Shortest Paths in and Colourings of the Unit Distance Graph

by

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Abstract

The unit-distance graph in n dimensions over the field $F \subseteq \mathbb{R}$, denoted by $U_{\mathbf{F}}^n$, is the graph G defined by $V(G) = \mathbf{F}^n$, and two vertices are adjacent if and only if they are at Euclidean distance 1. Generally $F = \mathbb{Q}$ or $\mathbb{R}$. In this essay, we show that between two vertices x, y of $U_{\mathbf{Q}}^n$ for $n \geq 4$ there is a path of length at most $\|x - y\| + c$, where c is a constant that depends on n, decreases as n increases, and is less than 20 for all n. We also explore some results about the chromatic number of $U_{\mathbf{Q}}^n$ and $U_{\mathbf{R}}^n$ for some small values of n.

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Chapter 1

Introduction

This essay serves as an introduction to the unit-distance graph and some problems associated with it. The *unit-distance graph* in n dimensions over the field $F \subseteq \{\mathbb{R}, \mathbb{Q}\}$, denoted by $U^n_{\mathbf{F}}$, is the graph G defined by $V(G) = \mathbf{F}^n$, and two vertices are adjacent if and only if they are at Euclidean distance 1.

Most of the interest in unit-distance graphs stems from colouring. The famous open problem in the area is to determine the chromatic number of $U^2_{\mathbf{R}}$. Jensen and Toft [9] attribute the problem to Edward Nelson in 1950. In 1992 Chilakamarri [5] surveyed the known results in the area. The best known lower bound for the chromatic number of $U^2_{\mathbf{R}}$ is four, due to Moser and Moser [8]. The best know upper boud is seven, due to Hadwiger and Debrunner [12].

There has been more success in determining the chromatic number of $U^n_{\mathbf{Q}}$ for small values of n. The chromatic numbers of $U^2_{\mathbf{Q}}$, $U^2_{\mathbf{Q}}$, and $U^4_{\mathbf{Q}}$ are known and good bounds are known for $U^5_{\mathbf{Q}}$, $U^6_{\mathbf{Q}}$, $U^7_{\mathbf{Q}}$, and $U^8_{\mathbf{Q}}$. The proofs for the chromatic numbers of $U^2_{\mathbf{Q}}$, $U^3_{\mathbf{Q}}$, and $U^4_{\mathbf{Q}}$ are included.

Another important result about the rational graphs is about connectivity; the rational unit-distance graphs are not necessarily connected. Chilakamarri [3] showed that $U^n_{\mathbf{Q}}$ is connected when $n \geq 5$ and is not connected when $n \leq 4$. We expand on this result by giving an explicit description of the components of the rational unit-distance graph in the cases when it is not connected.

We also look at the problem of determining the length of the shortest path between two points of $U^n_{\mathbf{Q}}$ that are in the same component when $n \geq 4$. For two points a, b, we are able to find a path with length $\|a - b\| + c$, where c is a small constant. We not only prove existence of paths of these lengths, we show how to construct them.

Chapter 2

Connectedness

A natural question that arises when studying the rational unit distance graphs is whether or not the graphs are connected. To understand the structure of the entire graph, it suffices to understand the structure of the components. Since $\mathbb{Q}^n$ is isomorphic under a rational translation, the components must all be translates of each other.

The proofs that $U^n_{\mathbb{Q}}$ is connected when $n \geq 5$ and non connected when $n \leq 4$ are not original. They are included here because they are elegant and because similar ideas are used later on. The characterization of connected components is a new result.

If e is an edge from a to b in $U^n_{\mathbb{Q}}$ or $U^n_{\mathbb{R}}$, it is convenient to think of it as a vector in $\mathbb{R}^n$. For example, in $U^4_{\mathbb{Q}}$, the edge from $a = (0, 1, 1/3, 0)$ to $b = (5/6, 1/2, 1/6, -1/6)$ would be denoted by vector $e = (5/6, -1/2, -1/6, -1/6)$, and the edge from b to a would be $-e$. Similarly, we refer to a path as a sequence of vectors.

When working with paths in $U^n_{\mathbb{Q}}$ or $U^n_{\mathbb{R}}$, there are two different concepts of length that we are concerned with. The first is the number of edges in a path, which we will refer to as the length of a path. The second concept is the distance between the two endpoints of the path in $\mathbb{R}^n$. We define the *Euclidean length of a path P* from c to d to be the vector $d - c \in \mathbb{R}^n$

2.1 $U^n_{\mathbb{Q}}$ is connected for $n \geq 5$

We show that $U^n_{\mathbb{Q}}$ is connected for $n \geq 5$ using the following lemma. The result and proof are originally due to Chilakamarri [3].

2.1.1 Lemma. *If $n \geq 5$ and $m \in \mathbb{N}$, then there exists a path in $U^n_{\mathbb{Q}}$ from $(0, 0, \ldots, 0)$ to $(\frac{1}{m}, 0, 0, \ldots, 0)$.*

Proof. It is a well-known result in number theory that any natural number can be written as the sum of four squares (see [2] for a proof). So given $m > 0$, we can find non-negative integers a_1, a_2, a_3, a_4 such that

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = 4m^2 - 1.$$

The length of each of the segments

$$S_1 = \left(\frac{1}{2m}, \frac{a_1}{2m}, \frac{a_2}{2m}, \frac{a_3}{2m}, \frac{a_4}{2m}, 0 \dots, 0\right)$$

$$S_2 = \left(\frac{1}{2m}, \frac{-a_1}{2m}, \frac{-a_2}{2m}, \frac{-a_3}{2m}, \frac{-a_4}{2m}, 0 \dots, 0\right)$$

is 1 and thus they are edges of $U_{\mathbb{Q}}^n$ for $n \geq 5$. If we take the path $S_1 S_2$, it travels from the origin to $(\frac{1}{m}, 0, 0, \dots, 0)$. $\square$

We are now equipped to prove our main theorem for this section.

2.1.2 Theorem. *$U_{\mathbb{Q}}^n$ is connected for $n \geq 5$.*

Proof. By Lemma 2.1.1, we can find a path P_m^1 from

$$(0, 0, \dots, 0) \text{ to } \left(\frac{1}{m}, 0, \dots, 0\right).$$

We can similarly find paths P_m^k from

$$(0, 0, \dots, 0) \text{ to } \left(0, \dots, 0, \frac{1}{m}, 0, \dots, 0\right)$$

where $\frac{1}{m}$ is in the k^{th} position. If we consider the path taken by combining a_i copies of P_m^i, we obtain a path from

$$(0, 0, \dots, 0) \text{ to } \left(\frac{a_1}{m}, \frac{a_2}{m}, \dots, \frac{a_n}{m}\right).$$

For any point p in $\mathbb{Q}^n$, we can express p in the above form by choosing m as the lowest common multiple of the denominators of each coordinate. Thus, we can find a path between $(0, 0, \dots, 0)$ and any other vertex of $U_{\mathbb{Q}}^n$, so the graph is connected. $\square$

2.2 $U_{\mathbb{Q}}^n$ is not connected for $n \leq 4$

The result and proof of this section are again due to Chilakamarri [3].

2.2.1 Theorem. *The graph $U_{\mathbb{Q}}^4$ is not connected.*

Proof. Consider the point $(\frac{1}{4}, 0, 0, 0)$. We will show that there is no path from the origin to this point. Suppose that such a path did exist. Let it be composed of edges

$$\left\{\left(\frac{a_i}{e_i}, \frac{b_i}{e_i}, \frac{c_i}{e_i}, \frac{d_i}{e_i}\right)\right\}_{i=1}^{n}$$

with fractions in lowest terms. For each $i \leq n$, we have

$$a_i^2 + b_i^2 + c_i^2 + d_i^2 = e_i^2$$

If e_i is a multiple of 4, considering this equation modulo 16 we get

$$a_i^2 + b_i^2 + c_i^2 + d_i^2 = 0$$

The only quadratic residues modulo 16 are 0, 1, 4 and 9. So the above equation can only hold if all terms are 0 or all terms are 4. Thus, e_i is not a multiple of 4. So for some odd β we must have

$$\sum_{i=1}^{n} \frac{a_i}{e_i} = \frac{\alpha}{\mathrm{lcm}(e_i)} = \frac{\alpha}{2\beta}.$$

We note that this can never be 1/4. □

Our desired result follows as a corollary to this theorem and its proof.

2.2.2 Corollary. *$U_{\mathbb{Q}}^n$ is not connected for $n \leq 4$.*

Proof. $U_{\mathbb{Q}}^4$ contains induced subgraphs isomorphic to $U_{\mathbb{Q}}^3$ and $U_{\mathbb{Q}}^2$ containg the point $(\frac{1}{4}, 0, 0, 0)$ and the origin. By the proof of Theorem 2.2.1 these points are not connected in $U_{\mathbb{Q}}^4$. They can not be connected in any induced subgraph of $U_{\mathbb{Q}}^4$, so $U_{\mathbb{Q}}^3$ and $U_{\mathbb{Q}}^2$ are also not connected. □

2.3 Connected Components

We mentioned earlier that Chilakamarri showed that $U_{\mathbb{Q}}^2$, $U_{\mathbb{Q}}^3$, and $U_{\mathbb{Q}}^4$ are not connected. Soon after this, Zaks [13] showed that these graphs all contain infinitely many connected components. Zaks showed this by finding an infinite set of points no two of which are in the same component.

We are interested in determining which points are in the same component. We do this by determining which points are in the same component as $(0, 0)$, and noting that all other components are translates of this component. The results and proofs are original.

2.3.1 Components of $U_{\mathbb{Q}}^2$

Our main goal in this section will be to determine a characterization for when two points of $U_{\mathbb{Q}}^2$ lie in the same component of the graph. We do this through a series of lemmas.

2.3.1 Lemma. *If $\frac{a}{m_1}$ and $\frac{b}{m_2}$ are in lowest terms, and $m_1 m_2$ has a prime divisor not congruent to 1 modulo 4, then there is no path from $(0, 0)$ to $(\frac{a}{m_1}, \frac{b}{m_2})$ in $U_{\mathbb{Q}}^2$.*

Proof. Consider a point $p = (\frac{c}{m_1}, \frac{d}{m_2})$ that consists of reduced fractions and let $n = \mathrm{lcm}(m_1, m_2)$. We can write $p = (\frac{a}{n}, \frac{b}{n})$ where $\gcd(a, b, n) = 1$. If there is a path from $(0, 0)$ to p then there is a set of edges $\{(\frac{a_i}{c_i}, \frac{b_i}{c_i})\}$ such that

$$a_i^2 + b_i^2 = c_i^2, \quad \sum_{i=1}^{m} \frac{a_i}{c_i} = \frac{a}{n}, \quad \sum_{i=1}^{m} \frac{b_i}{c_i} = \frac{b}{n}.$$

If we consider them in lowest terms, we see that (a_i, b_i, c_i) form a primitive Pythagorean triple.

We now require two well-known results from number theory. Firstly, any primitive pythagorean triple is of the form

$$(2st, s^2 - t^2, s^2 + t^2), \quad \gcd(s, t) = 1, \quad s \not\equiv t (\mathrm{mod}\ 2).$$

Secondly, -1 is not a quadratic residue modulo q if q is of the form $4k + 3$. Proofs of these results can be found in almost any text on elementary number theory, such as [2]. So considering:

$$a_i^2 + b_i^2 = c_i^2 \ (\mathrm{mod}\ q)$$

for q a prime of the form $4k + 3$, we see that if q divides c_i, then

$$a_i^2 \equiv -b_i^2 \ (\mathrm{mod}\ q)$$

and thus b_i^2 and $-b_i^2$ are both quadratic residues modulo q. If b_i is invertible modulo q, then -1 would be a residue modulo q. Since -1 is not a residue modulo q, we must have that a_i and b_i are multiples of q. Since we assumed each triple is primitive, we have that q does not divide c_i.

Thus, we have c_i is a product of primes congruent to 1 modulo 4 for each i. Notice that n divides the lowest common multiple of $c_1, c_2, \ldots, c_m$, so n is a product of primes congruent to 1 modulo 4. □

2.3.2 Lemma. *If (a, b, c) is a primitive pythagorean triple, then so is $(a^2 - b^2, 2ab, c^2)$.*

Proof. One can easily verify that $(a^2 - b^2, 2ab, c^2)$ is a pythagorean triple.

If $\gcd(2ab, c^2) > 1$ then $\gcd(a, c) > 1$ or $\gcd(b, c) > 1$ since c is odd. Since (a, b, c) is primitive, $\gcd(a, c) = \gcd(b, c) = 1$. So $(a^2 - b^2, 2ab, c^2)$ is also primitive. □

2.3.3 Lemma. *If p is a prime congruent to 1 modulo 4 and $k \in \mathbb{N}$ then there exists a primitive pythagorean triple with p^k dividing the last term.*

Proof. For a prime p congruent to 1 modulo 4, we always have that -1 is a quadratic residue modulo p (see, for example, [2] for a proof) . So there exists $a < p$ such that $a^2 \equiv -1 \bmod p$. We see also that $(p - a)^2 \equiv -1 \bmod p$, and that either a or $p - a$ is even. Assume that a is even.

Construct the primitive pythagorean triple $(a^2 - 1, 2a, a^2 + 1)$. We observe that $p | (a^2 + 1)$. By a repeated application of Lemma 2.3.2, we can generate a primitive pythagorean triple with any power of two dividing the last term.

2.3.4 Lemma. *If p a prime congruent to 1 modulo 4, then there exists a path in $U_{\mathbb{Q}}^2$ from $(0,0)$ to $(p^{-k},0)$, $\forall k \in \mathbb{N}$.*

Proof. By Lemma 2.3.3, we can find a primitive pythagorean triple (a,b,c) for which p^k divides c and with $c > a > b$. We define edges E and F as:

$$E = \left(\frac{a}{c}, \frac{b}{c}\right), \quad F = \left(\frac{a}{c}, \frac{-b}{c}\right).$$

We construct two paths P_1 and P_2 as:

$$P_1 = \{E, F, (-1,0)\}, \quad P_2 = \{(1,0)\} \cup -P_1.$$

The Euclidean lengths of these paths are:

$$P_1 = \left(\frac{2a-c}{c}, 0\right), \quad P_2 = \left(\frac{2c-2a}{c}, 0\right).$$

We see that

$$\gcd(2c-2a, 2a-c) = \gcd(a,c) = 1,$$

so there exist integers e, d such that

$$(2c-2a)e + (2a-c)d = 1.$$

Thus, the path $P_3 = eP_2 \cup dP_1$ has Euclidean length $(\frac{1}{c}, 0)$. Since p^k divides c, the path $\frac{c}{p^k}P_3$ exists and has Euclidean length $(\frac{1}{p^k}, 0)$. □

2.3.5 Lemma. *If $\gcd(a,b) = 1$ and there exist paths from $(0,0)$ to $(\frac{1}{a}, 0)$ and $(\frac{1}{b}, 0)$, then there exists a path from$(0,0)$ to $(\frac{1}{ab})$.*

Proof. There are integers c, d such that

$$cb + ad = \gcd(a,b) = 1,$$

Hence,

$$\frac{c}{a} + \frac{d}{b} = \frac{1}{ab}.$$

So we take c copies of the first path and d copies of the second path to get the desired path. □

2.3.6 Theorem. *If n is a product of primes congruent to 1 modulo 4, then there exists a path in $U_{\mathbb{Q}}^2$ from $(0,0)$ to $(\frac{a}{n}, \frac{b}{n})$.*

Proof. From Lemma 2.3.4 and Lemma 2.3.5, there exists a path P_1 from $(0,0)$ to $(\frac{1}{n}, 0)$. By symmetry, we can also find a path P_2 from $(0,0)$ to $(0, \frac{1}{n})$. The path formed by taking a copies of P_1 and b copies of P_2 has the desired Euclidean length. □

We can now prove the main result of the section.

2.3.7 Theorem. *Two points $p = (\frac{a_1}{n}, \frac{b_1}{n})$ and $q = (\frac{a_2}{n}, \frac{b_2}{n})$ in $U^2_{\mathbb{Q}}$ are in the same component if and only if when $\frac{a_2-b_2}{n}$ and $\frac{a_1-b_1}{n}$ are expressed in lowest terms, the denominators are each a product of primes congruent to* 1 *modulo* 4.

Proof. For any two points $p = (\frac{a_1}{n}, \frac{b_1}{n})$ and $q = (\frac{a_2}{n}, \frac{b_2}{n})$ in $U^2_{\mathbb{Q}}$, p and q are connected if and only if $(0,0)$ is connected to $(p-q)$. By Theorem 2.3.6, $(0,0)$ is connected to $p-q$ when $\frac{a_2-b_2}{n}$ and $\frac{a_1-b_1}{n}$ are expressed in lowest terms and the denominators are each a product of primes congruent to 1 modulo 4. □

2.3.2 Components of $U^3_{\mathbb{Q}}$

We now determine the components of $U^3_{\mathbb{Q}}$. Based on the work on $U^2_{\mathbb{Q}}$, we see that there is a path from the origin to any point $(\frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n})$ with n a product of primes congruent to 1 (mod 4). We show that this extends to n odd.

2.3.8 Lemma. *In $U^3_{\mathbb{Q}}$, for f_1, f_2, f_3 not all even, a vertex of the form $(\frac{f_1}{2n}, \frac{f_2}{2n}, \frac{f_3}{2n})$ is not in the same component as the origin for any value of n.*

Proof. Consider a point $p = (\frac{p_1}{n}, \frac{p_2}{n}, \frac{p_3}{n}) \in U^3_{\mathbb{Q}}$. If there is a path from $(0,0,0)$ to p, then there is a set of edges:

$$\left\{ \frac{a_i}{d_i}, \frac{b_i}{d_i}, \frac{c_i}{d_i} \right\}_{i=1}^{m}$$

such that

$$a_i^2 + b_i^2 + c_i^2 = d_i^2$$

$$\sum_{i=1}^{m} \frac{a_i}{d_i} = \frac{p_1}{n}, \quad \sum_{i=1}^{m} \frac{b_i}{d_i} = \frac{p_2}{n}, \quad \sum_{i=1}^{m} \frac{c_i}{d_i} = \frac{p_3}{n}.$$

If we consider (a_i, b_i, c_i, d_i) in lowest terms, and examine the first relation modulo 4, we see that the only possible solution is d^2 congruent to 1 modulo 4, two of a^2, b^2, c^2 are congruent to 0 modulo 4, and the other is congruent to 1 modulo 4.

The remaining relations show us that when the $\frac{p_i}{n}$ is in lowest terms, n divides the lowest common multiple of the d_is. This implies that n is odd. Since we have considered an arbitrary path, any point reachable from the origin must be expressable as $(\frac{p_1}{n}, \frac{p_2}{n}, \frac{p_3}{n})$ with n an odd number. If we were to express this point with denominator $2n$, then all the numerators would be even. Thus our result holds. □

2.3.9 Lemma. *If p is a prime congruent to* 3 *modulo* 4, *then there exists a Pythagorean quadruble (a, b, c, d) such that p divides d and* $\gcd(d, abc) = 1$

Proof.

Suppose $p \equiv 3 \pmod 8$. Then, by [2], p is the sum of three squares and not the sum of two squares. So there exist m, n, r such that $m^2 + n^2 + r^2 = p$. It is easy to verify that the set $(2rm, 2rn, r^2 - m^2 - n^2, r^2 + m^2 + n^2)$ is a Pythagorean quadruple. Clearly p divides $r^2 + m^2 + n^2$. Since p is odd and since m, n, and r are all non-zero and less than p, $2rm$ and $2rn$ are not divisible by p. If $r^2 - m^2 - n^2$ were divisible by p then $r^2 - m^2 - n^2 + r^2 + m^2 + n^2 = 2r^2$ is divisible by p. But r is strictly bewteen 0 and p, so $2r^2$ is not divisible by p. Thus, $r^2 - m^2 - n^2$ is not divisible by p.

Suppose $p \equiv 7 \pmod 8$. Consider $n = pq$, where $q = 7$, or $q = 23$ and $p \neq q$. Then by [2] n is the sum of three squares and not the sum of two squares. So there exist l, m, r such that $l^2 + m^2 + r^2 = n$. Suppose p divides l. Then p divides $m^2 + r^2$. If p divides m, then p divides r, and p^2 divides $l^2 + m^2 + r^2$, but p^2 does not divide n. So p does not divide m or r. Then we have $(\frac{m}{r})^2 \equiv -1$ modulo p. But since p is of the form $4k + 3$, -1 is not a quadratic residue modulo q. This is a contradiction, so p does not divide l, m, or r. Similarly, q does not divide l, m, or r.

The quadruple $(2lr, 2mr, r^2 - l^2 - m^2, r^2 + l^2 + m^2)$ is Pythagorean and p divides the last term. As in the first case, p does not divide $2lr$ and $2mr$ since p does not divide l, m or r. Also, p does not divide $r^2 - l^2 - m^2$ since p does not divide r. The same result holds for q. □

2.3.10 Lemma. *If (a, b, c, d) is a Pythagorean quadruple with an odd prime p dividing the last term and d has no common factors with a, b, or c, then $(a^2 + b^2 - c^2, 2ac, 2bc, d^2)$ has the same property.*

Proof. Since p divides d, p divides d^2. Consider a prime q dividing d. Since q does not divide a, b, or c, q does not divide $2ac$ or $2bc$. If q divides $a^2 + b^2 - c^2$, then q divides $d^2 - (a^2 + b^2 - c^2) = 2c^2$. But q does not divide c, so it does not divide $2c^2$. So the result holds. □

2.3.11 Lemma. *Given p a prime of the form $4k+3$, there exists a Pythagorean quadruple (a, b, c, d) with p^k dividing d and p not dividing a, b and c*

Proof. By Lemma 2.3.9, there exists a Pythagorean quadruple (e, f, g, h) with p dividing h and p not dividing e, f, and g. By repeatedly applying Lemma 2.3.10, we can use this quadruple to find one with the desired properties. □

2.3.12 Lemma. *If p is a prime of the form $4k + 3$, then there exists a path in $U^3_{\mathbb{Q}}$ from the origin to $(p^{-k}, 0, 0)$.*

Proof. By Lemma 2.3.11, we can find a Pythagorean quadruple (a, b, c, d) such that p^k divides d and d has no common factor with a. Since (a, b, c, d) is Pythagorean,

$$E_1 = \left(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}\right), \quad E_2 = \left(\frac{a}{d}, -\frac{b}{d}, -\frac{c}{d}\right)$$

are edges of $U^3_{\mathbb{Q}}$. We construct two paths P_1 and P_2 as:

$$P_1 = E \cup F, \quad P_2 = (1,0) \cup -P_1.$$

The Euclidean lengths of these paths are:

$$P_1 = \left(\frac{2a}{d}, 0, 0\right), \quad P_2 = \left(\frac{d-2a}{d}, 0, 0\right).$$

We see that

$$\gcd(2a, d-2a) = \gcd(d, 2a) = 1,$$

so there exist integers e,f such that

$$(2a)e + (d-2a)f = 1.$$

Thus, the path $P_3 = eP_1 \cup fP_2$ has Euclidean length $(\frac{1}{d}, 0, 0)$. Since p^k divides d, the path $\frac{d}{p^k}$ exists and has Euclidean Length $(\frac{1}{p^k}, 0, 0)$. □

2.3.13 Theorem. *A point $(\frac{a}{n}, \frac{b}{n}, \frac{c}{n})$ in $U^3_{\mathbb{Q}}$, expressed in lowest terms, is in the same component as the origin if and only if n is odd.*

Proof.
The only if direction follows directly from Lemma 2.3.8.

If a, b are relatively prime and there exists a path A from $(0,0,0)$ to $(a^{-1}, 0, 0$ and a path B from $(0,0,0)$ to $(b^{-1}, 0, 0)$, then there exist integers c and d such that $ac + bd = 1$. So the path $cB \cup aD$ has Euclidean length $((ab)^{-1}, 0, 0)$.

By Lemma 2.3.12 and Lemma 2.3.3, for any distinct odd primes p and q and integers j and k, there exists a path P from $(0,0,0)$ to $(p^{-j}, 0, 0)$ and a path Q from $(0,0,0)$ to $(q^{-k}, 0, 0)$.

Thus, for any odd n, there is a path from $(0,0,0)$ to $(n^{-1}, 0, 0)$, and the result follows. □

2.3.3 Components of $U^4_{\mathbb{Q}}$

As a simple extension of the results for $U^3_{\mathbb{Q}}$, the origin of $U^4_{\mathbb{Q}}$ is path connected to all points of the form $(\frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n}, \frac{a_4}{n})$, where n is odd. In four dimensions, the edge $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ has length 1 and so some points with even denominators will also be in the same component.

2.3.14 Lemma. *If the point $(\frac{b_1}{4n}, \frac{b_2}{4n}, \frac{b_3}{4n}, \frac{b_4}{4n})$ is in the same component as the origin, then $b_1, b_2, b_3,$ and b_4 are even..*

Proof. Suppose we can reach a point P such that:

$$P = \left(\frac{b_1}{4n}, \frac{b_2}{4n}, \frac{b_3}{4n}, \frac{b_4}{4n}\right).$$

The we can find a set of edges

$$\left\{\left(\frac{c_i}{4m}, \frac{d_i}{4m}, \frac{e_i}{4m}, \frac{f_i}{4m}\right)\right\}_{i=1}^{r}$$

where each edge has length 1 and the sum of all the edges is P. In particular, we have

$$c_i^2 + d_i^2 + e_i^2 + f_i^2 = 0 \pmod{16}.$$

The residues modulo 16 are $0, 1, 4$ and 9. The equation can only hold if all four terms are 0 or all four terms are 4. Thus, every number is even, so we can write P as:

$$\left(\frac{b_1/2}{2n}, \frac{b_2/2}{2n}, \frac{b_3/2}{2n}, \frac{b_4/2}{2n}\right)$$

where n is odd. Thus, we can not reach any points with denominator a multiple of 4. □

2.3.15 Theorem. *A point is in the same component as the origin in $U_{\mathbb{Q}}^4$ if and only if it is of the form $(\frac{a_1}{2n}, \frac{a_1}{2n}, \frac{a_1}{2n}, \frac{a_1}{2n})$, where a_1, a_2, a_3, a_4 all have the same parity and n is odd.*

Proof. We again consider a point P of the form

$$\left(\frac{b_1}{2n}, \frac{b_2}{2n}, \frac{b_3}{2n}, \frac{b_4}{2n}\right)$$

and a set of edges

$$\left\{\left(\frac{c_i}{2m}, \frac{d_i}{2m}, \frac{e_i}{2m}, \frac{f_i}{2m}\right)\right\}_{i=1}^{r}$$

where each edge has length 1 and the sum of all the edges is P. In this case, we have that:

$$c_i^2 + d_i^2 + e_i^2 + f_i^2 = 0 \pmod{4}.$$

This happens if and only if for each i we have $\{c_i, d_i, e_i, f_i\}$ are all odd or all even. Thus $\{b_1, b_2, b_3, b_4\}$ are all even or all odd.

The points where $\{b_1, b_2, b_3, b_4\}$ are all even are just points with odd denominators, so we can reach them. If we take any point $Q = (\frac{b_1}{2n}, \frac{b_2}{2n}, \frac{b_3}{2n}, \frac{b_4}{2n})$ where b_1, b_2, b_3, b_4 are all odd, then there is a path from the origin to

$$H = Q + \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \left(\frac{b_1+n}{2n}, \frac{b_2+n}{2n}, \frac{b_3+n}{2n}, \frac{b_4+n}{2n}\right)$$

since $b_i + n$ is even. We know there is a path from the origin to H, so there is also one from the origin to Q since $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is an edge. □

Chapter 3

Shortest Paths

Given two points p, q in the same component of the unit distance graph we would like to find the length of the shortest path p and q. Since all edge lengths have Euclidean length one, the triangle inequality tells us that we need at least $\lceil \|p-q\| \rceil$ edges to get between the two points.

Knowing this, the optimal solution would be to find a path with $\lceil \|p-q\| \rceil$ edges between p and q. This is not possible in all cases, for example if $\|p-q\| < 1$ then there is not a single edge between them. The next natural question to ask is whether $\lceil \|p-q\| \rceil + 1$ is possible. If not, what about $+2$ or $+3$? For dimension $d > 3$, we find a constant ϵ_d such that there is a path of length $\lceil \|p-q\| \rceil + \epsilon_d$.

Our method for finding a shortest path between two points relies on two important properties. The first property is that the set of rational points on the unit hypersphere in $\mathbb{R}^n$ is dense. The second property is that the set of rational points with Euclidean distance at most 2 from a point p has bounded path distance from p in $U_{\mathbb{Q}}^n$.

We will show that the first property holds for any dimension and that the second property holds for all dimensions larger than 3. The question of whether ϵ_2 and ϵ_3 exist remains open and answering it will determine whether the points with distance at most 2 from a point have bounded path distance in $U_{\mathbb{Q}}^2$ and $U_{\mathbb{Q}}^3$ respectively.

Define the function $C(x)$ for $x \geq 4, x \in \mathbb{Z}$ by

$$C(x) = \begin{cases} 18 & \text{if } x = 4 \\ 10 & \text{if } x = 5 \\ 6 & \text{if } x = 6 \text{ or } 7 \\ 4 & \text{if } x \geq 8 \end{cases}$$

What we will show is that if $n \geq 4$ and p, q are two points in the same component of $U_{\mathbb{Q}}^n$, then there exists a path of length at most $\lceil \|p-q\| \rceil + C(n) - 1$ from p to q.

3.1 Rational Points on Hyperspheres

3.1.1 Theorem. *The set of rational points on the unit sphere in dimension at least 2 is dense.*

Proof.

Let q be the point $q = (1, 0, \ldots, 0)$ and let $L(a_1, \ldots, a_n)$ be the line through q given by the equation

$$a_1(x_1 - 1) = a_2x_2 = a_3x_3 = \cdots = a_nx_n.$$

The unit sphere has equation

$$\sum_{i=1}^{n} x_i^2 = 1.$$

Substituting in values for $x_2, x_3, \ldots, x_n$ from the equation for $L(a_1, \ldots, a_n)$, we have

$$\sum_{i=2}^{n} \left(\frac{a_1}{a_n}(x_1 - 1)\right)^2 + x_1^2 = 1.$$

When the a_is are rational, this is just a quadratic equation with rational coefficients and we can see that one of the roots is 1. Thus, the other root must also be a rational number. If x_1 is rational and all a_is are rational then all the x_is are rational. So the intersection of $L(a_1, \ldots, a_n)$ with the unit sphere occurs at two rational points when all of the a_is are rational.

If we consider a point $p = (p_1, \ldots, p_n)$ on the unit sphere, we want to find a rational point on the unit sphere close to it. Consider a rational point $a = (a_1, \ldots, a_n)$ where

$$1 \geq (a_1/p_i) > 1 - \frac{1}{\epsilon}, \quad \|p - a\| < \frac{1}{\epsilon}.$$

If we consider the line $L(q, a)$ through q and a, it has equation

$$\frac{x_1 - 1}{a_1 - 1} = \frac{x_2}{a_2} = \frac{x_3}{a_3} = \cdots = \frac{x_n}{a_n}.$$

Taking the intersection of $L(q, a)$ with the unit sphere, we get the following equation for the x_1 coordinate:

$$x_1^2 + \sum_{i=2}^{n} \left(\left(\frac{x_1 - 1}{a_1 - 1}\right)^2 a_i^2\right) = 1$$

We know one solution is $x_1 = 1$, and solving for the other value, we get

$$x_1 = \frac{\sum_{i=1}^{n}(a_i^2) - 2a_1^2 + 2a_1 - 1}{\sum_{i=1}^{n}(a_i^2) - 2a_1 + 1}.$$

We know that $\sum_{i=1}^{n}(a_i^2) \leq 1$ by our choice of the a_is, so let $\sum_{i=1}^{n}(a_i^2) = 1 - \delta$, where $\delta \leq \epsilon$. Then we have

$$x_1 = \frac{2a_1 - 2a_1^2 - \delta}{2 - 2a_1 + \delta}$$

As a tends to p, δ tends to 0, so x_1 tends to a_1 and thus tends to p_1. For the other coordinates, we have

$$x_i = (x_1 - 1)\frac{a_i}{a_1 - 1}$$

Since x_1 is tending to p_1, x_i is tending to p_i, so the rational points are dense. □

3.2 Bounded Balls

3.2.1 Lemma. *Let p be a point in $U_{\mathbb{Q}}^n$ for $n \geq 5$ such that $|p| \leq 2$ and p has at most 4 non-zero coordinates. Then there exists a path from the orign to p of length at most 2.*

Proof. Assume without loss of generality that the last 4 coordinates are 0 and so

$$p = \left(\frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_{n-4}}{m}, 0, 0, 0, 0\right)$$

is such a point. Then

$$a_1^2 + a_2^2 + \ldots + a_{n-4}^2 \leq 4m^2$$

and so $4m^2 - a_1^2 - \ldots - a_{n-4}^2$ is a non-negative integer so it can be written as the sum of four squares, say b_1, b_2, b_3, b_4. Then

$$a_1^2 + a_2^2 + \ldots + a_{n-4}^2 + b_1^2 + b_2^2 + b_3^2 + b_4^2 = 4m^2$$

and so

$$\left(\frac{a_1}{2m}, \ldots, \frac{a_{n-4}}{2m}, \frac{b_1}{2m}, \frac{b_2}{2m}, \frac{b_3}{2m}, \frac{b_4}{2m}\right)$$

and

$$\left(\frac{a_1}{2m}, \ldots, \frac{a_{n-4}}{2n}, -\frac{b_1}{2m}, -\frac{b_2}{2m}, -\frac{b_3}{2m}, -\frac{b_4}{2m}\right)$$

are each edges of length 1. Their combined Euclidean length is p, so they form the desired path. □

3.2.2 Lemma. *Let $n \geq 5$ and p be a point in $U_{\mathbb{Q}}^n$ that is contained in the ball of radius 2 centered at the origin. Then there is a path from the origin to p with length at most ten when $n = 5$, at most six when $n = 6$ or 7 and at most four when $n \geq 8$.*

Proof. Any point in n dimensions in the ball of radius two centered at the origin can be expressed as the sum of $n/(n-4)$ points that satisfy the criteria of the lemma by projecting the vector into $n-4$ dimensions. Calculating $n/(n-4)$ for the various values of n gives the desired result.

We see immediately that this can not be extended to four dimensions, since we only have four coordinates. In this case, we can make use of the characterization from Burton [2] that any number not of the form $4^m(8k+7)$ can be written as the sum of three squares.

3.2.3 Lemma. *Let a_1, a_2, a_3, a_4, n be odd and $a_1^2 + a_2^2 + a_3^3 + a_4^2 < 4n^2$. Then there is path in $U_{\mathbb{Q}}^4$ of length eight from the origin to $p = (\frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n}, \frac{a_4}{n})$.*

Proof. We see that $d_i = 4n^2 - a_i^2$ is odd for any i, so as long as d_i is not of the form $8k+7$, it can be expressed as the sum of three squares.

$$4n^2 \equiv 4 \pmod 8, \quad a_i^2 \equiv 1 \pmod 8$$

So d_i is congruent to 3 modulo 8 and can be written as the sum of three squares. So we can find b_1, b_2, b_3 such that

$$\left(\frac{a_1}{2n}, \frac{b_1}{2n}, \frac{b_2}{2n}, \frac{b_3}{2n}\right), \quad \left(\frac{a_1}{2n}, -\frac{b_1}{2n}, -\frac{b_2}{2n}, -\frac{b_3}{2n}\right)$$

each have length 1 and their union is a two edge path from the origin to $(\frac{a_1}{n}, 0, 0, 0)$. We repeat this process for each of the other coordinates and find a path of length eight from the origin to p. □

3.2.4 Lemma. *Let m be odd and $b_1^2 + b_2^2 + b_3^2 + b_4^2 \leq 4m^2$ and m is odd. Then there exists a path in $U_{\mathbb{Q}}^4$ from $q = (\frac{b_1}{m}, \frac{b_2}{m}, \frac{b_3}{m}, \frac{b_4}{m})$ to a point $p = (\frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n}, \frac{a_4}{n})$ with a_1, a_2, a_3, a_4, n all odd and $a_1^2 + a_2^2 + a_3^3 + a_4^2 \leq 4n^2$.*

Proof. We construct two edges E and F. Define

$$E = \left(\frac{b_1}{2|b_1|}, \frac{b_2}{2|b_2|}, \frac{b_3}{2|b_3|}, \frac{b_4}{2|b_4|}\right)$$

with the convention that $\frac{0}{|0|} = 1$. We define F coordinatewise in terms of E. The i^{th} coordinate of F is the same as the i^{th} coordinate of E when b_i is even and is the negative of it when b_i is odd.

$E + F$ is a $\{-1, 0, 1\}$-valued vector that is 0 when b_i is odd, $+1$ when b_i is even and negative, and -1 when b_i is even and non-negative. So $r = q + E + F$ is a point with all the numerators odd. However, r may not be in the ball of radius 2 around the origin. The coordinates of r that differ from q have norm at most 1 by construction.

Let B_j be the $\{0, 1\}$-vector with 1 in the j^{th} position. For each coordinate i of r that is larger than 1 and less than 2, we take $r - 2B_i$ to get a point that still has all numerators odd. We can similarly alter r for coordinates between

-1 and -2. Since r differs from q on at least 1 coordinate, we need do this for at most 3 coordinates.

This construction gives us a path of length 8. However, notice that if at least 2 of the coordinates changed then the path has length at most 6, and if only 1 coordinate changes, then $E + F = B_j$ for some j, so it can be reduced to length 7. $\square$

3.2.5 Lemma. *Let $b_1^2+b_2^2+b_3^2+b_4^2 \leq 16m^2$ and m be odd. Then there is a path in $U_{\mathbb{Q}}^4$ of length three from $q = (\frac{b_1}{2m}, \frac{b_2}{2m}, \frac{b_3}{2m}, \frac{b_4}{2m})$ to a point $p = (\frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n}, \frac{a_4}{n})$ with n odd and $a_1^2 + a_2^2 + a_3^3 + a_4^2 \leq 4n^2$.*

Proof. Let E be any vector with each coordinate equal to $\frac{1}{2}$ or $-\frac{1}{2}$. Then for any of the 16 such vectors E, $q+E$ is a point with odd denominator. However, $q + E$ may not be in the ball of radius 2 around the origin. We can find one such point $r = q + E$ such that each of the coordinates are between -2 and 2 inclusive.

We define two $\{\frac{1}{2}, -\frac{1}{2}\}$ vectors H and F coordinatewise. F and H differ in the i^{th} coordinate if r is between -1 and 1 in the i^{th} coordinate. F and H are both $\frac{1}{2}$ if the i^{th} coordinate of r is between -1 and -2, and are both $-\frac{1}{2}$ otherwise.

Then $r + F + H$ has all coordinates at most 1 and each coordinate can be expressed as a rational number with odd denominator. Thus we have found the desired path. $\square$

We can combine all of these results to get a bound on the entire ball.

3.2.6 Lemma. *Let p be a point in $U_{\mathbb{Q}}^4$ that is containted in the ball of radius 2 centered at the origin. Then there is a path from the origin to p with length at most* 18.

Proof.

This follows directly from Lemmas 3.2.3, 3.2.4, and 3.2.5. $\square$

If we try to extend this idea to three dimensions, we are concerned with $n^2 - a^2$ being the sum of two squares. This is dependent on the factors of $n^2 - a^2$ and so is dependdnt on the choice of n.

3.3 Bounded Path Lengths

We are now equipped to prove our main theorem. The function $C(n)$ was defined in the introduction of the chapter.

3.3.1 Theorem. *Let p, q be two points in the same component of $U_{\mathbb{Q}}^n$, where $n \geq 4$. Then there exists a path of length at most $\lceil \|p - q\| \rceil + C(n) - 1$ from p to q.*

Proof. Let p and q be two distinct points in the same component of $U_{\mathbb{Q}}^n$. If $\|p - q\| \leq 2$, then there is a path of length at most $C(n)$ between them by Lemma 3.2.2, or Lemma 3.2.6.

If $r = \|p - q\|$ is an integer, then the points

$$r_a = \frac{a}{r}p + \frac{r-a}{r}p$$

are rational combinations of p and q for a an integer between 0 and r and are thus rational points. Also,

$$\|r_{a+1} - r_a\| = 1.$$

Thus, we have a path from p to q of length $\|p - q\|$.

If r is rational but not an integer, r_a are still rational points, and $r_{\lceil r \rceil - 2}$ is in the ball of radius 2 around p and the result follows.

When r is not a rational number, the point

$$s = \frac{1}{r}p + \frac{r-1}{r}q$$

is not in $U^n_{\mathbb{Q}}$. However, by Theorem 3.1.1, we can find a rational point s' on the unit sphere that is arbitrarily close to s.

Let

$$r' = \lceil r - 2 \rceil \text{ and } p' = q + r'(s' - q).$$

By construction, $\|s' - q\| = 1$, so there exists a path from q to p' of length r'. For some e, we have

$$\begin{aligned} p' &= q + r'(s + e - q) \\ &= q + r'\left(\frac{1}{r}p + \frac{r-1}{r}q + e - q\right) \\ &= q + \frac{r'}{r}p - \frac{r'}{r}q + r'e \\ &= \frac{r'}{r}p - \frac{r-r'}{r}q + r'e \end{aligned}$$

By construction, the distance from $\frac{r'}{r}p - \frac{r-r'}{r}q$ to q is less than 2 and we can make $r'e$ as small as we need, so p' is in the ball of radius 2 around p.

So by the results in section 3.2, there is a path from p' to p of length at most $C(n)$. Thus for this case we have a path from p to q of length at most $\lceil \|p - q\| \rceil + C(n) - 2$. □

Chapter 4

Colouring

The interest in unit distance graphs stems from trying to determine the chromatic number of $U^2_{\mathbb{R}}$. The problem is simple to state but the answer is elusive. In fact, for most unit distance graphs, the exact value of the chromatic number is still unknown.

The first major result was in 1972 by Larman and Rogers [11]. They showed that $\chi(U^n_{\mathbb{R}}) \leq (3+o(1))^n$. The result seems quite large, being exponential in n. However, in 1981, Frankl and Wilson [7] proved that $U^n_{\mathbb{Q}} \geq (1.2)^n(1+o(1))$. This shows that $\chi(U^n_{\mathbb{R}})$ grows exponentially, and puts Larman and Rogers' result into perspective.

4.1 Rational Colouring

The results in this section are not new. The two-colourability of $U^3_{\mathbb{Q}}$ and $U^2_{\mathbb{Q}}$ was originally shown by Johnson [10]. The idea behind the proof given here is similar, but is done so as not to require group theory.

The result for $U^4_{\mathbb{Q}}$ was first shown by Benda and Perles [1]. Their paper was not published until 2000, though it was circulating during the 1970s.

4.1.1 Theorem. *$U^3_{\mathbb{Q}}$ and $U^2_{\mathbb{Q}}$ are bipartite and thus 2-colourable.*

Proof. It is sufficient to prove the result for $U^3_{\mathbb{Q}}$ since $U^2_{\mathbb{Q}}$ is an induced subgraph of $U^3_{\mathbb{Q}}$.

Suppose we have a cycle in $U^3_{\mathbb{Q}}$, with edge set

$$\left\{\left(\frac{a_i}{d}, \frac{b}{d}, \frac{c_i}{d}\right)\right\}_{i=1}^{n}.$$

Then we have that

$$a_i^2 + b_i^2 + c_i^2 = d^2.$$

Examining this modulo 4, we see that if d is even, then a_i, b_i, c_i are even for all i. If we consider this in lowest terms, we may assume that d is odd. If we examine the equation modulo eight, we see that exactly one of a_i, b_i, c_i must be odd.

Since the edges form a cycle, we must have that

$$\sum_{i=1}^{n}(a_i, b_i, c_i) = (0, 0, 0).$$

This implies that

$$\sum_{i=1}^{n}(a_i + b_i + c_i) = 0.$$

Examining this modulo 2, we have that $\sum_{i=1}^{n} 1 = 0$ or equivalently $n \equiv 0$. Thus n is even and the cycle must have even length. Since there was nothing special about the cycle we chose, all cycles must have even length and the graph must be bipartite. $\square$

The graphs are clearly not 1-colourable as they contain edges.

4.1.2 Theorem. *$U_{\mathbb{Q}}^4$ has chromatic number 4.*

Proof. Consider the set of points

$$\left\{(0,0,0,0), (1,0,0,0), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right\}.$$

Each is a distance of 1 from all of the others, so the chromatic number must be at least 4.

Given a component of $U_{\mathbb{Q}}^4$ we can partition the vertex set into two classes A and B. Consider the component C that contins $(0,0,0,0)$. Let

$$A = \left\{\left(\frac{a}{n}, \frac{b}{n}, \frac{c}{n}, \frac{d}{n}\right) \mid n \text{ is odd}\right\}$$

$$B = V(C) \cap V(A)^c$$

By the characterization from Theorem 2.3.15,

$$\begin{aligned} B &= \left\{\left(\frac{a}{2n}, \frac{b}{2n}, \frac{c}{2n}, \frac{d}{2n}\right) \mid n, a, b, c, d \text{ are odd}\right\} \\ &= A + \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \end{aligned}$$

Consider a cycle in A,

$$\left\{\left(\frac{a_i}{e}, \frac{b_i}{e}, \frac{c_i}{e}, \frac{d_i}{e}\right)\right\}_{i=1}^{n}.$$

For these to all be edges, we have that

$$a_i^2 + b_i^2 + c_i^2 + d_i^2 = e^2 \text{ for each } i.$$

Examining this modulo 8, we have three possibilities.

$$\begin{array}{lr} e^2 \equiv 1 & \text{and one of } \{a_i, b_i, c_i, d_i\} \text{ is odd for each } i \\ e^2 \equiv 0 & \text{and } \{a_i, b_i, c_i, d_i\} \text{ are all even for each } i \\ e^2 \equiv 4 & \text{and } \{a_i, b_i, c_i, d_i\} \text{ are all odd or all even for each } i \end{array}$$

If there is one i where they are all odd, this is an edge between A and B, so there can be no such i. Thus, we see that if e is even, $\{a_i, b_i, c_i, d_i\}$ are even for each i. So we can assume e is odd.

So we have exactly one of $\{a_i, b_i, c_i, d_i\}$ is odd for each i. Since the edges form a cycle, we have that

$$\sum_{i=1}^{n} \left(\frac{a_i + b_i + c_i + d_i}{e_i} \right) = 0$$

$$\sum_{i=1}^{n} (a_i + b_i + c_i + d_i) = 0 \pmod 2$$

$$\sum_{i=1}^{n} 1 = 0 \pmod 2$$

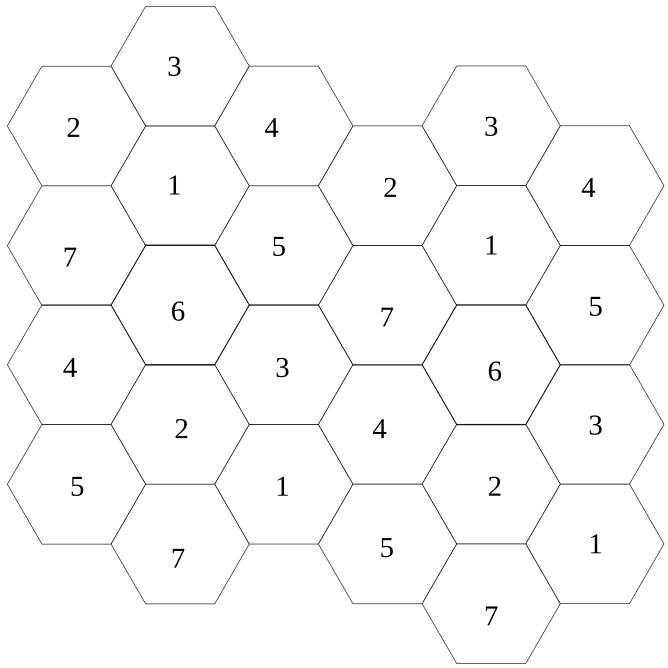
Thus, we see that n is even. So A and B are each bipartite. We can colour A with two colours and B with two colours and get a proper 4-colouring of the component. We can do this for each component since they are all translates of each other. Thus $U_{\mathbb{Q}}^4$ has chromatic number 4. □

Joseph Zaks [14] has provided lower bounds on the chromatic numbers for the next few cases. He shows $\chi(U_{\mathbb{Q}}^5) \geq 5$, $\chi(U_{\mathbb{Q}}^6) \geq 7$, $\chi(U_{\mathbb{Q}}^7) \geq 9$, and $\chi(U_{\mathbb{Q}}^8) \geq 10$. Chilakamarri [4] improved on this slightly by showing $\chi(U_{\mathbb{Q}}^5) \geq 6$.

4.2 Real Colouring

Determining the chromatic number of the real graph has proven to be a difficult problem. Current bounds for the two dimensional case are a lower bound of 4 and an upper bound of 7, which are shown below. However, the answer to the question might not be as simple as "it is either 4, 5, 6, or 7." Falconer [6] showed that if the colour classes form measurable sets of $\mathbb{R}^2$, then at least five colours are needed. Since the construction of non-measurable sets requires the axiom of choice, we might have the answer turn out to depend on whether or not we accept the axiom of choice.

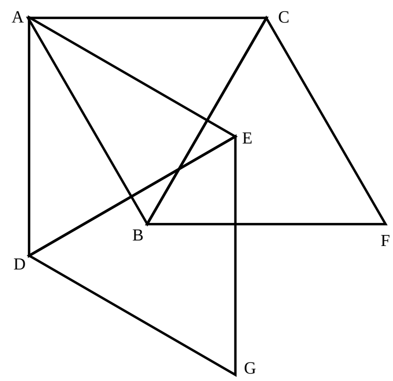
We can tile the plane with hexagons as below to obtain a proper 7-colouring of the graph. The result is originally due to Hadwiger and Debrunner [8].



For each point inside a hexagon, colour that point with the number inside the hexagon. For each point on an edge or vertex, colour it with the lowest colour of the hexagons incident to it. If the side length of the hexagon is slightly less than one half, no two points in or on the boundary of a single hexagon are at a distance one from each other. Also, the distance between any two hexagons of the same colour is greater than one, so we have a proper colouring of the plane with seven colours.

We could also tile the plane with squares instead of hexagons, and obtain a proper 9-colouring of the plane. This can be extended to cubes in three dimensions, for a proper 27 colouring of $\mathbb{R}^3$. We can not extend this directly to arbitrary dimensions because in high dimension, the diagonal of the cube gets large compared to the side length.

We can construct a small subgraph of the plane that is not three colourable. This was shown by Moser and Moser [12]. Consider the diagram below:



The two diamond shapes can be rotated about point A so that points F and G are one unit apart. If we then try to 3-colour the graph starting with vertex A, we see that veticies F and G must have the same colour as A.

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