Hamilton Connected or Hamilton Laceable Generalized Petersen Graphs

by

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Abstract

The generalized Petersen graphs $\mathrm{GP}(n,k)$ where introduced by Watkins in search for additional examples of non-Hamiltonian vertex-transitive graphs. Alspach and Qin showed that Cayley graphs for certain groups are Hamiltonian using the fact that $\mathrm{GP}(4m, 2m-1)$ is Hamilton laceable (it is bipartite, and any two vertices on different sides of the bipartition are joined by a Hamilton path).

For $k = 1$, 2, and 3, we completely determine which pairs of vertices in $\mathrm{GP}(n,k)$ are joined by Hamilton paths. We also provide a general approach for proving that $\mathrm{GP}(n,k)$ is Hamilton connected or Hamilton laceable.

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Chapter 1

Introduction

The principle motivation for studying Hamiltonicity in the generalized Petersen graph are the following conjectures.

Conjecture 1.1. *Vertex transitive graphs are Hamiltonian.*

Conjecture 1.2 (Lovász' Conjecture)**.** *Cayley graphs are Hamiltonian.*

Since the Petersen graph is a counterexample to the first conjecture, it is natural to wonder if the generalized Petersen graphs share this property. It was found that the Petersen graph remains the only vertex-transitive generalized Petersen graph that is not Hamiltonian.

Generalized Petersen graphs have been used to prove Lovász' conjecture on Cayley graphs, for certain groups. In particular, Hamilton connectedness of certain generalized Petersen graphs was needed.

1.1 Background

For integers n, k with $1 \leq k < n$, Watkins [12] defines the *generalized Petersen graph* $\mathrm{GP}(n,k)$ to be the graph with vertex set $\{u_i, v_i \mid 0 \leq i \leq n-1\}$ and edge set $\{u_iu_{i+1}, v_iv_{i+k}, u_iv_i \mid 0 \leq i \leq n-1\}$, where the subscript arithmetic is done modulo n. We will only consider the generalized Petersen graphs that are cubic; hence we assume that $n \neq 2k$.

Since these graphs mimic a structural property of the Petersen graphs, it was of interest to see if any of the vertex-transitive generalized Petersen graphs would be non-Hamiltonian. This problem was studied in two parts. Robertson [11] and Bondy [5] both proved that $\mathrm{GP}(n,2)$ is Hamiltonian if and only if $n \not\equiv 5 \pmod 6$. They conjectured that these were the only non-Hamiltonian generalized Petersen graphs. Bondy also proved that GP$(n, 3)$ is Hamiltonian for all $n \neq 5$. In solving the conjecture of Bondy and Robertson, Bannai [6] showed that GP(n, k) is Hamiltonian when n and k are relatively prime, and GP(n, k) is not isomorphic to GP$(n, 2)$, with $n \equiv 5 \pmod 6$ and Alspach [1] completed the proof of this conjecture. For the other part of the problem, Frucht, Graver, and Watkins [8] showed that GP(n, k) is vertex-transitive if and only if $k^2 \equiv \pm 1 \pmod n$ or $(n, k) = (10, 2)$. Together this implied that the Petersen graph is the only non-Hamiltonian, vertex-transitive generalized Petersen graph.

Although no new vertex-transitive, non-Hamiltonian graphs were found in this family of graphs, the hamiltonicity of the generalized Petersen graphs was used in solving the conjecture on vertex transitive graphs for other families of graphs, as shown in [2].

Progress on the Lovász conjecture has used a property stronger than the graph being Hamiltonian. A *Hamilton uv-path* is a path with ends u and v that contains all the vertices of the graph. A graph G is *Hamilton connected* if, for every pair u, v of vertices, G has a Hamilton uv-path. Some generalized Petersen graphs are bipartite and, therefore, cannot be Hamilton connected. A bipartite graph G is *Hamilton laceable* if, for any two vertices u and v on different sides of the bipartition, G has a Hamilton uv-path.

Alspach and Qin [4] proved that GP$(4m, 2m - 1)$ is Hamilton laceable and used this to show that Cayley graphs on certain groups are Hamiltonian, settling Lovász' conjecture for these groups.

Chen and Quimpo [7] consider which Cayley graphs are Hamilton connected or Hamilton laceable. They showed that a connected Cayley graph of valency at least three on an Abelian group is Hamilton connected, unless it is bipartite in which case it is Hamilton laceable. It is possible that the generalized Petersen graph will play a similar role in determining graphs which are Hamilton connected or Hamilton laceable as it did in the search for Hamilton cycles. For this reason we see the following developments in solving this problem.

First we will determine when the generalized Petersen graph is bipartite, as presented in [3].

Theorem 1.3. *GP(n, k) is bipartite if and only if n is even and k is odd.*

Alspach and Lui [3] prove the following:

- if n is odd, then $\mathrm{GP}(n, 1)$ is Hamilton connected;
- if n is even, then $\mathrm{GP}(n, 1)$ is Hamilton laceable;
- if n is odd and $n \not\equiv 5 \pmod 6$, then $\mathrm{GP}(n, 2)$ is Hamilton connected;
- if n is odd and $n \neq 5$, then $\mathrm{GP}(n, 3)$ is Hamilton connected; and
- if n is even, and $n \neq 2, 6$, then $\mathrm{GP}(n, 3)$ is Hamilton laceable.

Mavraganis [9] proves that:

- if n is odd, then $\mathrm{GP}(n, 1)$ is Hamilton connected; and
- if n is even, then $\mathrm{GP}(n, 1)$ is Hamilton laceable,

using general path structures. She determines almost all of the pairs of vertices in $\mathrm{GP}(2m, 2)$ that are joined by Hamilton paths.

Alspach and Qin [4] prove that $\mathrm{GP}(4m, 2m - 1)$ is Hamilton laceable.

Pensaert [10] takes a different approach, proving that, for $k \geq 3$, $\mathrm{GP}(3k + 1, k)$ is Hamilton connected if k is even, and Hamilton laceable if k is odd. He also makes the conjecture

Conjecture 1.4. *For $n \geq 3k$:*

- *if n is even and k is odd, then $GP(n, k)$ is Hamilton laceable; and*
- *for all other combinations of parities of n and k, $GP(n, k)$ is Hamilton connected.*

This conjecture and the examples given by Pensaert allude to a potentially very interesting property in the generalized Petersen graph, namely:

Conjecture 1.5. *If k is even, then for j even, no Hamilton u_0u_j-path exists in $GP(3k, k)$.*

Although unproven, the following examples demonstrate that this property likely holds.

- Hamilton u_0u_2 and u_0u_4-paths do not exist in $\mathrm{GP}(6, 2)$.
- $\mathrm{GP}(9, 3)$ is Hamilton connected.

- For j even, Hamilton u_0u_j-paths do not exist in $\mathrm{GP}(12, 4)$, $\mathrm{GP}(18, 6)$, $\mathrm{GP}(24, 8)$, or $\mathrm{GP}(30, 10)$.

- For j even, Hamilton u_0u_j-paths do exist in $\mathrm{GP}(15, 5)$, $\mathrm{GP}(21, 7)$.

Hence, we make the following conjecture.

Conjecture 1.6. *For $k > 2$ and $n > 2k$:*

- *if n is even and k is odd, then $GP(n, k)$ is Hamilton laceable;*

- *if $n = 3k$ and k is even, then, for j even, no Hamilton u_0u_j-path exists; and,*

- *for all other combinations of n and k, $GP(n, k)$ is Hamilton connected.*

1.2 Properties of GP(n,k)

In this section, we present two automorphisms of $\mathrm{GP}(n, k)$, initially described by Watkins [12], that will simplify our work. In general we are interested, for $x, y \in \{u, v\}$ and $i, j \in \mathbb{Z}_n$, whether there is a Hamilton x_iy_j-path in $\mathrm{GP}(n, k)$.

The first automorphism is the rotational symmetry $T : \mathbb{Z}_n \to \mathbb{Z}_n$ defined by $T(i) = i + 1$. This symmetry shows that there is a Hamilton x_iy_j-path if and only if there is a Hamilton x_0y_{j-i}-path.

The second automorphism is the reflective symmetry $R : \mathbb{Z}_n \to \mathbb{Z}_n$ defined by $R(i) = n - i$. This symmetry shows that there is a Hamilton x_0y_j-path if and only if there is a Hamilton x_0y_{n-j}-path.

Thus, to show $\mathrm{GP}(n, k)$ is Hamilton connected we only need to determine for which $j \in \mathbb{Z}_n$ and $j \leq \lfloor \frac{n}{2} \rfloor$ there is a:

- Hamilton u_0u_j-path;

- Hamilton u_0v_j-path; and

- Hamilton v_0v_j-path.

To show $\mathrm{GP}(n, k)$ is Hamilton laceable, we only need to determine for which $j \in \mathbb{Z}_n$, $j \leq \lfloor \frac{n}{2} \rfloor$ and j odd there is a:

- Hamilton u_0u_j-path;

- Hamilton $v_0 v_j$-path; and
- for j even, a Hamilton $u_0 v_j$-path.

R also describes an isomorphism between $\mathrm{GP}(n, k)$ and $\mathrm{GP}(n, n-k)$. Hence we may assume that $k \leq \lfloor \frac{n}{2} \rfloor$.

1.3 Organization of Essay

This essay is divided into three chapters. In Chapter 2 we will be considering the $k = 1$ and $k = 2$ cases. For $k = 1$, we will present the different proofs provided by Alspach and Lui [3] and Mavraganis [9]. For $k = 2$, we will prove that $\mathrm{GP}(n, 2)$ is Hamilton connected if and only if $n \equiv 1, 3 \pmod 6$, and will completely determine the existence and nonexistence of Hamilton paths for all other values of n. In Chapter 3, we will show that $\mathrm{GP}(n, 3)$ is Hamilton connected if and only if $n > 5$ is odd and it is Hamilton laceable if and only if $n \geq 4$ is even and $n \neq 6$. In Chapter 4, we will generalized the ideas we present. Initially we will take a brief look at $\mathrm{GP}(n, 4)$ and then we will provide a general approach for proving that $\mathrm{GP}(n, k)$ is Hamilton connected or laceable, completing ideas of Yehua Wei.

Chapter 2

GP(n,k), k=1,2

2.1 Introduction

In this chapter, for $k = 1, 2$, we completely determine which pairs of vertices in GP(n, k) are joined by Hamilton paths.

2.2 GP(n,1)

In this section, we treat the case $k = 1$ by proving the following theorem.

Theorem 2.1. *The graph GP(n, 1), $n \geq 3$, is Hamilton connected, unless it is bipartite in which case it is Hamilton laceable.*

This theorem is proven by both Alspach and Lui and Mavraganis. We will state both proofs.

Proof by Alspach and Lui [3]. GP(n, 1) is a connected Cayley graph on an Abelian group. By the Chen-Quimpo theorem [7] it is Hamilton connected, unless it is bipartite in which case it is Hamilton laceable. □

Proof by Mavraganis [9]. For n even and j odd, $j \in \mathbb{Z}_n$, there exists the Hamilton u_0u_j-path

$$u_0u_{n-1}\cdots u_{j+1}v_{j+1}v_{j+2}\cdots v_0v_1u_1u_2v_2v_3\cdots v_{j-1}v_ju_j,$$

For j even, there exists the Hamilton u_0v_j-path

$$u_0u_{n-1}\cdots u_{j+1}v_{j+1}v_{j+2}\cdots v_0v_1u_1u_2v_2v_3\cdots u_{j-1}u_jv_j.$$

Since there exists an automorphism interchanging u_i and v_i, $\mathrm{GP}(n,1)$ is Hamilton laceable for n even.

For n odd and j even, $j \in \mathbb{Z}_n$, there exists the Hamilton u_0u_j-path

$$u_0u_{n-1}\cdots u_{j+1}v_{j+1}v_{j+2}\cdots v_0v_1u_1u_2v_2v_3\cdots v_{j-1}v_ju_j$$

and the Hamilton u_0v_j-path

$$u_0u_1\cdots u_{j+1}v_{j+1}v_{j+2}u_{j+2}u_{j+3}\cdots v_{n-1}v_0v_1\cdots v_j.$$

By reflective symmetry there exist Hamilton u_0u_j and u_0v_j-paths for j even. Therefore, for n odd, $\mathrm{GP}(n,1)$ is Hamilton connected. $\square$

2.3 GP(n,2)

In the following sections, we determine all the pairs of vertices in $\mathrm{GP}(n,2)$ that are joined by Hamilton paths. In particular, we prove the following.

Theorem 2.2. *Let $n \geq 5$ be an integer. Let $x, y \in \{u, v\}$ and $i, j \in \mathbb{Z}_n$.*

1. *The graph $GP(n,2)$ is Hamilton connected if and only if $n \equiv 1, 3 \pmod 6$.*
2. *Suppose $n \equiv 0 \pmod 6$. There is a Hamilton x_iy_j-path in $GP(n,2)$ if and only if one of the following holds:*
   (a) *$\{x, y\} = \{u, v\}$;*
   (b) *$x = u$, $y = u$, and $j - i \not\equiv \pm 2 \pmod n$; and*
   (c) *$x = v$, $y = v$, and $j \not\equiv i \pmod 6$.*
3. *Suppose $n \equiv 2 \pmod 6$. There is a Hamilton x_iy_j-path in $GP(n,2)$ if and only if one of the following holds:*
   (a) *$x = u$, $y \in \{u, v\}$; and*
   (b) *$x = v$, $y = v$, and $j - i \not\equiv 4 \pmod 6$.*
4. *Suppose $n \equiv 4 \pmod 6$. There is a Hamilton x_iy_j-path in $GP(n,2)$ if and only if one of the following holds:*
   (a) *$x = u$, $y = u$, and $j - i \not\equiv \pm 2 \pmod n$;*

(b) $x = u$, $y = v$, $j - i \not\equiv \pm 1 \pmod{n}$, *and* $j - i \not\equiv 2 \pmod 6$; *and*

(c) $x = v$, $y = v$, *and* $j - i \not\equiv 0, 4 \pmod 6$.

5. *Suppose* $n \equiv 5 \pmod 6$. *There is a Hamilton* $x_i y_j$*-path in* $GP(n, 2)$ *if and only if one of the following holds:*

(a) $x = u$, $y = u$, *and* $j - i \not\equiv \pm 1 \pmod 6$;

(b) $x = u$, $y = v$, *and* $j \neq i$; *and*

(c) $x = v$, $y = v$, *and* $j - i \not\equiv 2, 3 \pmod 6$.

The existence of Hamilton paths will be proved by induction on n, using an operation that we will call an (i, j)-expansion. We also prove the nonexistence of Hamilton paths in $\mathrm{GP}(n, 2)$ by induction on n, using an (i, j)-reduction operation.

2.4 Expanding in GP(n,2)

The existence of the Hamilton paths will be shown inductively. In order to do so we introduce the following general terms. For $i \in \mathbb{Z}_n$, the *cut*, C_i in $\mathrm{GP}(n, k)$, is the set of edges $\{u_{i-1}u_i, v_{i-k}v_i, v_{i-k+1}v_{i+1}, \ldots, v_{i-1}v_{i+k-1}\}$. (The indices are read modulo n.) We will show in this section how any cut may be used to convert a Hamilton path in $\mathrm{GP}(n, 2)$ into a Hamilton path in $\mathrm{GP}(n+6, 2)$. This is a modification of the methods used by Alspach and Lui [3] and Mavraganis [9].

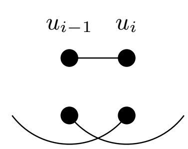

Figure 2.1: A cut C_i in $\mathrm{GP}(n, 2)$.

For $i \in \mathbb{Z}_n$ and $j \in \mathbb{Z}$, where j is a positive multiple of k, the (i, j)-*expansion of* $GP(n, k)$ defines a new graph, denoted $G^k_{\prec i,j}$, obtained from $\mathrm{GP}(n, k)$ by deleting the edges in C_i and adding the $2j$ vertices $\{u'_0, v'_0, \ldots, u'_{j-1}, v'_{j-1}\}$ at C_i as well as the edges: $u_{i-1}u'_0$; $u'_{j-1}u_i$; for $0 \leq \ell \leq j - 1$, the edges $u'_\ell v'_\ell$; for $0 \leq \ell \leq j - 2$, the edges $u'_\ell u'_{\ell+1}$; for $0 \leq \ell < k$, the edges

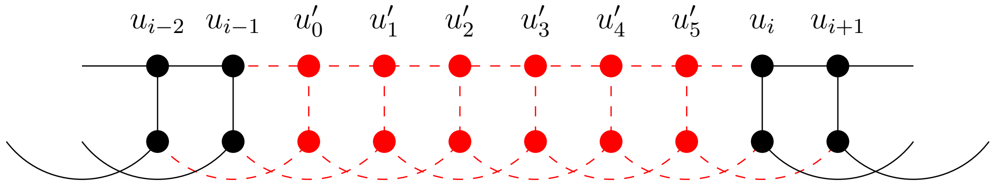

Figure 2.2: $(i,6)$-expansion in $\mathrm{GP}(n,2)$

$v_{i-k+\ell}v'_\ell$ and $v'_{j-k+\ell}v_{i+\ell}$; and, for $0 \le \ell \le j-1-k$, the edges $v'_\ell v'_{\ell+k}$. Note: $G^k_{\prec_{i,j}}$ is isomorphic to $\mathrm{GP}(n+j,k)$.

For convenience, we will use the following notation. Let n and k be positive integers with $n > 2k$. For $i, j \in \mathbb{Z}_n$, let $V_{i,j}$ denote the set of vertices $\{u_i, \ldots, u_{i+j}, v_i, \ldots, v_{i+j}\}$ in $\mathrm{GP}(n,k)$. Let $S_{i,j}$ be the subgraph of $\mathrm{GP}(n,k)$ consisting of the edges incident with any vertex of $V_{i,j}$ and all their incident vertices. Extremally, $v_{i-k}, v_{i-k+1}, \ldots, v_{i-2}, v_{i+j+1}, v_{i+j+2}, \ldots, v_{i+j+k-1}$ are all in $S_{i,j}$. An (i,j)-*strand* of a path P in $\mathrm{GP}(n,k)$ is a component of $P \cap S_{i,j}$. The following theorem is key to the induction.

Theorem 2.3. *Let $x, y \in \{u, v\}$. If there exists a Hamilton x_0y_j-path in $GP(n,2)$, then there exists a Hamilton x_0y_j-path in $GP(n+6,2)$.*

Proof. Let P be a Hamilton x_0y_j-path in $\mathrm{GP}(n,2)$. We will show that, for $j < i \le n$, we can apply an $(i,6)$-expansion so that P becomes a Hamilton path in $G^2_{\prec i,6}$. Since $j < i$, j is the same index in both $\mathrm{GP}(n,2)$ and $\mathrm{GP}(n+6,2)$. Of the eight cases, we consider four that are representative of the rest.

If no edge of P is in C_i, then at least one of $u_{i-1}v_{i-1}$ and u_iv_i is in P. Assume the latter. Then replace u_iv_i in P with the path

$$u_iu'_5v'_5v'_3v'_1u'_1u'_0v'_0v'_2u'_2u'_3u'_4v'_4v_i$$

to create the Hamilton x_0y_j-path in $G^2_{\prec i,6}$.

If $|C_i \cap P| = 1$, then $C_i \cap P = u_{i-1}u_i$, $v_{i-1}v_{i+1}$, or $v_{i-2}v_i$. Suppose $C_i \cap P = u_{i-1}u_i$. Then we can replace $u_{i-1}u_i$ with

$$u_{i-1}u'_0v'_0v'_2v'_4u'_4u'_3u'_2u'_1v'_1v'_3v'_5u'_5u_i$$

to create a Hamilton x_0y_j-path in $G^2_{\prec_{i,6}}$. A similar argument holds for $C_i \cap P = v_{i-1}v_{i+1}$, or $v_{i-2}v_i$.

If $|C_i \cap P| = 2$, then $C_i \cap P = \{u_{i-1}u_i, v_{i-1}v_{i+1}\}$, $\{u_{i-1}u_i, v_{i-2}v_i\}$, or $\{v_{i-1}v_{i+1}, v_{i-2}v_i\}$. Suppose $C_i \cap P = \{u_{i-1}u_i, v_{i-1}v_{i+1}\}$. Then we can replace $u_{i-1}u_i$ and $v_{i-1}v_{i+1}$ with

$$u_{i-1}u'_0v'_0v'_2v'_4u'_4u'_5u_i \text{ and } v_{i-1}v'_1u'_1u'_2u'_3v'_3v'_5v_{i+1},$$

respectively, to create a Hamilton x_0y_j-path in $G^2_{\prec i,6}$. A similar argument holds if $C_i \cap P = \{u_{i-1}u_i, v_{i-2}v_i\}$, or $\{v_{i-1}v_{i+1}, v_{i-2}v_i\}$.

If $|C_i \cap P| = 3$, then $C_i \cap P = C_i$. We can replace $u_{i-1}u_i$, $v_{i-2}v_i$, and $v_{i-1}v_{i+1}$ with

$$u_{i-1}u'_0u'_1 \dots u'_5u_i,\ v_{i-2}v'_0v'_2v'_4v_i, \text{ and } v_{i-1}v'_1v'_3v'_5v_{i+1},$$

respectively, to create a Hamilton x_0y_j-path in $G^2_{\prec i,6}$. □

In the rest of this section we will prove inductively that all the necessary paths exist as defined in the main theorem. For the base cases, we need to exhibit appropriate Hamilton paths. An effective tool is the *Posa exchange*: if $P = x_0x_1 \cdots x_n$ is a Hamilton path and x_n is adjacent to x_i with $0 \le i < n-1$, then $(P - x_ix_{i+1}) + x_nx_i$ is a Hamilton x_0x_{i+1}-path. We will combine this method with the reflective symmetry of the graph to find all the necessary paths. Since the induction argument for each case is very similar we will prove two of the cases, $n \equiv 0 \pmod 6$ and $n \equiv 1 \pmod 6$, in detail. For the other values of n we will provide the details for the base cases.

Theorem 2.4. *If $n \equiv 0 \pmod 6$ and $j \in \mathbb{Z}_n$, then in $GP(n, 2)$ a Hamilton path exists for the pairs (u_0, u_j) for all $j \ne 2, n-2$, (u_0, v_j) for all j, and (v_0, v_j) for all $j \not\equiv 0 \pmod 6$.*

Proof. We proceed by induction of n. Let $n = 6$ and $n = 12$ be the base cases. From the Hamilton path

$$P = u_0v_0v_2v_4u_4u_5v_5v_1v_3u_3u_2u_1$$

we can do (consecutively) the sequence of Posa exchanges using the edges u_1v_1 (to get the Hamilton u_0v_3-path), v_3v_5 (to get the Hamilton u_0v_1-path), and v_1v_3 (to get the Hamilton u_0u_3-path). Starting with P again, we do the sequence of Posa exchanges using u_0u_1 and v_0v_4 to get the Hamilton u_0v_0 and u_0v_2-paths. From the Hamilton path

$$P' = v_0u_0u_0v_5v_3u_3u_4v_4v_2u_2u_1v_1$$

the Posa exchange using the edge v_1v_5 gives a Hamilton v_0v_3-path and the consecutive Posa exchanges using the edges v_1v_3 and u_2u_3 gives the Hamilton v_0v_2-path. By reflective symmetry, the theorem holds for $n = 6$.

The preceding paragraph and Theorem 2.3 imply that, in $\mathrm{GP}(12, 2)$ there are Hamilton paths for the pairs (u_0, u_j), for all $j \in \{1, 3, 5\}$, (u_0, v_j), for all $j \in \{0, \ldots, 5\}$, and (v_0, v_j), for all $j \in \{1, 2, 3, 4, 5\}$. Given the Hamilton path

$$u_0v_0v_{10}v_8u_8u_7v_7v_9u_9u_{10}u_{11}v_{11}v_1u_1u_2v_2v_4v_6u_6u_5v_5v_3u_3u_4,$$

we can obtain Hamilton u_0v_6- and u_0u_6-paths by a sequence of Posa exchanges using the edges u_4v_4, v_6v_8, u_8u_9, v_9v_{11}, $u_{11}u_0$, v_0v_2, and v_4v_6. Therefore, the theorem holds for $n = 12$.

Suppose the theorem holds for $\mathrm{GP}(n, 2)$ where $n \equiv 0 \pmod 6$ and $n \geq 12$. Let $x, y \in \{u, v\}$ and $j \in \mathbb{Z}_{n+6}$ be such that the theorem asserts the existence of a Hamilton x_0y_j-path. If $j \in \mathbb{Z}_n$, then, except for $(x_0, y_j) = (u_0, u_{n-2})$, the inductive hypothesis and Theorem 2.3 imply the existence of the Hamilton x_0y_j-path in $\mathrm{GP}(n + 6, 2)$.

In the case $(x_0, y_j) = (u_0, u_{n-2})$, the vertex u_{n-2} is symmetric to u_8 in $\mathrm{GP}(n + 6, 2)$. Thus the Hamilton u_0u_{n-2}-path exists by reflective symmetry and the inductive hypothesis.

For $j \in \{n, n + 1, \ldots, n + 5\}$, we use reflective symmetry to get all the asserted Hamilton x_0y_j-paths. (For example, a Hamilton u_0v_n-path exists, since there exists a Hamilton u_0v_6-path, but no Hamilton v_0v_n-path exists, since no Hamilton v_0v_6-path exists.) Hence, all the paths defined in the theorem exist in $\mathrm{GP}(n + 6, 2)$, as required. □

Theorem 2.5. *If $n \equiv 1 \pmod 6$, then $GP(n, 2)$ is Hamilton connected.*

Proof. We proceed by induction on n. Let $n = 7$ be the base case. From the Hamilton path

$$P = u_0v_0v_5u_5u_6v_6v_1v_3u_3u_4v_4v_2u_2u_1,$$

the Posa exchange using the edge u_0u_1 gives the Hamilton u_0v_0-path. Starting with P again, we consecutively do the Posa exchanges using the edges u_1v_1 (to get to v_3), v_3v_5 (to get to u_5), u_4u_5 (to get to v_4), v_4v_6 (to get to v_1), v_1v_3 (to get to u_3), and u_2u_3 (to get to v_2). From the Hamilton path

$$v_0u_0u_1u_2v_2v_4u_4u_3v_3v_5u_5u_6v_6v_1$$

a Hamilton v_0v_5 and v_0v_4-path can be obtained by the sequence of Posa exchanges using the edges v_1v_3, v_0v_5, u_0u_6, v_4v_6, u_4u_5, and u_6v_6. By reflective and rotational symmetry, there exists a Hamilton path between any pair of vertices in the graph. Therefore GP$(7, 2)$ is Hamilton connected.

Suppose the theorem holds for $n \equiv 1 \pmod 6$, where $n \geq 7$. Then in GP$(n + 6, 2)$, there exists a Hamilton path for all pairs of vertices contained in $V_{0,n-1}$, by Theorem 2.3 and the inductive hypothesis. In GP$(n+6, 2)$, each vertex contained in $V_{n,5}$ is symmetric to a vertex contained in $V_{0,5}$. Therefore, by the inductive hypothesis and reflective and rotational symmetry, a Hamilton path exists for all pairs of vertices in GP$(n+6, 2)$, as required. □

For the following theorems we will only provide the proof for the base cases.

Theorem 2.6. *If $n \equiv 2 \pmod 6$ and $j \in \mathbb{Z}_n$, then in GP$(n, 2)$ a Hamilton path exists for the pairs (u_0, u_j) for all j, (u_0, v_j) for all j, and (v_0, v_j) for all $j \not\equiv 4 \pmod 6$.*

Proof of base case. From the Hamilton path

$$u_0v_0v_2v_4v_6u_6u_7v_7v_1v_3v_5u_5u_4u_3u_2u_1$$

in GP$(8, 2)$, we can do (consecutively) the Posa exchanges using the edges u_1v_1 (to get to v_3), v_3u_3 (to get to u_4), u_4v_4 (to get to v_6), v_0v_6, v_2u_2 (to get to u_3), u_3u_4, u_5u_6, u_7u_0 (to get to v_0), v_0v_2 (to get to v_4), v_4v_6 (to get to u_6), and u_6u_7 (to get to v_7). Reflective symmetry completes the task. Let P' be the Hamilton path

$$v_0u_0u_7v_7v_5v_3u_3u_4u_5u_6v_6v_4v_2u_2u_1v_1.$$

The consecutive Posa exchanges from P' using the edges v_1v_7, v_5u_5, and u_4v_4 give the Hamilton v_0v_5 and v_0v_6-paths. By reflective symmetry all the necessary paths exist. □

Theorem 2.7. *If $n \equiv 3 \pmod 6$, then GP$(n, 2)$ is Hamilton connected.*

Proof of base case. From the Hamilton path

$$u_0u_1u_2v_2v_0v_7v_5u_5u_4v_4v_6u_6u_7u_8v_8v_1v_3u_3$$

in GP$(9, 2)$, we can do the consecutive Posa exchanges using the edges u_3u_4 (to get to v_4), v_4v_2 (to get to v_0), v_0u_0 (to get to u_1), u_1v_1 (to get to v_8),

v_8v_6, u_6u_5 (to get to u_4), u_4v_4 (to get to v_2), v_2v_0, v_7u_7, u_6v_6, v_8v_1 (to get to v_3), v_3v_5, u_5u_4, and v_4v_2 (to get to u_2). From the Hamilton path

$$v_0u_0u_1u_2v_2v_4v_6u_6u_5u_4u_3v_3v_5v_7u_7u_8v_8v_1,$$

we can do the consecutive Posa exchanges using the edges v_1u_1, u_2u_3 and u_4v_4 (to get to v_6), v_6v_8, u_8u_0, and u_1u_2 (to get to v_2), v_2v_0, u_0u_1, and u_2v_2 (to get to v_4). By reflective and rotational symmetry, $\mathrm{GP}(9,2)$ is Hamilton connected. □

Theorem 2.8. *If $n \equiv 4 \pmod 6$ and $j \in \mathbb{Z}_n$, then in $GP(n,2)$ a Hamilton path exists for the pairs (u_0, u_j) for all $j \neq 2, n-2$, (u_0, v_j) for all $j \neq 1, n-1$ and $j \not\equiv 2 \pmod 6$, and (v_0, v_j) for all $j \not\equiv 0, 4 \pmod 6$.*

Proof of base case. Let P be the Hamilton path

$$u_0v_0v_2v_4v_6v_8u_8u_9v_9v_1v_3v_5v_7u_7u_6u_5u_4u_3u_2u_1$$

in $\mathrm{GP}(10,2)$. From P, we can do a sequence of Posa exchanges using the edges u_1u_0 (to get to v_0), v_0v_8 (to get to v_6), v_6u_6 (to get to u_7), u_7u_8, u_9u_0, u_1v_1 (to get to v_3), and v_3u_3 (to get to u_4). From the Hamilton path

$$v_0v_8u_8u_7u_6v_6v_4v_2u_2u_1u_0u_9v_9v_7v_5u_5u_4u_3v_3v_1,$$

we can do the consecutive Posa exchanges using the edges v_1v_9 (to get to v_7), v_7u_7, u_6u_5, and u_4v_4 (to get to v_2). The existence of the paths

$$u_0u_9v_9v_1u_1u_2v_2v_0u_8u_8u_7v_7v_5v_3u_3u_4v_4v_6u_6u_5,$$

$$u_0v_0v_2u_2u_1v_1v_3u_3u_4v_4v_6v_8u_8u_9v_9v_7u_7u_6u_5v_5, \text{ and}$$

$$v_0v_8v_6v_4v_2u_2u_1u_0u_9u_8u_7u_6u_5u_4u_3v_3v_1v_9v_7v_5$$

and reflective symmetry completes the task.

Also, in $\mathrm{GP}(16,2)$ a Hamilton u_0u_8-path exists. □

Theorem 2.9. *If $n \equiv 5 \pmod 6$ and $j \in \mathbb{Z}_n$, then in $GP(n,2)$ a Hamilton path exists for the pairs (u_0, u_j) for all $j \neq 1, n-1$, (u_0, v_j) for all $j \neq 0$, and (v_0, v_j) for all $j \not\equiv 2, 3 \pmod 6$.*

Proof of base cases. For GP$(5,2)$, the Petersen graph, the Hamilton path

$$v_0u_0u_1u_2v_2v_4u_4u_3v_3v_2$$

exists. From the Hamilton path

$$u_0v_0v_2v_4u_4u_3v_3v_1u_1u_2$$

we can do a sequence of Posa exchanges using the edges u_2v_2 (to get to v_4) and v_4v_1 (to get to v_3). Reflective symmetry completes the task.

In GP$(11,2)$, the preceding paragraph and Theorem 2.3 imply that there exists a Hamilton path for the pairs (u_0,u_j) for $j=2,3$, (u_0,v_j) for all $j\in\{1,2,3,4\}$, and (v_0,v_j) for $j=1,4$. The existence of the paths

$$u_0u_1v_1v_3u_3v_2v_2v_0v_9u_9u_{10}v_{10}v_8u_8u_7v_7v_5u_5u_6v_6v_4u_4,$$

$$u_0u_{10}v_{10}v_8u_8v_9v_9v_0v_2u_2u_1v_1v_3u_3u_4v_4v_6u_6u_7v_7v_5u_5,$$

$$u_0v_0v_2v_4u_4u_5u_6v_6v_8u_8u_7v_7v_9u_9u_{10}v_{10}v_1u_1u_2u_3v_3v_5, \text{ and}$$

$$v_0v_2u_2u_3v_3v_1u_1u_0u_{10}v_{10}v_8u_8u_9v_9v_7u_7v_6v_6v_4u_4u_5v_5,$$

and reflective symmetry imply that the necessary paths exist in GP$(11,2)$. □

Now that we have established which Hamilton paths exist, we look at the paths that do not exist.

2.5 Reducing in GP(n,2)

In this section we will complete the proof of Theorem 2.2. We do this in two steps. First we show that, for large n, an (i,j)-reduction of GP(n,k) can be applied while maintaining the Hamilton path. For i and $j\in\mathbb{Z}_n$, an *(i,j)-reduction of GP(n,k)*, denoted $G^k_{\succ i,j}$, is the graph obtained from GP(n,k) by deleting the edges $u_\ell v_\ell$ for all $\ell\in\{i,\ldots,i+j-1\}$ and contracting the edges $u_\ell u_{\ell+1}$ and $v_{\ell-1}v_{\ell+k-1}$ for all $\ell\in\{i,\ldots,i+j-1\}$. Thus the vertex set of $G^k_{\succ i,j}$ is $V_{0,i-1}\cup V_{i+j,n-i+j}$. Note: $G^k_{\succ i,j}$ is isomorphic to GP$(n-j,k)$. Since this is an inductive argument, in the second step we show that these paths do not exist in the base cases.

In order to accomplish the first step, we need to understand what the necessary conditions are for applying an (i,j)-reduction of $\mathrm{GP}(n,k)$. The following definitions will help us.

For some $i, j \in \mathbb{Z}_n$, each pair of edges

$$\{u_{i-1}u_i, u_{i+j-1}u_{i+j}\}\,\{v_{i-k}v_i, v_{i+j-k}v_{i+j}\}\,\{v_{i-k+1}v_{i+1}, v_{i+j-k+1}v_{i+j+1}\}, \ldots,$$

$$\{v_{i-1}v_{i+k-1}, v_{i+j-1}v_{i+j+k-1}\}$$

is defined to be (i,j)-*congruent*.

For a Hamilton path P, the cuts C_i and C_{i+j} are P-*congruent* if, for each (i,j)-strand Q of P, $Q \cap C_i$ is (i,j)-congruent to $Q \cap C_{i+j}$. (See figure 2.3) This is the main concept needed in our inductive argument, which the following theorem develops.

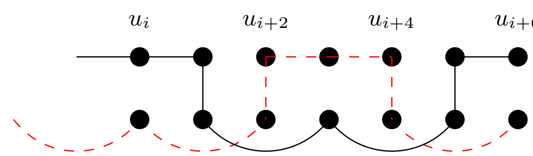

Figure 2.3: C_i and C_{i+6} are P-congurent in $\mathrm{GP}(n,2)$.

Theorem 2.10. *Let $x, y \in \{u, v\}$, $j \in \mathbb{Z}_n$, and P be a Hamilton x_0y_j-path in $GP(n,2)$. Suppose there exists an i such that $x_0, y_j \notin V_{i,5}$. If $|C_i \cap P| \neq 0$ and C_i and C_{i+6} are P-congruent, then there is a Hamilton x_0y_j-path in $G^2_{\succ_{i,6}}$.*

Proof. Let Q be an $(i,6)$-strand of P. Then Q contains exactly one edge of C_i and one edge of C_{i+6}, and these edges are $(i,6)$-congruent. In $G^2_{\succ_{i,6}}$ the strand Q becomes one of the edges $u_{i-1}u_{i+6}$, $v_{i-1}v_{i+7}$ or $v_{i-2}v_{i+6}$, as defined by the ends of Q. Since this holds for each strand, there is a Hamilton x_0y_j-path in $G^2_{\succ_{i,6}}$. For example, see Figure 2.4 □

The following lemmas describe conditions that guarantee that two cuts are P-congruent in $\mathrm{GP}(n,2)$.

Lemma 2.11. *Let $x, y \in \{u, v\}$, $j \in \mathbb{Z}_n$, and P be a Hamilton x_0y_j-path in $GP(n,2)$. Suppose that there exists an i such that:*

1. $x_0, y_j \notin V_{i,5}$;

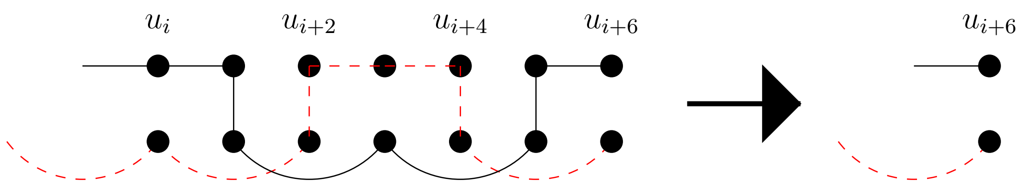

Figure 2.4: Example of an $(i, 6)$-reduction in $\mathrm{GP}(n, 2)$.

2. $C_i \cap P = \{u_{i-1}u_i\}$*; and*

3. $|C_{i+6} \cap P| = 1$.

Then:

1. $C_{i+6} \cap P = \{u_{i+5}u_{i+6}\}$*; and*

2. if the ends of P *are not contained in* $V_{i-1,7}$*, then* C_{i-1} *and* C_{i+5} *are* P*-congruent, as are* C_{i+1} *and* C_{i+7}*.*

Proof. Since $C_i \cap P = u_{i-1}u_i$, P contains the edges $u_{i-1}u_i$, u_iv_i, v_iv_{i+2}, $u_{i+2}u_{i+1}$, $u_{i+1}v_{i+1}$, and $v_{i+1}v_{i+3}$. If the edge $u_{i+2}v_{i+2} \in P$, then the edges $u_{i+3}v_{i+3}$, $u_{i+3}u_{i+4}$, $u_{i+4}v_{i+4}$, $v_{i+4}v_{i+6}$, $u_{i+5}u_{i+6}$, $u_{i+5}v_{i+5}$, and $v_{i+5}v_{i+7}$ are all contained in P, so $|C_{i+6} \cap P| = 3$, a contradiction. Thus $u_{i+2}u_{i+3}$ and $v_{i+2}v_{i+4} \in P$. If $u_{i+3}v_{i+3} \in P$, then P has a cycle. Thus $u_{i+3}u_{i+4}$ and $v_{i+3}v_{i+5} \in P$. If $u_{i+4}u_{i+5} \in P$, then $u_{i+5}v_{i+5}$ is not in P, as otherwise P has a cycle, and we see that $|C_{i+6} \cap P| = 3$, a contradiction. Therefore $u_{i+4}v_{i+4} \in P$ and $C_{i+6} \cap P = u_{i+5}u_{i+6}$. Hence C_i and C_{i+6} are P-congruent.

By hypothesis, $C_{i-1} \cap P = v_{i-3}v_{i-1}$, $C_{i+1} \cap P = v_iv_{i+2}$, $C_{i+5} \cap P = v_{i+3}v_{i+5}$, and $C_{i+7} \cap P = v_{i+6}v_{i+8}$. Therefore C_{i-1} and C_{i+5} are P-congruent, as are C_{i+1} and C_{i+7}. □

Lemma 2.12. *Let* $x, y \in \{u, v\}$, $j \in \mathbb{Z}_n$*, and* P *be a Hamilton* x_0y_j*-path in* $GP(n, 2)$*. Suppose that there exists an* i *such that* $x_0, y_j \notin V_{i,5}$*. If, for all* $\ell \in \{i, \dots, i+6\}$, $|C_\ell \cap P| = 2$*, then* C_i *and* C_{i+6} *are* P*-congruent.*

Proof. If $C_i \cap P = \{u_{i-1}u_i, v_{i-2}v_i\}$, then, as each C_ℓ has two edges in P, the $(i, 6)$-strands are

$$u_{i-1}u_iu_{i+1}v_{i+1}v_{i+3}v_{i+5}u_{i+5}u_{i+6} \text{ and } v_{i-2}v_iv_{i+2}u_{i+2}u_{i+3}u_{i+4}v_{i+4}v_{i+6}.$$

Thus $C_{i+6} \cap P = \{u_{i+5}u_{i+6}, v_{i+4}v_{i+6}\}$, and C_i and C_{i+6} are P-congruent. A similar argument holds if $C_i \cap P = \{u_{i-1}u_i, v_{i-1}v_{i+1}\}$ or $\{v_{i-2}v_i, v_{i-1}v_{i+1}\}$. □

Lemma 2.13. *Let $x, y \in \{u, v\}$, $j \in \mathbb{Z}_n$, and P be a Hamilton x_0y_j-path in GP$(n, 2)$. Suppose that there exists an i such that $x_0, y_j \notin V_{i,5}$. If $|C_\ell \cap P| = 3$ for all $\ell \in \{i, \ldots, i+6\}$, then C_i and C_{i+6} are P-congruent.*

Proof. Since $|C_\ell \cap P| = 3$ for all $\ell \in \{i, \ldots, i+6\}$, C_i, C_{i+2}, C_{i+4} and C_{i+6} are all P-congruent. □

Naturally it is possible that a path does not have any of the above properties. We wish to minimize the value of n needed to guarantee that one of the above conditions applies. Thus, we look at ways of manipulating the structure of the path. Suppose P and P' are paths in GP$(n, 2)$ and let i and j be such that neither P nor P' has any ends in $V_{i,j}$. Then P and P' are (i, j)-*equivalent* if outside of $S_{i,j}$, P and P' are the same, and, for each (i, j)-strand Q of P, there exists an (i, j)-strand Q' of P' so that Q and Q' have equivalent ends. We can use this to adjust a path to allow for an (i, j)-reduction.

There is an additional property of paths that simplifies our argument.

Remark Let $x, y \in \{u, v\}$, $0 \le j \le \lfloor \frac{n}{2} \rfloor$, and P be a Hamilton x_0y_j-path in GP(n, k). If P crosses a cut C_i, where $j < i \le n$, an even (odd) number of times, then any cut in this range is crossed an even (odd) number of times by P. This is because, for any ℓ and m, where $j < \ell$, $m \le n$, $|C_\ell \cap P| + |C_m \cap P|$ must be even.

This means that we have two cases to consider. We will first consider the properties needed to guarantee a repeat in the even case.

The following lemma describes how the position of a cut that is not crossed by any edge of a Hamilton path can be shifted.

Lemma 2.14. *Let $x, y \in \{u, v\}$, $j \in \mathbb{Z}_n$, and P be a Hamilton x_0y_j-path in GP$(n, 2)$. Suppose there exists an i such that $x_0, y_j \notin V_{i-1,4}$. If $C_i \cap P = \emptyset$, then there exists a Hamilton x_0y_j-path P' such that $C_{i+3} \cap P' = \emptyset$ and P' is $(i-1, 5)$-equivalent to P.*

Proof. Since $C_i \cap P = \emptyset$, we have that $|C_\ell \cap P| = 2$ for all $\ell \in \{i-1, i+1, \ldots, i+4\}$. Thus $S_{i-1,5}$ contains exactly two strands of P, namely,

$$u_{i-2}u_{i-1}v_{i-1}v_{i-3} \text{ and } u_{i+4}u_{i+3}u_{i+2}v_{i+2}v_iu_iu_{i+1}v_{i+1}v_{i+3}v_{i+5}.$$

Let P' be the path in GP$(n, 2)$ obtained from P by replacing the two $(i-1, 5)$-strands of P with

$$u_{i-1}u_iv_iv_{i+2}u_{i+2}u_{i+1}v_{i+1}v_{i-1}v_{i-3} \text{ and } u_{i+4}u_{i+3}v_{i+3}v_{i+5}.$$

Then P and P' are $(i-1,5)$-equivalent Hamilton paths. $\square$

This lemma can be applied multiple times, leading to the following corollary.

Corollary 2.15. *Let $x, y \in \{u, v\}$, $j \in \mathbb{Z}_n$, and P be a Hamilton x_0y_j-path in GP$(n,2)$, for $j \neq 0$. Suppose there is some i so that $j < i \leq n$ and $C_i \cap P = \emptyset$. Then there is a Hamilton x_0y_j-path P' that is $(j, n-j)$-equivalent to P, for which there is an $i' \in \{n-2, n-1, n\}$, so that $C_{i'} \cap P' = \emptyset$.*

Proof. If $i \in \{n-2, n-1, n\}$, then $P = P'$. Otherwise, we can apply Lemma 2.14 to obtain a Hamilton path P' that is $(j, n-j)$-equivalent to P with $C_{i'} \cap P' = \emptyset$, where $i' \equiv i \pmod 3$ and $i' \in \{n-2, n-1, n\}$. $\square$

This leads to the following claim, which describes the conditions needed to guarantee that an (i,j)-reduction can be applied. In this claim we choose j in such a way that the indices of the path remain the same in both GP$(n,2)$ and $GP(n-6,2)$.

Claim 2.16. *Let $x, y \in \{u, v\}$, $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$, and P be a Hamilton x_0y_j-path in GP$(n,2)$. If $|C_{j+1} \cap P|$ is even and $n - j \geq 10$, then there is a Hamilton x_0y_j-path in $G^2_{\succ_{j+1,6}}$.*

Proof. Since $|C_{j+1} \cap P|$ is even, $|C_\ell \cap P|$ is even for all $\ell \in \{j+1, \ldots, n\}$. If there exists an i, where $j < i \leq n$, such that $|C_i \cap P| = 0$, then i is unique. Corollary 2.15 implies that there exists an $(j, n-j)$-equivalent Hamilton path, where $i' \in \{n-2, n-1, n\}$ and $|C_{i'} \cap P| = 0$.

Thus $(i'-1) - j \geq (n-3) - j \geq 7$, and Lemma 2.12 applies for $S_{j+1,5}$. By Theorem 2.10, there is a Hamilton x_0y_j-path in $G^2_{\succ_{j+1,6}}$.

If no such i exists, then Lemma 2.12 applies for $S_{j+1,5}$ and, by Theorem 2.10, there is a Hamilton x_0y_j-path in $G^2_{\succ_{j+1,6}}$. $\square$

The odd case is more complicated than the even case. The following lemma, similar to the even case, shows that we can shift the position of a cut crossed by one strand of P.

Lemma 2.17. *Let $x, y \in \{u, v\}$, $j \in \mathbb{Z}_n$, and P be a Hamilton x_0y_j-path in GP$(n,2)$. Suppose there exists an i such that $x_0, y_j \notin V_{i,5}$. If $|C_i \cap P| = 3$, $|C_{i+1} \cap P| = 1$, and $|C_{i+5} \cap P| = 3$, then there exists an $(i,6)$-equivalent Hamilton x_0y_j-path P', such that, for $\ell \in \{i+3, i+4, i+5\}$, $|C_\ell \cap P'| = 1$, and for $\ell \in \{i, i+1, i+2\}$, $|C_\ell \cap P'| = 3$.*

Proof. Since $|C_i \cap P| = 3$, $|C_{i+1} \cap P| = 1$, and $|C_{i+5} \cap P| = 3$, we have that $S_{i,6}$ contains precisely three strands of P, namely,

$$u_{i-1}u_iv_iv_{i-2},\ v_{i-1}v_{i+1}u_{i+1}u_{i+2}v_{i+2}v_{i+4}v_{i+6}, \text{ and } u_{i+6}u_{i+5}u_{i+4}u_{i+3}v_{i+3}v_{i+5}v_{i+7}.$$

Let P' be the path in $\mathrm{GP}(n, 2)$ obtained from P by replacing the three $(i, 6)$-strands of P with

$$u_{i-1}u_iu_{i+1}u_{i+2}v_{i+2}v_iv_{i-2},\ v_{i-1}v_{i+1}v_{i+3}u_{i+3}u_{i+4}v_{i+4}v_{i+6}, \text{ and } u_{i+6}u_{i+5}v_{i+5}v_{i+7}$$

Then P and P' are $(i, 6)$-equivalent Hamilton paths in $\mathrm{GP}(n, 2)$. □

Lemma 2.18. *Let $x, y \in \{u, v\}$, $j \in \mathbb{Z}_n$, and P be a Hamilton x_0y_j-path in GP$(n, 2)$. Suppose there exists an i such that $x_0, y_j \notin V_{i,4}$. If $|C_i \cap P| = 3$, $|C_{i+1} \cap P| = 1$, and $|C_{i+5} \cap P| = 1$, then there exists an $(i, 5)$-equivalent Hamilton x_0y_j-path P', such that $|C_{i+5} \cap P'| = 1$ and, for $\ell \in \{i, \ldots, i+4\}$, $|C_\ell \cap P'| = 3$.*

Proof. Since $|C_i \cap P| = 3$, $|C_{i+1} \cap P| = 1$, and $|C_{i+5} \cap P| = 1$, we have that $S_{i,5}$ contains precisely two strands of P, namely,

$$u_{i-1}u_iv_iv_{i-2} \text{ and } v_{i-1}v_{i+1}u_{i+1}u_{i+2}v_{i+2}v_{i+4}u_{i+4}u_{i+3}v_{i+3}v_{i+5}.$$

Let P' be the path in $\mathrm{GP}(n, 2)$ obtained from P by replacing the two $(i, 5)$-strands of P with

$$u_{i-1}u_iu_{i+1}u_{i+2}u_{i+3}u_{i+4}v_{i+4}v_{i+2}v_iv_{i-2} \text{ and } v_{i-1}v_{i+1}v_{i+3}v_{i+5}.$$

Then P and P' are $(i, 5)$-equivalent Hamilton paths in $\mathrm{GP}(n, 2)$. □

As in the even case, we can iteratively apply these lemmas to obtain the following corollaries.

Corollary 2.19. *Let $x, y \in \{u, v\}$, $j \in \mathbb{Z}_n$, and P be a Hamilton x_0y_j-path in GP$(n, 2)$, where $|C_{j+1} \cap P| = 3$ and $|C_n \cap P| = 3$. Suppose there is some i so that $j < i \leq n$ and $C_i \cap P = v_{i-2}v_i$. Then there is a Hamilton x_0y_j-path P' that is $(j, n-j)$-equivalent to P for which there is an $i' \in \{n-4, n-3\}$, so that $C_{i'} \cap P' = v_{i'-2}v_{i'}$.*

Proof. If $i \in \{n-4, n-3\}$, then $P = P'$. Otherwise, by Lemma 2.18 we can assume that i is unique, which implies that $C_m \cap P = C_m$ for all $m \in \{j+1, \ldots, n\}$ where $m \neq i, i+1$ or $i+2$. Thus we can apply Lemma 2.17 to obtain a Hamilton path P' that is $(j, n-j)$-equivalent to P with $C_{i'} \cap P' = v_{i'-2}v_{i'}$, where $i' \equiv i \pmod 2$ and $i' \in \{n-4, n-3\}$. □

Corollary 2.20. *Let $x, y \in \{u, v\}$, $j \in \mathbb{Z}_n$, and P be a Hamilton x_0y_j-path in $GP(n, 2)$, where $|C_{j+1} \cap P| = 1$ and $|C_n \cap P| = 3$. Then there is a Hamilton x_0y_j-path P' that is $(j, n-j)$-equivalent to P, where, for all $i \in \{j+4, \ldots, n\}$, $|C_i \cap P'| = 3$.*

Proof. Since $|C_{j+1} \cap P| = 1$, $C_{j+1} \cap P = u_ju_{j+1}$, $v_{j-1}v_{j+1}$, or v_jv_{j+2}. Suppose $C_{j+1} \cap P = v_{j-1}v_{j+1}$. Then $|C_{j+2} \cap P| = |C_{j+3} \cap P| = 1$ and $|C_{j+4} \cap P| = 3$. If $|C_i \cap P| = 3$ for all $i \in \{j+4, \ldots, n\}$, then $P = P'$. Otherwise, we can apply Lemma 2.18 to obtain a Hamilton x_0x_j-path P' that is $(j, n-j)$-equivalent to P, where $|C_i \cap P'| = 3$ for all $i \in \{j+4, \ldots, n\}$. $\square$

We can now describe the conditions needed to guarantee that an (i, j)-reduction can be applied. As in the even case, we choose j in such a way that the indices of the path remain the same in both $\mathrm{GP}(n, 2)$ and $GP(n-6, 2)$.

Claim 2.21. *Let $x, y \in \{u, v\}$, $j \in \mathbb{Z}_n$, and P be a Hamilton x_0y_j-path in $GP(n, 2)$. Assume by reflective symmetry that $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$. If $|C_{j+1} \cap P|$ is odd and $n - j \geq 12$, then there is a Hamilton x_0y_j-path in $G^2_{\succ i,6}$.*

Proof. Since $|C_{j+1} \cap P|$ is odd, $|C_\ell \cap P|$ is odd for all $\ell \in \{j+1, \ldots, n\}$.

If there exists an i such that Lemma 2.13 applies, then, by Theorem 2.10, there is a Hamilton x_0y_j-path in $G^2_{\succ i,6}$. If no such i exists, then between any set of six consecutive cuts there exists a cut crossed by just one strand of P. By symmetry, there are three distinct cases: $|C_{j+1} \cap P| = 1$ and $|C_n \cap P| = 3$; $|C_{j+1} \cap P| = 3$ and $|C_n \cap P| = 3$; and $|C_{j+1} \cap P| = 1$ and $|C_n \cap P| = 1$.

If $|C_{j+1} \cap P| = 1$ and $|C_n \cap P| = 3$, then, by Corollary 2.20, we can obtain a $(j, n-j)$-equivalent Hamilton path where $|C_\ell \cap P| = 3$ for all $\ell \in \{j+4, \ldots, n\}$. Since $n - (j+4) \geq 8$, Lemma 2.13 applies and, by Theorem 2.10, there is a Hamilton x_0y_j-path in $G^2_{\succ j+4,6}$.

If $|C_{j+1} \cap P| = 3$ and $|C_n \cap P| = 3$, then by Corollary 2.19 we can obtain a $(j, n-j)$-equivalent Hamilton path where $|C_\ell \cap P| = 3$ for all $\ell \in \{j+1, \ldots, n-5\}$. Since $(n-5) - (j+1) = n - j - 6 \geq 6$, Lemma 2.13 applies and there is a Hamilton x_0y_j-path in $G^2_{\succ j+1,6}$, by Theorem 2.10.

If $|C_{j+1} \cap P| = 1$ and $|C_n \cap P| = 1$, then $C_{i+j} \cap P = u_iu_{i+1}$, $v_{i-1}v_{i+1}$, or v_iv_{i+2}. These three cases are interconnected so we will assume that $C_{j+1} \cap P = v_{j-1}v_{j+1}$ and make slight adjustments where necessary to account for the other possibilites. Since $C_{j+1} \cap P = v_{j-1}v_{j+1}$, we have that $C_{j+2} \cap P = u_{j+1}u_{j+2}$, $C_{j+3} \cap P = v_{j+2}v_{j+4}$ and $C_{j+4} \cap P = C_{j+4}$. (Note that the other two possiblities are accounted for at C_{j+2} and C_{j+3}.) The path may cross C_{j+5} with one or three strands.

If $C_{j+5} \cap P = v_{j+3}v_{j+5}$, then $C_{j+8} \cap P = C_{j+8}$ and we can apply the reverse of Lemma 2.18 at $j+8$ so that the cuts C_ℓ, $\ell = j+4, j+5, \ldots, j+8$ are all crossed by three strands of the path. If $C_{j+9} \cap P = v_{j+8}v_{j+10}$, then $C_{j+12} \cap P = C_{j+12}$. Assuming $n - j = 12$, the first case applies. (For the other two possibilities of $C_{j+1} \cap P$, we can assume that $j = j+1$ or $j+2$, respectively. Thus we have that $C_{j+13} \cap P$ could be $v_{j+12}v_{j+14}$, since $|C_n \cap P| = 1$. We can apply the reverse of Lemma 2.18 at $j+12$ to obtain nine consecutive cuts C_ℓ, $\ell = j+4, j+5, \ldots, j+13$, that are all crossed by three strands of the path. Thus Lemma 2.13 applies at $S_{j+4,6}$, and by Theorem 2.10, there is a Hamilton x_0y_j-path in $G^2_{\succ j+4,6}$.) If $C_{j+9} \cap P = C_{j+9}$, then Lemma 2.13 applies for $S_{j+4,6}$ and by Theorem 2.10 there is a Hamilton x_0y_j-path in $G^2_{\succ j+4,6}$.

If $C_{j+5} \cap P = C_{j+5}$, then $C_{j+6} \cap P = C_{j+6}$. If $C_{j+7} \cap P = v_{j+6}v_{j+8}$, then Lemma 2.11 applies at $S_{j+1,6}$ and by Theorem 2.10, there is a Hamilton x_0y_j-path in $G^2_{\succ j+1,6}$. (For the other possibilities, Lemma 2.11 applies at $S_{j+2,6}$ and $S_{j+3,6}$, respectively.) If $C_{j+7} \cap P = C_{j+7}$, then $C_{j+8} \cap P = C_{j+8}$ and the same cases occur at C_{j+9} as above. $\square$

Now that we have established the conditions for guaranteeing that an (i,j)-reduction can be applied, we prove the nonexistence of the paths as described in the Theorem 2.2. Since the ideas are the same for each case we will only prove the $n \equiv 0 \pmod 6$ case in detail. Recall, for $n \equiv 0 \pmod 6$ there are two kinds of paths which do not exist. We prove them separately in the following lemmas.

Lemma 2.22. *If $n \equiv 0 \pmod 6$, then no Hamilton u_0u_2 or u_0u_{n-2}-path exists in $GP(n,2)$.*

Proof. We proceed by induction on n, with the base cases being $n = 6$ and $n = 12$.

For $n = 6$, suppose by way of contradiction that there exists a Hamilton u_0u_2-path P in GP$(6,2)$. There are two possible initial edges at u_0, by symmetry.

If the edge u_0v_0 is in P, then the edges u_1v_1, u_1u_2, v_0v_2, v_2v_4, u_4v_4, u_3u_4, and u_4u_5 are all in P. Hence the degree of u_4 is three, a contradiction.

If the edge u_0u_5 is in P, then the subpaths $u_0u_5v_5$, $u_2u_1v_1$, and $v_3u_3u_4v_4v_0-v_2u_2$ are contained in P. Hence the degree of u_2 in P is two, a contradiction.

Therefore no Hamilton u_0u_2-path exists in GP$(6, 2)$, and by reflective symmetry no Hamilton u_0u_4-path exists. Thus, the theorem holds for $n = 6$.

For $n = 12$, suppose P is a Hamilton u_0u_2-path in GP$(12, 2)$.

If $|C_0 \cap P|$ is even, then $n-2 = 10$ and by Lemma 2.16, P can be reduced to a Hamilton u_0u_2-path in GP$(6, 2)$. But no Hamilton u_0u_2-path exists in GP$(6, 2)$, and therefore $|C_0 \cap P|$ is odd.

If $|C_0 \cap P|$ is odd, then there are two possible pairs of initial and final edges for P, by symmetry.

If the edge u_0v_0 is in P, then the subpaths $u_8u_9v_9$, $v_8v_{10}u_{10}u_{11}v_{11}$, $u_0v_0v_2v_4$, $v_1u_1u_2$, and $v_3u_3u_4$ are contained in P. The edge v_1v_{11} is in P, since $|C_0 \cap P|$ is odd. Thus the edges v_3v_5 and v_7v_9 are also in P. If u_4v_4 is in P, then the cycle $u_7u_8u_9v_9v_7u_7$ exists. Thus u_4u_5 and v_4v_6 are in P, and the cycle $u_3u_4 \cdots u_9v_9v_7v_5v_3u_3$ exists. Therefore P does not contain the edge u_0v_0.

If $u_0u_{11} \in P$, then the subpaths $v_{10}v_0v_2v_4$, $v_1u_1u_2$, and $v_3u_3u_4$ are contained in P. By hypothesis the edge $v_1v_{11} \in P$ and the edges v_3v_5, $v_{11}v_9$, and $u_{11}u_{10}$ are also contained in P. If $u_4v_4 \in P$, then the cycle $u_4u_3v_3v_5u_5u_6v_6-$ $v_8 \cdots v_4u_4$ exists. Thus the edge $u_4u_5 \in P$, and the edges v_4v_6, v_5v_7 and $u_5u_6 \in P$. If $u_6v_6 \in P$, then P is the path $u_0u_{11}u_{10}u_9v_9v_{11}v_1u_1u_2$, which is not Hamiltonian. Thus $u_6u_7 \in P$ and the cycle $v_0v_2 \cdots v_{10}v_0$ exists. Therefore no Hamilton u_0u_2-path exists in GP$(12, 2)$, and by reflective symmetry no Hamilton u_0u_{10}-path exists. Thus, the theorem holds for $n = 12$.

Suppose the theorem holds for $n \geq 12$, where $n \equiv 0 \pmod 6$. If there exists a Hamilton u_0u_2-path P in GP$(n + 6, 2)$, then $(n + 6) - 2 \geq 16$ and P can be reduced to a Hamilton u_0u_2-path in GP$(n, 2)$ by Claims 2.16 and 2.21. This contradicts the inductive hypothesis, and therefore no Hamilton u_0u_2-path exists in GP$(n + 6, 2)$. Also, by reflective symmetry no Hamilton u_0u_{n+4}-path exists. Therefore, by induction, the theorem holds. $\square$

Lemma 2.23. *No Hamilton v_0v_j-path exists in GP$(n, 2)$, where $n, j \equiv 0 \pmod 6$.*

Proof. We proceed by induction on n, with base case $n = 12$.

For $n = 12$, suppose by way of contradiction that P is a Hamilton v_0v_6-path in GP$(12, 2)$. By symmetry there are four possible pairs of initial and final edges for P.

If the edges $v_0u_0, v_6u_6 \in P$, then the subpaths $u_{10}v_{10}v_8u_8$ and $u_2v_2v_4u_4$ are contained in P. Assume without loss of generality that $u_0u_1 \in P$, then for all odd $j \in \mathbb{Z}_{12}$ the edges u_jv_j and $u_{j-1}u_j$ are in P. The edge $v_3v_5 \notin P$, as otherwise a cycle exists. Therefore the edges v_1v_3, v_5v_7, and $v_9v_{11} \in P$ and the cycle $u_{10}u_{11}v_{11}v_9u_9u_8v_8v_{10}u_{10}$ exists.

If the edges v_0u_0 and $v_6v_4 \in P$, then the subpaths $u_{10}v_{10}v_8u_8$, $u_7u_6u_5u_4u_3$, $u_2v_2v_4v_6$ and $v_7v_5v_3$ are contained in P. Since $|C_7 \cap P|$ is even, $|C_0 \cap P|$ is even as well. If $|C_0 \cap P| = 0$, then the cycle $u_{10}u_{11}v_{11}v_9u_9u_8v_8v_{10}u_{10}$ exists. Otherwise, if $|C_0 \cap P| = 2$, then the cycle $u_3 \cdots u_7v_7v_5v_3u_3$ exists.

If the edges v_0v_2 and $v_6v_4 \in P$, then the subpaths $u_{11}u_0u_1$, $v_0v_2u_2$, $u_4v_4v_6$, $u_5u_6u_7$, and $u_8v_8v_{10}u_{10}$ are contained in P. If $|C_0 \cap P|$ is odd, then $|C_7 \cap P|$ is odd and the edges $v_{11}v_1$ and v_5v_7 are not in P. Hence the edges v_1v_3 and $v_3v_5 \in P$, as well as the edges u_2u_3 and u_3u_4. Therefore $P = v_0v_2u_2u_3u_4v_4v_6$. If $|C_0 \cap P|$ is even, then the edges v_1v_{11} and v_5v_7 are in P. If the edge u_1v_1 is in P, then $u_{11}v_{11} \notin P$ and the cycle $u_{10} \cdots u_1v_1v_{11}v_9u_9u_8v_8v_{10}u_{10}$ exists. Thus u_1u_2 is in P. If $u_{11}v_{11} \in P$, then P is not a Hamilton path. Therefore the edges $u_{11}u_{10}$ and $v_{11}v_9$ are in P, which forces the cycle $u_5u_6u_7v_7v_5u_5$.

If the edges v_0v_2 and $v_6v_8 \in P$ then the subpaths $u_{11}u_0u_1u_2u_3$, $v_{11}v_1v_3$, $v_0v_2v_4u_4$, $u_5u_6u_6u_8u_9$, $v_5v_7v_9$, and $v_6v_8v_{10}u_{10}$ are contained in P. This implies that $|C_0 \cap P| = 2$ and $|C_7 \cap P| = 3$, a contradiction.

Therefore no Hamilton v_0v_6-path exists in $\mathrm{GP}(12, 2)$.

Suppose the theorem holds for $n \geq 12$ where $n \equiv 0 \pmod 6$. By way of contradiction, we assume that there exists a Hamilton v_0v_j-path P in $\mathrm{GP}(n+6, 2)$, for $j \leq \frac{(n+6)}{2}$ and $j \equiv 0 \pmod 6$. Note that $(n+6)-j \geq \frac{(n+6)}{2}$. If $n \geq 18$, then $(n+6) - j \geq 12$. If $n = 12$, $j \leq \frac{(n+6)}{2}$, and $j \equiv 0 \pmod 6$, then $j \leq 6$, and again $(n+6) - j \geq 12$. Thus P can be reduced by Claims 2.16 and 2.21 to a Hamilton v_0v_j-path in $\mathrm{GP}(n, 2)$. This contradicts the inductive hypothesis, hence no Hamilton v_0v_j-path exists in $\mathrm{GP}(n+6, 2)$, as required. □

Lemma 2.24. *For $n \equiv 2 \pmod 6$ and $j \equiv 4 \pmod 6$, no Hamilton v_0v_j-path exists in $GP(n, 2)$.*

This follows, since no Hamilton v_0v_4-path exists in $\mathrm{GP}(8, 2)$ or in $\mathrm{GP}(14, 2)$.

Lemma 2.25. *For $n \equiv 4 \pmod 6$, $j \equiv 2 \pmod 6$, and $\ell \equiv 0, 4 \pmod 6$, no Hamilton u_0u_2, u_0u_{n-2}, u_0v_1, u_0v_{n-1}, u_0v_j, or v_0v_ℓ-path exists in $GP(n, 2)$.*

This follows, since no Hamilton u_0u_2, u_0v_1, u_0v_2, or v_0v_4-path exists in $\mathrm{GP}(10, 2)$ and no Hamilton u_0v_8 or v_0v_6-path exists in $\mathrm{GP}(16, 2)$.

Lemma 2.26. *For $n \equiv 5 \pmod 6$, no Hamilton path exists for pairs of adjacent vertices; for $j \equiv 2, 3 \pmod 6$, no Hamilton v_0v_j-path exists in $GP(n, 2)$.*

This follows since $\mathrm{GP}(n,2)$ is Hamiltonian if and only if $n \not\equiv 5 \pmod 6$. Also, no Hamilton v_0v_3-path exists in $\mathrm{GP}(11,2)$ and no Hamilton v_0v_8-path exists in $\mathrm{GP}(17,2)$.

This completes the proof of Theorem 2.2.

Chapter 3

GP(n,3)

3.1 Introduction

In this chapter, we prove the following result.

Theorem 3.1. *GP*$(n,3)$ *is Hamilton connected if and only if* n *is odd and* $n > 5$. *It is Hamilton laceable if and only if* $n \geq 4$ *is even and* $n \neq 6$.

This was proved by Alspach and Lui [3]. Our proof is different: we use an $(i,12)$-expansion, whereas Alspach and Liu used a variant of an $(i,6)$-expansion.

In the first section we develop the ground work for the $(i,12)$-expansion argument and in the second section we provide the proof of Theorem 3.1.

3.2 Expanding in GP(n,3)

As for GP$(n,2)$, we will be using cuts and the (i,j)-expansion operation to describe the inductive step. Recall that, for $i \in \mathbb{Z}_n$, the cut C_i in GP$(n,3)$ is the set of edges $\{u_{i-1}u_i, v_{i-3}v_i, v_{i-2}v_{i+1}, v_{i-1}v_{i+2}\}$. The $(i,12)$-expansion of GP$(n,3)$ is the graph obtained from GP$(n,3)$ by deleting the edges in C_i and adding the vertices $\{u'_0, v'_0, \ldots, u'_{11}, v'_{11}\}$ at C_i, as well as the edges: $u_{i-1}u'_0$; $u'_{11}u_i$; for $0 \leq \ell \leq 11$, the edges $u'_\ell v'_\ell$; for $0 \leq \ell \leq 10$, the edges $u'_\ell u'_{\ell+1}$; for $0 \leq \ell < 3$, the edges $v_{i-3+\ell}v'_\ell$ and $v'_{9+\ell}v_{i+\ell}$; and, for $0 \leq \ell \leq 8$, the edges $v'_\ell v'_{\ell+3}$. The following theorem establishes that an expansion can occur at any cut while maintaining the Hamilton path.

Theorem 3.2. *Let $x, y \in \{u, v\}$ and $j \in \mathbb{Z}_n$. If there exists a Hamilton x_0y_j-path in $GP(n, 3)$, then there exists a Hamilton x_0y_j-path in $G^3_{\prec_{i,12}}$.*

Proof. Let P be a Hamilton x_0y_j-path in $\mathrm{GP}(n, 3)$. We will show that for $j < i \leq n$, we can apply an $(i, 12)$-expansion so that there is a Hamilton x_0y_j-path in $G^3_{\prec i,12}$. Of the 16 cases, we consider six that are representative of the rest.

If no edge of P is in C_i, then at least one of $u_{i-1}v_{i-1}$ and u_iv_i is in P. Assume the latter. In this case, replace u_iv_i in P with the path

$$u_iu'_{11}v'_{11}v'_8v'_5v'_2u'_2u'_3\cdots u'_{10}v'_{10}v'_7v'_4v'_1u'_1u'_0v'_0v'_3v'_6v'_9v_i$$

to create the Hamilton x_0y_j-path in $G^3_{\prec i,12}$.

If $|C_i \cap P| = 1$, then $C_i \cap P$ is either $u_{i-1}u_i$ or $v_{i-3}v_i$ or $v_{i-2}v_{i+1}$ or $v_{i-1}v_{i+2}$. Suppose $C_i \cap P = u_{i-1}u_i$. Then $u_{i-1}u_i$ can be replaced with the path

$$u_{i-1}u'_0v'_0v'_3u'_3u'_4v'_4v'_1u'_1u'_2v'_2v'_5u'_5u'_6v'_6v'_9u'_9u'_{10}v'_{10}v'_7u'_7u'_8v'_8v'_{11}u'_{11}u_i$$

to obtain the Hamilton x_0y_j-path in $G^3_{\prec_{i,12}}$ A similar argument holds if $C_i \cap P = v_{i-3}v_i$, $v_{i-2}v_{i+1}$, or $v_{i-1}v_{i+2}$.

If $|C_i \cap P| = 2$, then $C_i \cap P$ is one of six possible combinations of edges in C_i.

Consider the case $C_i \cap P = \{u_{i-1}u_i, v_{i-3}v_i\}$. Then the edges $u_{i-1}u_i$ and $v_{i-3}v_i$ can be replaced with the paths

$$u_{i-1}u'_0u'_1v'_1v'_4u'_4u'_3u'_2v'_2v'_5u'_5u'_6u'_7v'_7v'_{10}u'_{10}u'_9u'_8v'_8v'_{11}u'_{11}u_i \text{ and } v_{i-3}v'_0v'_3v'_6v'_9v_i,$$

respectively, to obtain the Hamilton x_0y_j-path in $G^3_{\prec_{i,12}}$. A similar argument holds if $C_i \cap P$ is either $\{u_{i-1}u_i, v_{i-1}v_{i+2}\}$ or $\{v_{i-3}v_i, v_{i-2}v_{i+1}\}$ or $\{v_{i-2}v_{i+1}, -v_{i-1}v_{i+2}\}$.

We also treat the case $C_i \cap P = \{u_{i-1}u_i, v_{i-2}v_{i+1}\}$. The edges $u_{i-1}u_i$ and $v_{i-2}v_{i+1}$ can be replaced by the paths

$$u_{i-1}u'_0v'_0v'_3u'_3u'_4v'_4v'_7u'_7u'_8v'_8v'_{11}u'_{11}u_i \text{ and } v_{i-2}v'_1u'_1u'_2v'_2v'_5u'_5u'_6v'_6v'_9u'_9u'_{10}v'_{10}v_{i+1},$$

respectively, to obtain the Hamilton x_0y_j-path in $G^3_{\prec_{i,12}}$. A similar argument holds for $C_i \cap P = \{v_{i-3}v_i, v_{i-1}v_{i+2}\}$.

If $|C_i \cap P| = 3$, then $C_i \cap P$ is one of four possible combination of edges in C_i. Suppose $C_i \cap P = \{u_{i-1}u_i, v_{i-3}v_i, v_{i-2}v_{i+1}\}$. Then the edges $u_{i-1}u_i$, $v_{i-3}v_i$, and $v_{i-2}v_{i+1}$ can be replaced with the paths

$$u_{i-1}u'_0u'_1u'_2v'_2v'_5v'_8v'_{11}u'_{11}u_i,\ v_{i-3}v'_0v'_3u'_3u'_4u'_5u'_6v'_6v'_9v_i,$$

$$\text{and } v_{i-2}v'_1v'_4v'_7u'_7u'_8u'_9u'_{10}v'_{10}v_{i+1},$$

respectively, to obtain the Hamilton x_0y_j-path in $G^3_{\prec_{i,12}}$. A similar argument holds for the other cases.

If $|C_i \cap P| = 4$, then $C_i \cap P = C_i$. The edges $u_{i-1}u_i$, $v_{i-3}v_i$, $v_{i-2}v_{i+1}$ and $v_{i-1}v_{i+2}$ can be replaced with the paths

$$u_{i-1}u'_0u'_1\cdots u'_{11}u_i,\ v_{i-3}v'_0v'_3v'_6v'_9v_i,\ v_{i-2}v'_1v'_4v'_7v'_{11}v_{i+1}, \text{ and } v_{i-1}v'_2v'_5v'_8v'_{10}v_{i+2},$$

respectively, to obtain the Hamilton x_0y_j-path in $G^3_{\prec_{i,12}}$. □

3.3 Base Cases

In this section we show the existence of the Hamilton paths for the twelve cases. We show the the first two cases in detail.

Theorem 3.3. *If* $n \equiv 0 \pmod{12}$*, then* $GP(n,3)$ *is Hamilton laceable.*

Proof. We proceed by induction on n, with base cases $n = 12$ and $n = 24$.

From the Hamilton path

$$u_0v_0v_3v_6v_9u_9u_8v_8v_{11}u_{11}u_{10}v_{10}v_7u_7u_6u_5v_5v_2u_2u_3u_4v_4v_1u_1$$

in $\mathrm{GP}(12,3)$ we can do a (consecutive) sequence of Posa exchanges using the edges u_1u_2 (to get the Hamilton u_0u_3-path), u_3v_3 (to get the Hamilton u_0v_6-path), v_6u_6 (to get to u_7), u_7u_8 (to get to v_8), v_8v_5, u_5u_4, v_4v_7 (to get to v_{10}), $v_{10}v_1$, and v_1u_1 (to get to v_0). From the Hamilton path

$$P = v_0u_0u_{11}u_{10}v_{10}v_7v_4u_4u_3v_3v_6v_9u_9u_8u_7u_6u_5v_5v_8v_{11}v_2u_2u_1v_1$$

we can do a Posa exchange using the edge v_1v_{10} to obtain the Hamilton v_0v_7-path. Starting with P we can do a sequence of Posa exchanges using the edges v_1v_4, u_4u_5, and u_6v_6 to obtain the Hamilton v_0v_9-path. Reflective and rotational symmetry complete the task.

In $\mathrm{GP}(24,3)$, the preceding paragraph and Theorem 3.2 imply that, for $0 \le j \le 11$, there exist Hamilton paths for the pairs (u_0,u_j) and (v_0,v_j), when j is odd, and (u_0,v_j), when j is even. This leaves the case u_0v_{12}, which is achieved by the path

$$u_0v_0v_3u_3u_4v_4v_1u_1u_2v_2v_5u_5u_6v_6v_9u_9u_{10}v_{10}v_7u_7u_8v_8v_{11}u_{11}u_{12}u_{13}v_{13}-$$

$$v_{16}v_{19}v_{22}u_{22}u_{23}v_{23}v_{20}v_{17}v_{14}u_{14}u_{15}\cdots u_{21}v_{21}v_{18}v_{15}v_{12}.$$

Therefore, by reflective and rotational symmetry, GP$(24,3)$ is Hamilton laceable.

Suppose the theorem holds for GP$(n,3)$, where $n \equiv 0 \pmod{12}$ and $n \geq 24$. Theorem 3.2 and the inductive hypothesis impy that in GP$(n+12,3)$, there exists a Hamilton path for all necessary pairs of vertices in $V_{0,n-1}$. Each vertex in $V_{n,11}$ is symmetric to a vertex contained in $V_{0,11}$. Therefore by the inductive hypothesis and reflective and rotational symmetry, a Hamilton path exists for all pairs of vertices on opposite sides of the bipartition in GP$(n+12,3)$, as required. $\square$

Theorem 3.4. *If $n \equiv 1 \pmod{12}$, then GP$(n,3)$ is Hamilton connected.*

Proof. We proceed by induction on n, with base case $n = 13$. From the Hamilton path

$$P = u_0v_0v_{10}u_{10}u_{11}u_{12}v_{12}v_9u_9u_8v_8v_{11}v_1v_4v_7u_7u_6v_6v_3u_3u_4u_5v_5v_2u_2u_1$$

in GP$(13,3)$, we can do a consecutive sequence of Posa exchanges using the edges u_1v_1 (to get to v_4), v_4u_4 (to get to u_3), u_3u_2 (to get to v_2), v_2v_{12} and v_9v_6 (to get to v_3). Starting with P we can also do a sequence of Posa exchanges using the edges u_1u_0 (to get to v_0), v_0v_3 (to get to v_6), v_6v_9 (to get to u_9), u_9u_{10} (to get to u_{11}), and $u_{11}v_{11}$ (to get to v_1). From the Hamilton path

$$P' = u_0v_0v_{10}u_{10}u_{11}u_{12}v_{12}v_2u_2u_1v_1v_{11}v_8u_8u_9v_9v_6v_3u_3u_4v_4v_7u_7u_6u_5v_5$$

we can do a Posa exchange using the edge v_5v_8 to get the Hamilton u_0u_8-path. Starting again with P' we can do a sequence of Posa exchanges using the edges v_5v_2, u_2u_3, v_3v_0, and $v_{10}v_7$ to obtain the Hamilton u_0u_7-path. From the Hamilton path

$$P'' = v_0v_{10}u_{10}u_{11}v_{11}v_8u_8u_9v_9v_{12}u_{12}u_0u_1u_2v_2v_5u_5u_4u_3v_3v_6u_6u_7v_7v_4v_1$$

we can do a sequence of Posa exchanges using the edges v_1u_1, u_2u_3, and u_4v_4 to obtain the Hamilton v_0v_7-path. From P'' we can also do the sequence of Posa exchanges using the edges v_1v_{11} (to get to v_8), v_8v_5 (to get to v_2), v_2v_{12}, $u_{12}u_{11}$, $v_{11}v_8$, v_5v_2, $v_{12}u_{12}$, u_0v_0 (to get to v_{10}) and $v_{10}v_7$ (to get to v_4). Reflective and rotational symmetry complete the task.

Suppose the theorem holds for $n \equiv 1 \pmod{12}$, where $n \geq 13$. Theorem 3.2 and the inductive hypothersis imply that in GP$(n+12,3)$, there exists a Hamilton path for all pairs of vertices contained in $V_{0,n-1}$. Each vertex contained in $V_{n,11}$ is symmetric to a vertex contained in $V_{0,11}$. Therefore, by the inductive hypothesis and reflective and rotational symmetry, GP$(n+12,3)$ is Hamilton connected, as required. □

Theorem 3.5. *If* $n \equiv 2 \pmod{12}$*, GP*$(n,3)$ *is Hamilton laceable.*

Proof of base case. From the Hamilton path

$$P = u_0v_0v_3v_6v_9u_9u_8v_8v_{11}u_{11}u_{10}v_{10}v_7u_7u_6u_5v_5v_2v_{13}u_{13}u_{12}v_{12}v_1v_4u_4u_3u_2u_1$$

in GP$(14,3)$ we can do a sequence of Posa exchanges using the edges u_1v_1 (to get to v_4) and v_4v_7 (to get to u_7). Starting with P we can also do a sequence of Posa exchanges using the edges u_1u_0 (to get to v_0), v_0v_{11} (to get to v_8), v_8v_5 (to get to u_5), u_5u_4, v_4v_7, $v_{10}v_{13}$ (to get to v_2) and v_2u_2 (to get to u_3). From the Hamilton path

$$v_0v_{11}u_{11}u_{12}v_{12}v_1u_1u_0u_{13}v_{13}v_{10}u_{10}u_9v_9v_6u_6u_5u_4v_4v_7u_7u_8v_8v_5v_2u_2u_3v_3$$

we can do a sequence of Posa exchanges using the edges v_3v_0, $v_{11}v_8$, u_8u_9 (to get to v_9), v_9v_{12} (to get to v_1), and v_1v_4 (to get to v_7). Reflective and rotational symmetry complete the task. □

Theorem 3.6. *If* $n \equiv 3 \pmod{12}$*, the GP*$(n,3)$ *is Hamilton connected.*

Proof of base case. From the Hamilton path

$$u_0v_0v_3v_6v_9v_{12}u_{12}u_{11}v_{11}v_8v_5u_5u_6u_7u_8u_9u_{10}v_{10}v_7v_4u_4u_3u_2v_2v_{14}u_{14}u_{13}v_{13}v_1u_1$$

in GP$(15,3)$ we can do the sequence of Posa exchanges using the edges u_1u_2 (to get to v_2), v_2v_5 (to get to u_5), u_5u_4 (to get to v_4), v_4v_1, u_1u_0 (to get to v_0), v_0v_{12} (to get to v_9), v_9u_9, $u_{10}u_{11}$, $v_{11}v_{14}$, v_2u_2 (to get to u_3), u_3v_3, v_6u_6 (to get to u_7), u_7v_7 (to get to v_{10}), $v_{10}v_{13}$ (to get to v_1), v_1u_1 (to get to u_2), u_2u_3 (to get to v_2), v_3v_6 (to get to u_6), u_6u_7, u_8v_8, $v_{11}u_{11}$, $u_{10}u_9$, v_9v_{12}, v_0u_0, u_1u_2, u_3v_3, v_6u_6, u_7u_8 (to get to v_8), v_8v_{11}, $v_{14}v_2$, and u_2u_3 (to get to u_4). Let P be the Hamilton path

$$v_0v_3u_3u_2v_2v_{14}v_{11}u_{11}u_{12}v_{12}v_9v_6u_6u_7v_7v_4u_4u_5v_5v_8u_8u_9u_{10}v_{10}v_{13}u_{13}u_{14}u_0u_1v_1.$$

From P we can do a sequence of Posa exchanges using the edges v_1v_4, u_4u_3, u_2u_1, v_1v_{13}, $u_{13}u_{12}$, $u_{11}u_{10}$ (to get to v_{10}), $v_{10}v_7$ (to get to v_4), v_4u_4, u_5u_6 (to get to v_6), v_6v_3, u_3u_2 (to get to v_2), and v_2v_5 (to get to v_8). Starting with P we can do a sequence of Posa exchanges using the edges v_1v_{13} and $u_{13}u_{12}$ to obtain the Hamilton v_0v_{12}-path. Reflective and rotational symmetry complete the task. □

Theorem 3.7. *If $n \equiv 4 \pmod{12}$, then $GP(n,3)$ is Hamilton laceable.*

Proof of base case. Let P be the Hamilton path

$$u_0v_0v_3u_3u_4u_5v_5v_8v_{11}u_{11}u_{10}u_9u_8u_7u_6v_6v_9v_{12}u_{12}u_{13}v_{13}v_{10}v_7v_4 -$$

$$v_1v_{14}u_{14}u_{15}v_{15}v_2u_2u_1$$

in GP$(16,3)$. From P be can do a sequence of Posa exchanges using the edges u_1u_0 (to get to v_0), v_0v_{13} (to get to u_{13}), $u_{13}u_{14}$ (to get to v_{14}), $v_{14}v_{11}$ (to get to v_8), v_8u_8 (to get to u_9), and u_9v_9 (to get to v_6). Starting from P again, we can do a sequence of Posa exchanges using the edges u_0v_1, $v_{14}v_{11}$ (to get to u_{11}), and $u_{11}u_{12}$ (to get to v_{12}). From the Hamilton path

$$v_0v_{13}u_{13}u_{14}v_{14}v_1u_1u_0u_{15}v_{15}v_{12}u_{12}u_{11}v_{11}v_8u_8u_7u_6v_6v_9u_9u_{10} -$$

$$v_{10}v_7v_4u_4u_5v_5v_2u_2u_3v_3,$$

we can do a sequence of Posa exchanges using the edges v_3v_0, $v_{13}v_{10}$, $u_{10}u_{11}$ (to get to v_{11}), $v_{11}v_{14}$ (to get to v_1), and v_1v_4 (to get to v_7). Reflective and rotational symmetry complete the task. □

Theorem 3.8. *If $n \equiv 5 \pmod{12}$, then $GP(n,3)$ is Hamilton connected.*

Proof of base case. From the Hamilton path

$$P = u_0v_0v_{14}u_{14}u_{15}u_{16}v_{16}v_{13}u_{13}u_{12}v_{12}v_{15}v_1v_4u_4u_3v_3v_6v_9u_9u_8 -$$

$$v_8v_{11}u_{11}u_{10}v_{10}v_7u_7u_6u_5v_5v_2u_2u_1$$

in GP$(17,3)$ we can do a sequence of Posa exchanges using the edges u_1v_1 (to get to v_4), v_4v_7 (to get to v_{10}), $v_{10}v_{13}$ (to get to u_{13}), $u_{13}u_{14}$ (to get to u_{15}), $u_{15}v_{15}$ (to get to v_1), v_1v_4, v_7v_{10} (to get to u_{10}), $u_{10}u_9$ (to get to u_8), u_8u_7 (to get to u_6), u_6v_6 (to get to v_3), v_3v_0 (to get to v_{14}), $v_{14}v_{11}$ (to get to v_8), v_8v_5 (to get to u_5), u_5u_4, v_4v_7, and u_7u_6 (to get to v_6). Starting from P we can do a sequence of Posa exchanges using the edges u_1u_0 (to get to v_0), v_0v_3, and u_3u_2 (to get to v_2). From the Hamilton path

$$v_0u_0u_1v_1v_{15}u_{15}u_{16}v_{16}v_2u_2u_3u_4v_4v_7u_7u_8u_9v_9v_{12}u_{12}u_{11}u_{10}v_{10}v_{13} -$$

$$u_{13}u_{14}v_{14}v_{11}v_8v_5u_5u_6v_6v_3$$

we can do a sequence of Posa exchanges using the edges v_3u_3, u_4u_5 (to get to v_5), v_5v_2, u_2u_1 (to get to v_1), v_1v_4 (to get to v_7), v_7v_{10}, $u_{10}u_9$ (to get to v_9), v_9v_6, u_6u_7, v_7v_4, v_1u_1, u_2v_2, v_5u_5, u_4u_3, v_3v_0, u_0u_{16}, $u_{15}u_{14}$, $u_{13}u_{12}$, $u_{11}v_{11}$, v_8u_8, u_7v_7 (to get to v_4), v_4v_1, u_1u_2 (to get to v_2), v_2v_5, v_8v_{11}, $u_{11}u_{12}$, $u_{13}u_{14}$, $u_{15}u_{16}$, u_0v_0, v_3u_3, u_4u_5, and v_5v_8 (to get to v_{11}). The existence of the Hamilton u_0v_5-path

$$u_0u_{16}u_{15}v_{15}v_{12}u_{12}u_{11}v_{11}v_8u_8u_7v_7v_{10}u_{10}u_9v_9v_6u_6u_5u_4v_4v_1u_1 -$$

$$u_2u_3v_3v_0v_{14}u_{14}u_{13}v_{13}v_{16}v_2v_5$$

and reflective an rotational symmetry imply that GP$(17, 3)$ is Hamilton connected. □

Theorem 3.9. *If $n \equiv 6 \pmod{12}$, the GP$(n, 3)$ is Hamilton laceable.*

Proof of base case. In GP$(18, 3)$, from the Hamilton path

$$P = u_0v_0v_3v_6v_9u_9u_{10}u_{11}v_{11}v_{14}u_{14}u_{15}v_{15}v_{12}u_{12}u_{13}v_{13}v_{10}v_7v_4 -$$

$$u_4u_3u_2u_1v_1v_{16}u_{16}u_{17}v_{17}v_2v_5v_8u_8u_7u_6u_5$$

we can do a sequence of Posa exchanges using the edges u_5v_5 (to get to v_8), v_8v_{11} (to get to v_{14}), $v_{14}v_{17}$ (to get to u_{17}), $u_{17}u_0$ (to get to v_0), v_0v_{15} (to get to u_{15}), $u_{15}u_{16}$ (to get to v_{16}), $v_{16}v_{13}$, and $v_{10}u_{10}$ (to get to u_9). Starting again from P we can do a sequence of Posa exchanges using the edges u_5u_4, u_3v_3 (to get to v_6), v_6u_6, u_5v_5, and v_8v_{11} (to get to u_{11}). From the Hamilton path

$$v_0v_3u_3u_2v_2v_5u_5u_4v_4v_7v_{10}u_{10}u_9v_9v_6u_6u_7u_8v_8v_{11}u_{11}u_{12}v_{12}v_{15}u_{15} -$$

$$u_{16}v_{16}v_{13}u_{13}u_{14}v_{14}v_{17}u_{17}u_0u_1v_1$$

we can do a sequence of Posa exchanges using the edges v_1v_{16} (to get to v_{13}), $v_{13}v_{10}$, $u_{10}u_{11}$ (to get to v_{11}), $v_{11}v_{14}$, $v_{17}v_2$, v_5v_8, $v_{11}u_{11}$, $u_{10}v_{10}$, v_7u_7, u_8u_9 (to get to v_9), v_9v_{12}, $u_{12}u_{13}$, and $u_{14}u_{15}$ (to get to v_{15}). □

Theorem 3.10. *If $n \equiv 7 \pmod{12}$, then GP$(n, 3)$ is Hamilton connected.*

Proof of base cases. In GP$(7, 3)$, from the Hamilton path

$$P = u_0u_1v_1v_5v_2u_2u_3v_3v_1v_4u_4u_5u_6v_6,$$

we can do a sequence of Posa exchanges using the edges v_6v_2 (to get to u_2), u_2u_1 (to get to v_1), to get to v_1v_4 (to get to u_4), u_4u_3 (to get to v_3), v_3v_6 (to get to v_2). Starting again with P we can do a sequence of Posa exchanges using the edges v_6v_3 (to get to v_0) and v_0u_0 (to get to u_1). The Hamilton paths

$$v_0u_0u_6v_6v_3u_3u_2u_1v_1v_4u_4u_5v_5v_2,$$

$$v_0u_0u_1v_1v_5v_2u_2u_3v_3v_6u_6u_5u_4v_4, \text{ and}$$

$$v_0u_0u_6u_5v_5v_2u_2u_1v_1v_4u_4u_4v_3v_6,$$

with reflective and rotational symmetry imply that $\mathrm{GP}(7,3)$ is Hamilton connected.

In $\mathrm{GP}(19,3)$, the preceding paragraph and Theorem 3.2 imply that we need the additional Hamilton x_0y_j-paths for $x,y \in \{u,v\}$ and $j \in \{7,8,9\}$. Given the Hamilton path

$$u_0u_{18}v_{18}v_2v_5v_8u_8u_7v_7v_{10}v_{13}u_{13}u_{12}v_{12}v_{15}u_{15}u_{14}v_{14}v_{17}u_{17}u_{16} -$$

$$v_{16}v_0v_3u_3u_2u_1v_1v_4u_4u_5u_6v_6v_9u_9u_{10}u_{11}v_{11}$$

we can do a sequence of Posa exchanges using the edges $v_{11}v_{14}$, $v_{17}v_1$, u_1u_0, $u_{18}u_{17}$, $v_{17}v_{14}$, $v_{11}v_8$ (to get to u_8), u_8u_9, v_9v_{12}, $v_{15}v_{18}$, v_2u_2, u_2u_3, u_3u_4, v_4v_7, $v_{10}u_{10}$ (to get to u_9), u_9v_9, v_6v_3, u_3u_2, v_2v_{18}, $v_{15}v_{12}$ (to get to v_9), v_9v_6, v_3u_3, u_4v_4 (to get to v_7), v_7v_{10}, $v_{13}v_{16}$, $u_{16}v_{15}$, and $u_{14}u_{13}$ (to get to u_{12}). From the Hamilton path

$$P = v_0u_0u_1v_1v_4u_4u_5u_6u_7v_7v_{10}v_{13}v_{16}u_{16}u_{15}u_{14}u_{13}u_{12}v_{12}v_{15}v_{18}u_{18} -$$

$$u_{17}v_{17}v_{14}v_{11}u_{11}u_{10}u_9u_8v_8v_5v_2u_2u_3v_3v_6v_9$$

we can do a sequence of Posa exchanges using the edges v_9u_9 and u_8u_7 to obtain a Hamilton v_0v_7-path. Starting with P we can do a sequence of Posa exchanges using the edges v_9v_{12}, $v_{15}u_{15}$, and $u_{14}v_{14}$ to obtain the Hamilton v_0v_{11}-path. □

Theorem 3.11. *If $n \equiv 8 \pmod{12}$, then $GP(n,3)$ is Hamilton laceable.*

Proof of base cases. In $\mathrm{GP}(8,3)$, from the Hamilton path,

$$u_0v_0v_3u_3u_2v_2v_5u_5u_4v_4v_7u_7u_6v_6v_1u_1,$$

we can do a sequence of Posa exchanges using the edges u_1u_0 (to get to v_0), v_0v_5 (to get to v_2), v_2v_7 (to get to v_4), v_4v_1 and v_6v_3 (to get to u_3). From the Hamilton path

$$v_0u_0u_7u_6u_5v_5v_2v_7v_4u_4u_3u_2u_1v_1v_6v_3,$$

we can obtain the Hamilton v_0v_7-path by a sequence of Posa exchanges using the edges v_3u_3 and u_2v_2.

In GP$(20, 3)$, from the Hamilton path

$$P = u_0v_0v_{17}u_{17}u_{16}v_{16}v_{13}u_{13}u_{12}v_{12}v_{15}u_{15}u_{14}v_{14}v_{11}u_{11}u_{10}v_{10} -$$

$$v_7v_4u_4u_5u_6u_7u_8v_8v_5v_2v_{19}u_{19}u_{18}v_{18}v_1u_1u_2u_3v_3v_6v_9u_9$$

we can do a Posa exchanges using the edge u_9u_{10} to obtain the Hamilton u_0u_{10}-path. Also starting with P we can do a sequence of Posa exchanges using the edges u_9u_8 (to get to v_8) and v_8v_{11} (to get to u_{11}). Also the Hamilton path

$$v_0u_0u_{19}u_{18}u_{17}v_{17}v_{14}u_{14}u_{15}u_{16}v_{16}v_{19}v_2v_5v_8v_{11}u_{11}u_{12}u_{13} -$$

$$v_{13}v_{10}u_{10}u_9u_8u_7v_7v_4u_4u_5u_6v_6v_3u_3u_2u_1v_1v_{18}v_{15}v_{12}v_9$$

exists. □

Theorem 3.12. *If* $n \equiv 9 \pmod{12}$*, then GP*$(n, 3)$ *is Hamilton connected.*

Proof of base cases. In GP$(9, 3)$, from the Hamilton path

$$P = u_0u_8v_8v_2v_5u_5u_4u_3v_3v_0v_6u_6u_7v_7v_4v_1u_1u_2,$$

we can do a sequence of Posa exchanges using the edges v_3v_6, v_0u_0, u_8u_7, v_7v_1, and v_4u_4 to get the Hamilton u_0u_3-path. Starting again with P we can do a sequence of Posa exchanges using the edges u_2v_2 (to get to v_5), v_5v_8 (to get to v_2), v_2v_5 (to get to u_5), u_5u_6 (to get to v_6), v_6v_3 (to get to v_0), v_0u_0 (to get to u_8),, u_8u_7, v_7v_1, v_4u_4 and u_5v_5 (to get to v_8). From the Hamilton path

$$v_0u_0u_8v_8v_5u_5u_4u_3v_3v_6u_6u_7v_7v_4v_1u_1u_2v_2,$$

we can do a sequence of Posa exchanges using the edges v_2v_5, u_5u_6 (to get to v_6), v_6v_0, u_0u_1, u_2u_3, u_4v_4, and v_7v_1 (to get to v_4). The existence of the Hamilton path

$$v_0u_0u_8u_7u_6v_6v_3u_3u_2u_1v_1v_7v_4u_4u_5v_5v_2v_8$$

implies that GP$(9,3)$ is Hamilton connected.

In GP$(21,3)$, given the Hamilton path

$$P = u_0u_1v_1v_{19}v_{16}v_{13}v_{10}u_{10}u_{11}\cdots u_{20}v_{20}v_{17}v_{14}v_{11}v_8u_8u_9v_9v_6u_6 -$$

$$u_7v_7v_4u_4u_5v_5v_2u_2u_3v_3v_0v_{18}v_{15}v_{12}$$

we can do a sequence of Posa exchanges using the edges $v_{12}u_{12}$ and $u_{13}v_{13}$ to obtain the Hamilton u_0v_{10}-path. Starting with P we can do a sequence of Posa exchanges using the edges $v_{12}v_9$, v_6v_3, u_3u_4, u_5u_6, u_7u_8 (to get to u_9), and u_9u_{10} (to get to u_{11}). From the Hamilton path

$$v_0u_0u_1v_1v_{19}u_{19}u_{20}v_{20}v_{17}u_{17}u_{18}v_{18}v_{15}u_{15}u_{16}v_{16}v_{13}v_{10}u_{10}u_{11}u_{12} -$$

$$u_{13}u_{14}v_{14}v_{11}v_8v_5v_2u_2u_3v_3v_6u_6u_5u_4v_4v_7u_7u_8u_9v_9v_{12}$$

we can obtain the Hamilton v_0v_{10}-path from a sequence of Posa exchanges using the edges $v_{12}u_{12}$ and $u_{13}v_{13}$. □

Theorem 3.13. *If* $n \equiv 10 \pmod{12}$*, the GP*$(n,3)$ *is Hamilton laceable.*

Proof of base cases. In GP$(10,3)$, from the Hamilton path

$$u_0u_9v_9v_2u_2u_3u_4u_5v_5v_8u_8u_7u_6v_6v_3v_0v_7v_4v_1u_1$$

we can do a sequence of Posa exchanges using the edges u_1u_2 (to get to u_3), u_3v_3 (to get to v_6), v_6v_9 (to get to v_2), v_2v_5 (to get to u_5), u_5u_6, u_7v_7, v_4u_4, u_3u_2, v_2v_9, v_6v_3, u_3u_4, and v_4v_7 (to get to v_0). From the Hamilton path

$$P = v_0v_7v_4u_4u_3v_3v_6v_9u_9u_0u_1u_2v_2v_5u_5u_6u_7u_8v_8v_1,$$

we can do a sequence of Posa exchanges using the edges v_1u_1, and u_2u_3 to obtain the Hamilton v_0v_3-path. Starting with P we can do a sequence of Posa exchanges using the edges v_1v_4 and u_4u_5 to get the Hamilton v_0v_5-path.

In GP$(22,3)$, the Hamilton path

$$v_0u_0u_1v_1v_{20}u_{20}u_{21}v_{21}v_2u_2u_3v_3v_6u_6u_7v_7v_4u_4u_5v_5v_8u_8u_9v_9v_{12}u_{12}u_{13} -$$

$$u_{14}v_{14}v_{17}u_{17}u_{16}u_{15}v_{15}v_{18}u_{18}u_{19}v_{19}v_{16}v_{13}v_{10}u_{10}u_{11}v_{11}$$

exists. Given the Hamilton path

$$u_0u_1v_1v_{20}\cdots v_8u_8u_9u_{10}v_{10}v_{13}\cdots v_3u_3u_2v_2v_5u_5u_4v_4v_7u_7u_6v_6 -$$

$$v_9\cdots v_{21}u_{21}u_{20}\cdots u_{11}$$

we can obtain a Hamilton u_0v_{10}-path by a Posa exchange using the edge $u_{11}u_{10}$. □

Theorem 3.14. *If $n \equiv 11 \pmod{12}$, then $GP(n,3)$ is Hamilton connected.*

Proof of base cases. In $\mathrm{GP}(11,3)$, from the Hamilton path

$$P = u_0v_0v_3u_3u_2u_1v_1v_4v_7u_7u_8v_8v_5v_2v_{10}u_{10}u_9v_9v_6u_6u_5u_4,$$

we can do a sequence of Posa exchanges using the edges u_4u_3, u_2v_2, v_5u_5, u_6u_7, v_7v_{10}, and v_2v_5 to the Hamilton u_0v_8-path. Starting from P we can also do a sequence of Posa exchanges using the edges u_4v_4 (to get to v_7), v_7v_{10} (to get to v_2), v_2u_2 (to get to u_1), u_1u_0 (to get to v_0), v_0v_8 (to get to v_5), v_5u_5 (to get to u_6), u_6u_7, v_7v_4, u_4u_3 (to get to u_2), u_2u_1 (to get to v_1), v_1v_9, u_9u_8, u_7v_7, and v_4u_4 (to get to u_3). From the Hamilton path

$$v_0u_0u_{10}v_{10}v_2u_2u_1v_1v_9u_9u_8v_8v_5u_5u_6u_7v_7v_4u_4u_3v_3v_6,$$

we can do a sequence of Posa exchanges using the edges v_6v_9, u_9u_{10} (to get to v_{10}), $v_{10}v_7$ (to get to v_4), v_4v_1 (to get to v_9), v_9u_9, u_8u_7, and u_6v_6 (to get to v_3).

In $\mathrm{GP}(23,2)$ from the Hamilton path

$$u_0v_0v_{20}u_{20}u_{19}v_{19}v_{16}u_{16}u_{15}v_{15}v_{18}u_{18}u_{17}v_{17}v_{14}u_{14}u_{13}v_{13}v_{10}v_7v_4 -$$

$$u_4u_5\cdots u_{11}v_{11}v_8v_5v_2v_{22}u_{22}u_{21}v_{21}v_1u_1u_2u_3v_3v_6v_9v_{12}u_{12},$$

we can do a Posa exchange using the edge $u_{12}u_{11}$ to obtain the Hamilton u_0v_{11}-path. The Hamilton path

$$v_0u_0u_{22}v_{22}v_2u_2u_1v_1v_{21}u_{21}u_{20}v_{20}v_{17}u_{17}u_{16}v_{16}v_{19}u_{19}u_{18}v_{18}v_{15}u_{15}u_{14} -$$

$$v_{14}v_{11}u_{11}u_{12}u_{13}v_{13}v_{10}u_{10}u_9u_8v_8v_5u_5u_6u_7v_7v_4u_4u_3v_3v_6v_9v_{12},$$

exists. □

Chapter 4

GP(n,k)

4.1 Introduction

In this final chapter, we develop an approach that can be used in general. We had hoped to be able to extend the ideas presented in Chapters 2 and 3, but were unable to. The first section of this chapter will look briefly at the $k = 4$ case, describing the problem we faced in trying to extend our original approach. In the next section we will present the necessary conditions for applying an (i, j)-expansion and an (i, j)-reduction in GP(n, k). We will show that for each k-value, there are a finite number of base cases needed to prove inductively the existence and nonexistence of Hamilton paths in GP(n, k).

4.2 A brief look at GP(n,4)

For $k = 2$ and 3 we showed the existence of Hamilton paths by proving the following: Given a Hamilton path in GP(n, k), an (i, j)-expansion could be applied at any cut in the path while retaining the Hamilton path. The main part of this argument was that for any cut in the original graph we can apply an $(i, 6)$-expansion or an $(i, 12)$-expansion. Unfortunately, for $k = 4$, we were unable to establish property.

Claim 4.1. *Let $x, y \in \{u, v\}$, $j \le \lfloor \frac{n}{2} \rfloor \in \mathbb{Z}_n$, and P be a Hamilton $x_0 y_j$-Hamilton path in GP$(n, 4)$. If $C_i \cap P = v_{i-2}v_{i+2}$, then no (i, ℓ)-expansion can occur in the graph.*

Proof. Since $C_i \cap P = v_{i-2}v_{i+2}$, the (i,ℓ)-strands of P are

$$v_{i-2}v_{i+2}u_{i+2}u_{i+3}v_{i+3}v_{i+7}u_{i+7}u_{i+8}v_{i+8}v_{i+12}\cdots, \text{ and}$$

$$\cdots v_{i+14}v_{i+10}v_{i+6}u_{i+6}u_{i+5}u_{i+4}v_{i+4}v_iu_iu_{i+1}v_{i+1}v_{i+5}v_{i+9}u_{i+9}u_{i+10}u_{i+11}v_{i+11}\cdots.$$

These strands will never combine into a single strand, so, for $\ell \geq 1$, each $C_{i+\ell}$ contains three edges of P. □

However, if the Hamilton x_0y_j-path P meets the cut C_i with $v_{i-2}v_{i+2}$, then there is still an $(i+1,20)$-expansion or $(i+1,20)$-reduction as shown in the following Lemma.

Lemma 4.2. *Let $x,y \in \{u,v\}$, $j \in \mathbb{Z}_n$, and P be a Hamilton x_0y_j-path in $GP(n,k)$. Suppose $k \geq 4$ is even and suppose there exists an i such that:*

1. *$j \notin \{i,\ldots,i+k(k+1)\}$;*
2. *for all $\ell \in \{i,\ldots,i+k(k+1)\}$, $|C_\ell \cap P| = \frac{k}{2}+1$; and*
3. *$C_i \cap P = \{u_{i-1}u_i, v_{i-k}v_i, v_{i-k+2}v_{i+2},\ldots,v_{i-2}v_{i+k-2}\}$.*

Then there is a Hamilton x_0y_j-path in $G^k_{\succ_{i,k(k+1)}}$ and $G^k_{\prec_{i,k(k+1)}}$.

Proof. Since $C_i \cap P = \{u_{i-1}u_i, v_{i-k}v_i, v_{i-k+2}v_{i+2},\ldots,v_{i-2}v_{i+k-2}\}$ and each cut is crossed by the same number of edges of P, the $(i,k(k+1))$-strands of P corresponding to the edges $u_{i-1}u_i$ and $v_{i-k}v_i$ remain distinct. These $(i,k(k+1))$-strands of P are of the form

$$u_{i-1}u_iu_{i+1}v_{i+1}v_{i+k+1}v_{i+2k+1}u_{i+2k+1}\ldots u_{i+k(k+1)-1}u_{i+k(k+1)} \text{ and}$$

$$v_{i-k}v_iv_{i+k}u_{i+k}u_{i+k+1}u_{i+k+2}v_{i+k+2}\ldots v_{i+k^2}v_{i+k(k+1)}.$$

The $(i,k(k+1))$-strands of P corresponding to the other edges are of the form

$$v_{i-k+2}v_{i+2}u_{i+2}u_{i+3}v_{i+3}v_{i+k+3}u_{i+k+3}\ldots u_{i+k^2+2}v_{i+k^2+2}v_{i+k(k+1)+2}$$

Therefore C_i and $C_{i+k(k+1)}$ are P-congruent and there exists a Hamilton x_0y_j-path in $G^k_{\succ_{i,k(k+1)}}$ and $G^k_{\prec_{i,k(k+1)}}$. □

For $k=4$ and $C_i \cap P = v_{i-2}v_{i+2}$ we have that $C_{i+1} \cap P = \{u_iu_{i+1}, v_iv_{i+4}, v_{i-2}v_{i+2}\}$ and for all $\ell \geq 1$, each $C_{i+\ell}$ contains three edges of P. Thus Lemma 4.2 applies at $i+1$ and we can apply an $(i+1,20)$-expansion or $(i+1,20)$-reduction in $\mathrm{GP}(n,4)$, to obtain a Hamilton x_0y_j path in $G^4_{\prec_{i+1,20}}$ or $G^4_{\succ_{i+1,20}}$, respectively. We have not undertaken a detailed analysis of the case $k=4$.

We will develop this idea of expanding and reducing in more detail in the following section.

4.3 GP(n,k)

Given a Hamilton path P in GP(n, k), we found that requiring P-congruence in order to apply an (i, j)-expansion or (i, j)-reduction is a stronger condition than what we need. Instead it is sufficient to require that there exist two matching cuts, which we define as follows.

Cuts C_i and C_{i+j} *match* with respect to C_h, where $h \leq i$, if there exists a bijection between the pairs of end edges of the $(h, i-h)$ and $(h, i+j-h)$-strands of P such that the pairs of ends are either equal, (i, j)-congruent, or one end in each pair is equal and the other ends are (i, j)-congruent. (See Figure 4.1).

If the pairs of edges are equal, then by definition they are all contained in C_h. This implies that no edge of the corresponding $(h, i-h)$-strand is in C_i. The same holds for C_{i+j}. If the pairs of edges are (i, j)-congruent, then by definition the ends of the $(h, i-h)$-strand are both contained in C_i and the ends of the $(h, i+j-h)$-strand are both contained in C_{i+j}. This implies that C_h contains no edge of the corresponding strands. If one end in each pair is equal and the other ends are (i, j)-congruent, then the equal edges are in C_h, and the (i, j)-congruent ends are in C_i and C_{i+j}.

For convenience we will denote the end edges of each (i, j)-strand of P as $t_\ell = \{e_{\ell,1}, e_{\ell,2}\}$.

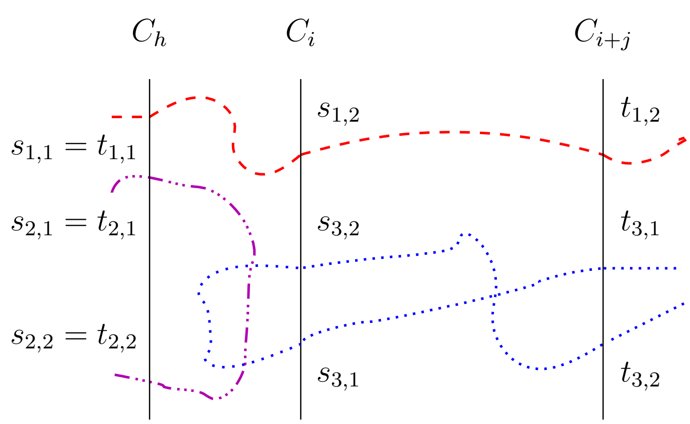

Figure 4.1: C_i and C_{i+j} match with respect to C_h. The pairs of end edges: s_1 and t_1 each have one end edge that is equal and the other is (i, j)-congruent; s_2 and t_2 are equal; and, s_3 and t_3 are (i, j)-congruent.

Lemma 4.3. *Let $x, y \in \{u, v\}$, $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$, and P be a Hamilton x_0y_j-path in $GP(n, k)$.*

Suppose there exists two cuts C_a and C_{a+b} such that $\lfloor \frac{n}{2} \rfloor + 1 \leq a$, $a+b \leq n$, and the cuts match with respect to $C_{\lfloor \frac{n}{2} \rfloor + 1}$.

Then there exists a Hamilton x_0y_j-path P' in $GP(n-b, k)$ and a Hamilton x_0y_j-path P'' in $GP(n+b, k)$.

Proof. Let $S = \{s_1, s_2, \ldots, s_\ell\}$ be the set of pairs of end edges of the $(\lfloor \frac{n}{2} \rfloor + 1, a - (\lfloor \frac{n}{2} \rfloor + 1))$-strands of P and let $T = \{t_1, t_2, \ldots, t_\ell\}$ be the set of pairs of end edges of the $(\lfloor \frac{n}{2} \rfloor + 1, a + b - (\lfloor \frac{n}{2} \rfloor + 1))$-strands of P.

Suppose we decompose the path into $(\lfloor \frac{n}{2} \rfloor + 1, a - (\lfloor \frac{n}{2} \rfloor + 1))$-strands of P, $R_1, R_2, \ldots, R_\ell$, and $(a, n - (a - (\lfloor \frac{n}{2} \rfloor + 1)))$-strands of P, $P_1, P_2, \ldots, P_{\ell+1}$. Then $P = P_1 t_{1,1} R_1 t_{1,2} P_2 t_{2,1} \cdots R_\ell t_{\ell,2} P_{\ell+1}$. See figure 4.3. Similarly, we can define the path as $P = P'_1 s_{1,1} R'_1 s_{1,2} P'_2 s_{2,1} \cdots R'_\ell s_{\ell,2} P'_{\ell+1}$, where $R'_1, R'_2, \ldots, R'_\ell$ are the $(\lfloor \frac{n}{2} \rfloor + 1, a + b - (\lfloor \frac{n}{2} \rfloor + 1))$-strands of P, and $P'_1, P'_2, \ldots, P'_{\ell+1}$ are the $(a + b, n - (a + b - (\lfloor \frac{n}{2} \rfloor + 1)))$-strands of P. See figure 4.4.

Let $f : T \to S$ be the bijection between S and T which shows that C_a and C_{a+b} match with respect to $C_{\lfloor \frac{n}{2} \rfloor + 1}$. For $i, j \in \{1, \ldots, \ell\}$ and $h \in \{1, 2\}$, the bijection acts on both indicies of $t_{i,h}$ so that for $f(t_i) = s_j$, $f(t_{i,1}) \in \{s_{j,1}, s_{j,2}\}$ and $f(t_{i,2}) \in \{s_{i,1}, s_{i,2}\}$. Also, if $f(t_i) = s_j$, then $f(R'_i) = R_j$, where the orientation of the path depends on the ends $f(t_{i,1})$ and $f(t_{i,2})$.

We show that we can apply an (a, b)-reduction or an $(a + b, b)$-expansion while maintaining the Hamilton path in the new graph.

In the (a, b)-reduction of GP(n, k), each edge in $C_a \cap P$, and its (a, b)-congruent edge in $C_{a+b} \cap P$, become the same edge. Therefore we obtain the Hamilton path $P' = P'_1 f(t_{1,1}) f(R'_1) f(t_{1,2}) P'_2 f(t_{2,1}) \cdots f(R'_\ell) f_(t_{\ell,2}) P'_{\ell+1}$ in $G^k_{\succ_{a,b}}$. See figure 4.5.

Let $Q_1, Q_2, \ldots, Q_m$ be the (a, b)-strands of P. Using the decomposition of the path as defined above, we can describe P in terms of t_i, s_i, P_i, Q_i and R'_i, since each R_i can be written in terms of s_i, Q_i and R'_i in a unique manner. See figure 4.6. In the $(a + b, b)$-expansion of the graph, we use a copy of the (a, b)-strands of P, denoted $\overline{Q_i}$, to extend the path. Let $Q = \{q_1, q_2, \ldots, q_\ell\}$ be the pairs of end edges of the $(\frac{n}{2} + 1, a + b - (\frac{n}{2} + 1))$-strands in $G^k_{\prec_{a+b,b}}$, defined in the same order as the pairs of end edges in T. Let $R''_1, R''_2, \ldots, R''_\ell$ denote the $(\frac{n}{2} + 1, a + b - (\frac{n}{2} + 1))$-strands, where each R''_i is written in terms of $f^{-1}(s_i)$, $\overline{Q_i}$, and R_i, in the same manner in which R_i is described by s_i,R'_i and Q_i. Since we are not changing the structure of the path outside of $S_{a,b}$ and the ends of the (a, b)-stands of the path remain the same. Then

$P'' = P'_1 q_{1,1} R''_1 q_{1,2} P'_2 q_{2,1} \cdots R''_\ell q_{\ell,2} P'_{\ell+1}$ is a Hamilton path in $G^k_{\prec_{a+b,b}}$. See figure 4.7. $\square$

Theorem 4.4. *Given $k > 0$, there exists an N_k and r_k such that if:*

1. *$n \geq N_k$; and*
2. *for $x, y \in \{u, v\}$, there exists a Hamilton x_0y_j-path in GP(n, k) with $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$;*

then there exists a Hamilton x_0y_j-path in GP$(n - r_kk, k)$ and a Hamilton x_0y_j-path in GP$(n + r_kk, k)$.

Proof. For some fixed i and variable j a multiple of k, let f_k be equal to the number of ways the end edges of (i, j)-strands of P can pair up. Let $r_k = LCM\{1, 2, \ldots, f_k+1\}$, $m_k \geq [f_k+1][r_k-1]+1$, and $N_k = 2m_kk[f_k+1]$. If $n \geq N_k$, then $\lfloor \frac{n}{2} \rfloor \geq m_kk[f_k + 1]$. Thus there are at least m_k pairs of cuts C_i and C_{i+j} which match and where j is a value between k and $k[f_k + 1]$. There exists a number a that is repeated $\frac{m_k}{f_k+1}$ times. So if we choose $\frac{r_k}{a}$ of the repeated a's to expand or reduce with, then n changes by $(\frac{r_k}{a}a)k = r_kk$. Thus by Lemma 4.3 we change the Hamilton path into a Hamilton path in GP$(n - r_kk, k)$ or into a Hamilton path in GP$(n + r_kk, k)$. $\square$

This theorem establishes that there are a finite number of base cases to consider, in order to prove, by induction on n, the existence or nonexistence of Hamilton paths in GP(n, k), for each value of k.

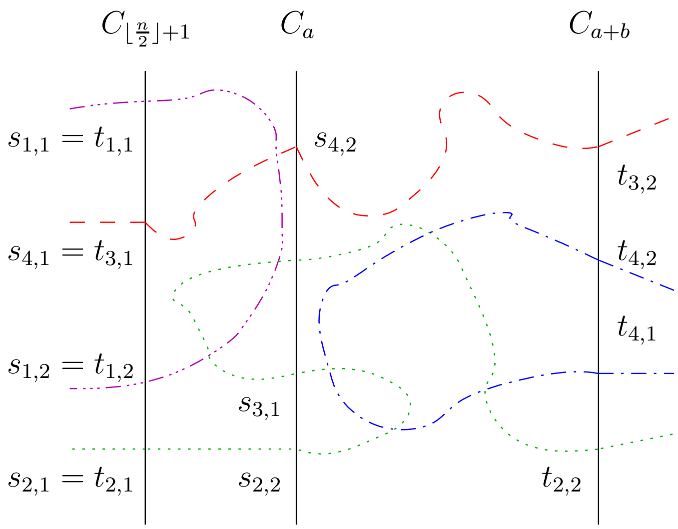

Figure 4.2: C_a and C_{a+b} match with respect to $C_{\frac{n}{2}+1}$ in $\mathrm{GP}(n,k)$.

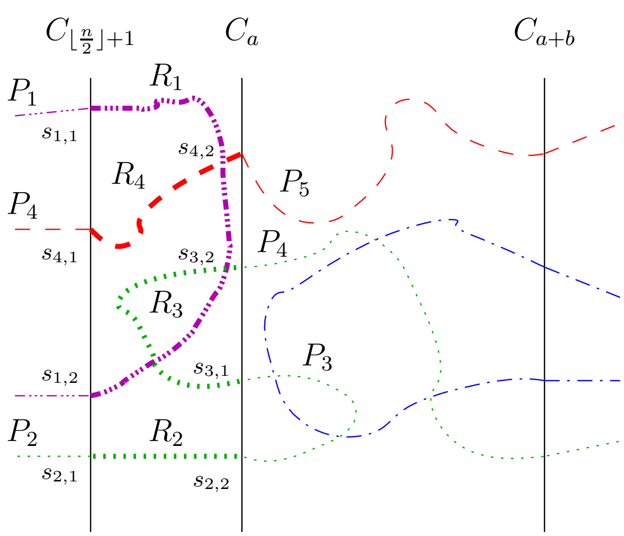

Figure 4.3: Decompose the path in terms of $(\lfloor\frac{n}{2}\rfloor+1, a-(\lfloor\frac{n}{2}\rfloor+1))$-strands of P, R_1, R_2, R_3, and R_4, and $(a, n-(a-(\lfloor\frac{n}{2}\rfloor+1)))$-strands of P, P_1, P_2, P_3, P_4, and P_5.

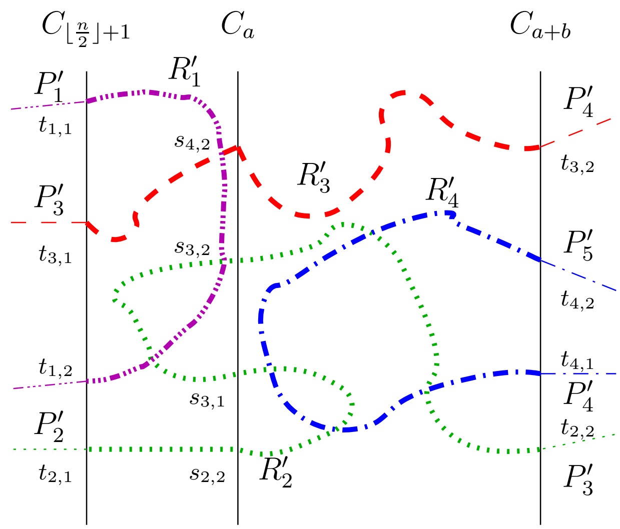

Figure 4.4: Decompose the path in terms of $(\lfloor \frac{n}{2} \rfloor+1, a+b-(\lfloor \frac{n}{2} \rfloor+1))$-strands of P, R'_1, R'_2, R'_3, and R'_4, and $(a+b, n-(a+b-(\lfloor \frac{n}{2} \rfloor+1)))$-strands of P, P'_1, P'_2, P'_3, P'_4, and P'_5.

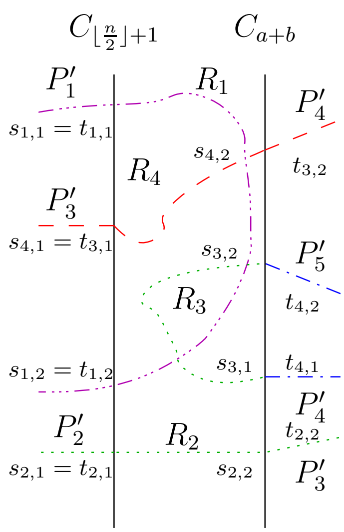

Figure 4.5: An (a, b)-reduction in $\mathrm{GP}(n, k)$, where the Hamilton path $P'_1 t_{1,1} R'_1 t_{1,2} P'_2 t_{2,1} \cdots R'_4 t_{4,1} P'_5$ becomes $P'_1[t_{1,1}/s_{1,1}]R_1[s_{1,2}/t_{1,2}]P'_2[t_{2,1}/s_{2,1}]R_2[s_{2,1}/t_{2,1}]P'_3[t_{3,1}/s_{4,1}]R_4[s_{4,2}/t_{3,1}]P'_4-$ $[t_{4,1}/s_{3,1}]R_3[s_{3,2}/t_{4,2}]P'_5$

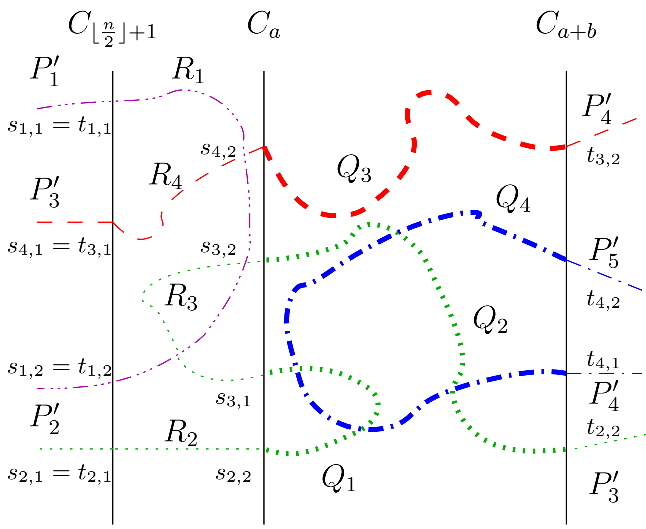

Figure 4.6: (a, b)-strands, Q_1, Q_2, Q_3, and Q_4 of P

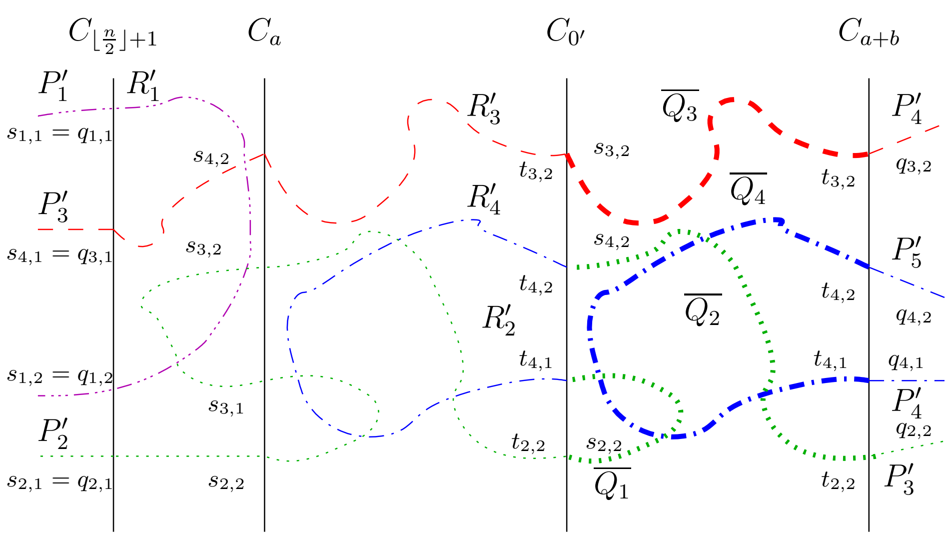

Figure 4.7: An (a, b)-expansion in $\mathrm{GP}(n, k)$ with Hamilton path $P'' = P'_1 q_{1,1} R''_1 q_{1,2} P'_2 q_{2,1} R''_2 q_{2,2} \cdots R''_\ell q_{\ell,2} P'_{\ell+1}$ where $R''_1 = R_1$, $R''_2 = R'_2 \overline{Q_1} R'_4 \overline{Q_2}$, $R''_3 = R'_3 \overline{Q_3}$, and $R''_4 = \overline{Q_4}$.

Chapter 5

Conclusion

We have provided a general approach for showing that $\mathrm{GP}(n, k)$ is Hamilton connected or Hamilton laceable. Since the cases $k = 1$, 2, and 3 have been dealt with, the next case to look at is $k = 4$. Working with some of the smaller values of n in $\mathrm{GP}(n, 4)$, we know that most of the necessary Hamilton paths exist. $\mathrm{GP}(12, 4)$ is a special case presented in Conjecture 1.5, where for j even, no Hamilton u_0u_j-path exists. General progress in Conjecture 1.5 or Conjecture 1.6 would be welcome.

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