Note that in every case, a straighten-($\frac{1}{2}$-$\frac{1}{2}$) operation will always decrease the number of $\frac{1}{2}$-$\frac{1}{2}$-bumps by at least one.

($\frac{1}{2}$-$\frac{1}{2}$) operation changes A to B:

A	B
Bump	Left jump
Across edge	Across edge
2-bump	Left 2-jump
Left 2-jump	2-across edge
Right 2-jump	Right 2-jump
$\frac{1}{2}$-bump	Bump
$\frac{1}{2}$-jump	Jump
$\frac{1}{2}$-$\frac{1}{2}$-bump	1-$\frac{1}{2}$-bump
1-$\frac{1}{2}$-bump	2-bump

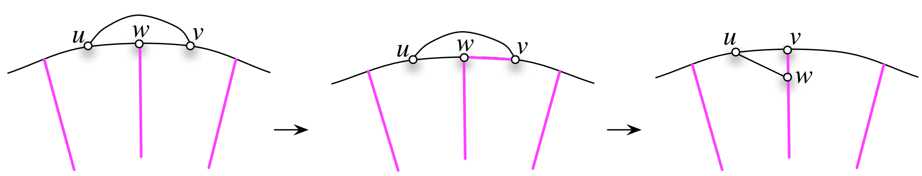

Figure 5.10: Straighten-$\frac{1}{2}$-$\frac{1}{2}$ operation.

For the straighten-(1-$\frac{1}{2}$) operation, suppose we have a 1-$\frac{1}{2}$-bump (u, v), where u is the vertex on the spoke, v is the midpoint of a rim branch and let w be the vertex on the spoke under (u, v) as shown on Figure 5.11. We can rearrange the picture by extending spoke S to include the edge (w, v), and by doing so we see that this set of ring configurations is isomorphic to the set of rim configurations with a jump from u to w. We call this rearrangement a *straighten-*(1-$\frac{1}{2}$) *operation.*

Observe that a straighten-(1-$\frac{1}{2}$) operation will not affect any configurations that are immediately to the left of (u, v). As we observed before, the only rim configurations that could be immediately to the right of (u, v) are a $\frac{1}{2}$-bump, a $\frac{1}{2}$-jump, a 1-$\frac{1}{2}$-bump or a $\frac{1}{2}$-$\frac{1}{2}$-bump, which would be transformed into a jump, a bump, a 2-bump or a 1-$\frac{1}{2}$-bump respectively.

We can conclude that a straighten-(1-$\frac{1}{2}$) operation will always decrease the number of $1\frac{1}{2}$-bumps by at least one, unless there was a $\frac{1}{2}$-$\frac{1}{2}$-bump immediately to the right of v.

Let G be a 5-covered graph using the eight possible rim configurations shown on Figure 5.1. Let G' be the graph obtained from applying straighten-($\frac{1}{2}$-$\frac{1}{2}$) operations until there are no $\frac{1}{2}$-$\frac{1}{2}$-bumps left, and then applying straighten-(1-$\frac{1}{2}$) operations until there are no 1-$\frac{1}{2}$-bumps left. It is clear that we can apply straighten-($\frac{1}{2}$-$\frac{1}{2}$) operations until there are no $\frac{1}{2}$-$\frac{1}{2}$-bumps left. Observe that applying straighten-(1-$\frac{1}{2}$) operations to a graph with no $\frac{1}{2}$-$\frac{1}{2}$-bumps will create no new 1-$\frac{1}{2}$-bumps, therefore these straighten-(1-$\frac{1}{2}$) operations will now always decrease the number of 1-$\frac{1}{2}$-bumps. Therefore G' is isomorphic to G and has no 1-$\frac{1}{2}$-bumps or $\frac{1}{2}$-bumps. $\square$

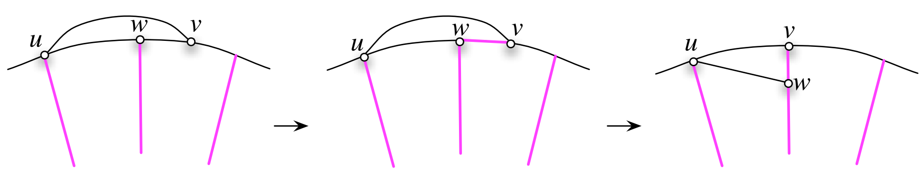

Figure 5.11: Straighten-1-$\frac{1}{2}$ operation.

Chapter 6

Edges that yield graphs that are not 2-crossing-critical

We make some straightforward observations about rim configurations that interact in ways that produce non-critical graphs. These observations are not relevant to any proof, but they enabled us to write a more efficient computer program to search through all the graphs that were considered in our search for 2-crossing-critical graphs. We can divide the two types of problems into two categories: *(i)* individual rim configurations that yield non-critical graphs when they appear covering a particular rim branch out of five consecutive covered rim branches, and *(ii)* pairs of rim configurations that cannot be placed next to each other when five consecutive covered rim branches.

Rim configuration positioning

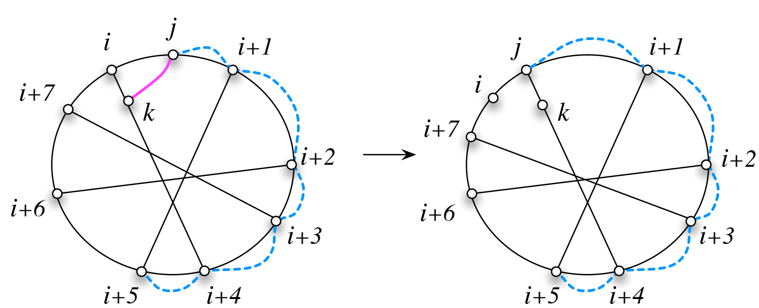

Figure 6.1: Illustration for Lemma 6.1

Lemma 6.1. *Let G be a 5-covered graph where $(i, i+1)$ has a rim configuration that begins in a right $\frac{1}{2}$-jump (or equivalently $(i+4, i+5)$ has a rim configuration that ends in a left $\frac{1}{2}$-jump). Then G is not 2-crossing-critical.*

Proof. Suppose that the rim configuration on rim branch $(i, i+1)$ begins in a right $\frac{1}{2}$-jump (j, k), where j is a vertex on rim branch $(i, i+1)$, and k is a vertex on spoke $(i, i+4)$, as shown in Figure 6.1. Then observe that we can delete edge (i, k) and obtain a graph that is equivalent to another 5-covered rim configuration by rerouting spoke $(i, k, i+4)$ to $(j, k, i+4)$. Therefore $G - \{(i, k)\}$ must have crossing number at least two, so G cannot be 2-crossing-critical. □

The next lemma treats the case of an across edge.

Lemma 6.2. *Let G be a 5-covered graph with a rim configuration in which rim branch $(i, i+1)$ (or $(i+4, i+5)$) is covered by an across edge, and $(i+4, i+5)$ (or respectively $(i, i+1)$) is covered by a rim configuration that covers only rim branch $(i+4, i+5)$ (or respectively $(i, i+1)$). Then G is not 2-crossing-critical.*

Proof. Observe that an across edge on rim branch $(j, j+1)$ is also an across edge for rim branch $(j+4, j+5)$. Then by Corollary 4.8 the across edge covers both $(j, j+1)$ and $(j+4, j+5)$. Therefore if there is an across edge rim configuration on rim branch $(i, i+1)$, and $(i+4, i+5)$ is covered by a rim configuration r that covers only rim branch $(i+4, i+5)$, r can be removed without uncovering $(i+4, i+5)$. □

Note that this argument does not hold if the rim configuration on $(i+4, i+5)$ (or respectively $(i, i+1)$) is covered by a rim configuration of length two on one side of the group of five consecutive rim branches.

Rim configuration interactions

Lemma 6.3. *A 5-covered graph with a left jump, or left $\frac{1}{2}$-jump immediately to the left of a right jump or right $\frac{1}{2}$-jump is not 2-crossing-critical.*

Proof. Suppose we have a rim configuration with a left jump immediately to the left or a right jump, such that both jumps are incident to a rim branch that meets the spoke at a vertex u. Suppose that the jump edges meet the rim branch at vertices v and w, and that $v = w$ or the vertices appear in the order u, v, w on the rim branch. Then observe that we could remove the edge (u, v) and obtain a different rim configuration by replacing the two jumps.

Two half jumps can be replaced by a $\frac{1}{2} - \frac{1}{2}$-bump, a left half bump and right bump or a left jump and a right half jump can be replaced by a $1 - \frac{1}{2}$-bump, as shown in Figure 6.2, and two jumps can be replaced by a 2-bump.

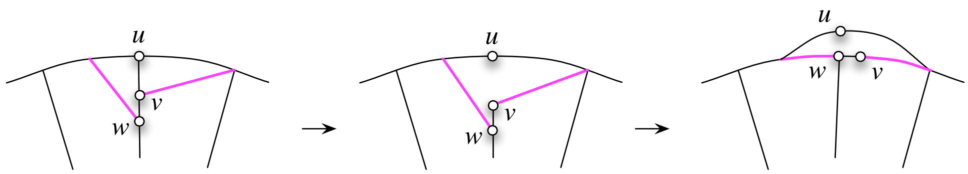

Figure 6.2: Illustration for Lemma 6.3

By Theorem 5.4 the rim configuration has crossing number at least two. Thus our original rim configuration cannot be in any 2-crossing-critical graph.

□

Lemma 6.4. *Let G be a 5-covered graph with a 2-jump. Then if there is a rim configuration of length one that only covers a rim branch that is covered by the 2 jump G is not 2-crossing-critical.*

Proof. Clearly removing the rim configuration of length 1 will not cause a rim branch not to be covered, as the 2-jump will still cover it. □

Lemma 6.5. *Let G be a 5-covered graph in which rim branch $(i, i+1)$ is covered by a left jump, and vertex $i+1$ has degree four in G (or $(i+4, i+5)$ is covered by a right jump, and $i+4$ has degree four). Then G is not 2-crossing-critical.*

Proof. Clearly removing the rim configuration of length 1 will not cause a rim branch not to be covered, as the 2-jump will still cover it. □

Chapter 7

The computer program

Two results were checked by computer: *(i)* The graphs obtained by our construction have no V_{10} minor, and *(ii)* there are 326 2-crossing-critical 5-covered graphs found using the construction.

Lemma 7.1. *A* 5*-covered graph does not have a* V_{10} *minor.*

This proof is computer-based.

We used a computer program developed by Oporowski, which we helped to debug in the process of checking for V_{10} minors.

It can be seen with some casework that there are no V_{10} minors that are subgraphs of $H \cong V_8 \cup e$ such that e is a green edge and $V_{10} \cap e \neq \emptyset$, for certain kinds of rim edges.

Theorem 7.2. *There are 326* 2*-crossing-critical* 5*-covered graphs using the green cycles that are shown in Figure 5.1.*

Our proof for Theorem 7.2 is computer-based.

For the computer program we used the `nauty` package developed by Brendan D. McKay. We used a subroutine based on `nauty` written by Drago Bokal that checks if a given graph is 2-crossing-critical. We give a description of how the graphs were generated, checked for 2-criticality, and counted.

To construct the 5-covered graphs that were tested for 2-criticality we constructed a base graph *grafica* that looks like a V_8 with a few subdivisions, as shown in Figure 7.1.

Array entry Rim[j] is used to represent which rim configuration is used for each of the five consecutive rim branches that are covered. Rim branch $(i+j, i+j+1)$ corresponds to Rim[j] for $j \in \{0, 1, 2, 3, 4\}$.

Array entry Spoke[j] will describe the interaction of the rim configurations covering branch $(i+j, i+j+1)$ and $(i+j+1, i+j+2)$ for $j \in \{0, 1, 2, 3\}$.

Suppose, for example, that we have a left jump (i, u) covering rim branch $(i, i+1)$ and a right jump $(v, i+1)$ covering rim branch $(i+1, i+2)$, with u and v vertices on spoke branch $(i+1, i+5)$. Then u and v can interact in three possible ways:

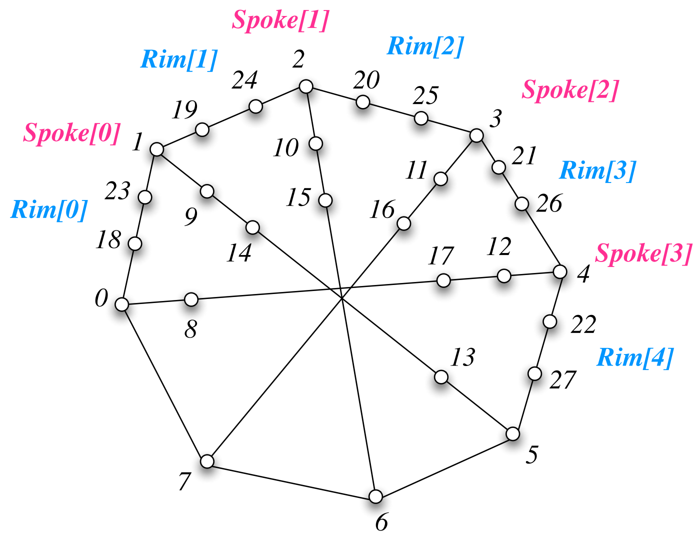

Figure 7.1: Base graph *grafica*.

1. Spoke branch $(i+j+1, i+j+5)$ is contained in subpath $(i+j+1, u, v)$ and $u \neq v$. We say that u *is above* v.
2. $u = v$. We say that u *and* v *meet on the spoke*.
3. Spoke branch $(i+j+1, i+j+5)$ is contained in subpath $(i+j+1, v, u)$ and $u \neq v$. We say that u *is below* v.

Notice that the only rim configurations that can have different kinds of interactions at the spoke are jumps and across edges. If we do not have one of these types of rim configurations then we can ignore the spoke values.

In general we assign values to array Spoke$[j]$ as follows: Let u be the vertex of the rim configuration of the rim branch that is represented by Rim$[j]$ and let v be the vertex of the rim configuration that is represented by Rim$[j+1]$.

- Spoke$[j]$ = **0**: Spoke branch $(i+1, i+5)$ is contained in subpath $(i+1, u, v)$ and $u \neq v$. We say that u *is above* v.
- Spoke$[j]$ = **1**: $u = v$. We say that u *and* v *meet on the spoke*.
- Spoke$[j]$ = **2**: Spoke branch $(i+1, i+5)$ is contained in subpath $(i+1, v, u)$ and $u \neq v$. We say that u *is below* v.

The green cycles that are used as rim configurations are encoded as follows:

- Rim$[j]$ = **0**: We add a bump to rim branch $(j, j+1)$ of *grafica*.
- Rim$[j]$ = **1**: We add an across edge to rim branch $(j, j+1)$ of *grafica*.
- Rim$[j]$ = **2**: We add a left jump to rim branch $(j, j+1)$ of *grafica*.
- Rim$[j]$ = **3**: We add a right jump to rim branch $(j, j+1)$ of *grafica*.
- Rim$[j]$ = **4**: We add a $\frac{1}{2}$-bump and a left $\frac{1}{2}$-jump to rim branch $(j, j+1)$ of *grafica* by adding edges $(j, j+23)$ and $(j+23, u)$ to it, such that u is a vertex on spoke branch $(j+1, j+5)$.
- Rim$[j]$ = **5**: We add a $\frac{1}{2}$-jump and a left $\frac{1}{2}$-bump to rim branch $(j, j+1)$ of *grafica* by adding edges $(u, j+18)$ and $(j+18, v)$ to it, such that u is a vertex on spoke branch $(j, j+4)$ and v is a vertex on spoke branch $(j+1, j+5)$.
- Rim$[j]$ = **6**: We do not add any edge to *grafica*.
- Rim$[j]$ = **7**: We add two $\frac{1}{2}$-jumps to rim branch $(j, j+1)$ of *grafica* by adding edges $(u, j+18)$ and $(j+19, j+1)$ to it, such that u is a vertex on spoke branch $(j, j+4)$.
- Rim$[j]$ = **8**: We add 2-bump covering rim branches $(j, j+1)$ and $(j+1, j+2)$ by adding edge $(j, j+2)$.
- Rim$[j]$ = **9**: We add left 2-jump covering rim branches $(j, j+1)$ and $(j+1, j+2)$ by adding edge (j, u), such that u is a vertex on spoke branch $(j+2, j+6)$.
- Rim$[j]$ = **10**: We add right 2-jump covering rim branches $(j, j+1)$ and $(j+1, j+2)$ by adding edge $(u, j+2)$, such that u is a vertex on spoke branch $(j+2, j+6)$.

Note that the $\frac{1}{2}$-bump edges are added to *grafica* in such a way that they do not generate parallel edges, as `nauty` can only deal with simple graphs.

We require the case Rim$[j] = 6$ for these situations:

1. Suppose we have a Rim$[j] = 6$ (a 2-jump). If the value of Rim$[j+1]$ corresponds to a rim configuration of length i (with $i \neq 2$) then we could remove the rim configuration of length i, and its corresponding rim branch would still be covered by the 2-jump. Hence the graph could not be 2-crossing-critical.
2. Suppose we have Rim$[j] = 1$ for $j = 0$ or $j = 4$. If $j = 0$ (analog. $j = 4$) and the value of Rim$[4]$ corresponds to a rim configuration of length i (with $i \neq 2$) then we could remove the rim configuration of length i, and its corresponding rim branch would still be covered by the across edge.

3. Suppose we have Rim[j] $= 1$ for $j = 1$ or $j = 3$. If $j = 1$ (analog. $k = 3$) and the value of Rim[0] $\neq 6$. Then observe that rim branches $(i, i+1)$, $(i+1, i+2)$, $(i+2, i+3)$, $(i+3, i+4)$, $(i+4, i+5)$, $(i+5, i+6)$ are covered, the first five are covered by the rim configuration assigned the branches by the values of the array Rim[], and $(i+5, i+6)$ is covered by the across edge on rim branch $(i+1, i+2)$. We could remove the rim configuration covering $(i, i+1)$ and the resulting graph would still have crossing number two.

To run through all the possible ways of assigning values to the arrays we run through all values of c between 0 and $11^5 3^4$. For every c value we assign values a and b as int $a = \lfloor \frac{c}{3^4} \rfloor$ and $b = c \pmod{3^4}$. We then assign Rim[j] $=$*j-th digit of a* $\pmod{11^5}$ and Spoke[j] $=$*j-th digit of b* $\pmod{3^4}$.

For every graph we generate this way we use the built in subroutine

isKCrossingCritical(graph,size of graph,K-value)

to check if it is 2-crossing-critical by writing

isKCrossingCritical(grafica,28,2) && !isKCrossing(grafica,28,1).

By doing this we can create list *goodcvalues* that lists all the values of c that correspond to 2-crossing-critical graphs.

To count the graphs we want create a sublist of *goodcvalues* that has no two graphs that are topologically isomorphic.

We wrote program `topologicalisomorphism` that maps a graph G to a graph G' by contracting edges adjacent to vertices of degree two that do not create parallel edges (as `nauty` only accepts simple graphs). We run all the graphs corresponding to values of list *goodcvalues* through `topologicalisomorphism` and run these new graphs through subroutine `pickg`, the `nauty` program that checks for isomorphism. The list that is returned by `pickg` has only non-topologically isomorphic 2-crossing-critical graphs, of which there are exactly 326.

Bibliography

[1] D. Bokal, B. Oporowski, R. B. Richter, G. Salazar, Characterizing 3-connected, 2-crossing-critical graphs. *preprint.*

[2] G. S. Bloom, J. W. Kennedy, and L. V. Quintas, On crossing numbers and linguistic structures. Graph theory (Łagów, 1981), 14–22, Lecture Notes in Math., 1018, Springer, Berlin, 1983.

[3] M. Kochol, Construction of crossing-critical graphs, Discrete Math. **66** (1987), 311–313.

[4] B. Richter, Cubic graphs with crossing number two, J. Graph Theory **12** (1988), no. 3, 363–374.

[5] J. Širáň, Infinite families of crossing-critical graphs with a given crossing number, Discrete Math. **48** (1984), no. 1, 129–132.

[6] Thomassen, C. The Jordan-Schönflies theorem and the classification of surfaces, American Mathematical Monthly 99, (1992), (2): 116130.