A Matrix Rank Problem

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1 Introduction

Suppose we are given a $V_r \times V_c$ matrix where not all the entries are known. The *maximum rank matrix completion problem* is the process of assigning values for these indeterminate entries from some set such that the rank of the matrix is maximum.

The main result of this paper is an $O(n^4)$ algorithm for solving the maximum rank matrix completion problem where $n = |V_r \cup V_c|$. This improves on an $O(n^9)$ algorithm of Geelen [6] by updating certain matrix factorizations between iterations. Similar algorithms are given for matching and path-matching problems.

A specific assignment of the unknown entries is called an *evaluation*. A *perturbation* of a matrix is the changing of a single non-zero entry to some other value.

$$X = \begin{pmatrix} z_{11} & z_{12} & 0 & 0 \\ z_{21} & z_{22} & z_{23} & 0 \\ 1 & 1 & z_{33} & z_{34} \\ 1 & 1 & z_{43} & z_{44} \end{pmatrix} \tag{1}$$

A matrix is called *mixed* if it contains both real entries and indeterminate entries where the indeterminates are independent of one another. Consider the mixed matrix X above. It can be verified that $rank(X) = 4$. Let $\tilde{X}$ be the evaluation of X obtained by setting all indeterminate entries z_{ij} equal to 1. Then $rank(\tilde{X}) = 3$. If we choose a perturbation of $\tilde{X}$ by changing any specific indeterminate entry, our new matrix will still have rank at most three. To see why, notice the first two columns are linearly dependent. The same can be said for the last two rows.

Later on we will develop polynomial-time algorithms that can be used to determine the rank of general mixed matrices and of skew-symmetric matrices that strictly contain indeterminate and zero entries. First there are various definitions that must be introduced.

Let Q be a $V_r \times V_c$ matrix where V_r (resp. V_c) denotes the indices of rows (resp. columns) of Q. $Q[R, C]$ denotes the submatrix of Q by taking $R \subseteq V_r$ and $C \subseteq V_c$. If Q is a $V_r \times V_c$ mixed matrix then for $X_r \subseteq V_r$ and $X_c \subseteq V_c$, we call $X_r \cup X_c$ a *cover* of Q if each indeterminate entry is contained in a row indexed by X_r or a column indexed by X_c. The following result was proved by Hartfiel and Loewy; see [8].

Theorem 1.1 *Let Q be a $V_r \times V_c$ mixed matrix. If $(V_r \backslash X_r) \cup (V_c \backslash X_c)$ is a cover of Q, then $rank(Q) \leq rank(Q)[X_r, X_c] + |V_r \backslash X_r| + |V_c \backslash X_c|$. Furthermore, there exists a cover $(V_r \backslash Y_r) \cup (V_c \backslash Y_c)$ such that $rank(Q) = rank(Q)[Y_r, Y_c] + |V_r \backslash Y_r| + |V_c \backslash Y_c|$.*

Let i be a row of X. Row i is called *avoidable* if there exists a row basis that avoids row i. Otherwise, we say i is *unavoidable.* Let $D_r(X)$ denote the set of avoidable rows of X. Analogously we can define the same terms for columns.

Let X and Y be $V_r \times V_c$ matrices. If $rank(X) > rank(Y)$, or $rank(X) = rank(Y)$ and $D_r(Y) \subseteq D_r(X)$ then we say $X \succeq Y$. If $rank(X) = rank(Y)$ and $D_r(X) = D_r(Y)$ then we say $X \approx Y$. If $X \succeq Y$ but $X \not\approx Y$ then we say X is *more independent* than Y and is denoted by $X \succ Y$.

Call a matrix *homogeneous* if it strictly contains zero entries and indeterminates that are independent of one another. Let $X(x_{ij} \to \alpha)$ be a perturbation on X by replacing entry x_{ij} with α. The following is a key result; see [4].

Theorem 1.2 *Let A and X be $V_r \times V_c$ matrices where A is of real entries and X is homogeneous. Let $\tilde{X}$ be an evaluation of X. Then either $rank(\tilde{X} + A) = rank(X + A)$ or there exists an indeterminate entry x_{ij} of X and $\alpha \in \{1, 2, \ldots, |V_r| + 1\}$ such that $\tilde{X}(x_{ij} \to \alpha) + A \succ \tilde{X} + A$.*

Thus we can solve the maximum rank matrix completion problem by repeatedly applying Theorem 1.2. After each perturbation either the rank goes up, or the set of avoidable rows increases. Therefore we require at most $O(|V_r|^2)$ perturbations. For each perturbation we have to consider each possible value $\alpha \in \{1, 2, \ldots, |V_r| + 1\}$ for at most $O(|V_r||V_c|)$ entries. Each of these evaluations requires $|V_r| + 1$ rank compuations to determine the rank and avoidable rows. But calculating the rank of a matrix requires $O(|V_r \cup V_c|^3)$ arithmetic operations. Let $|V_r \cup V_c| = n$. Then, Theorem 1.2 provides a deterministic algorithm for solving the maximum rank matrix completion problem in $O(n^9)$-time.

In the algorithm we are required to determine the rank and the set of avoidable rows of a $V_r \times V_c$ matrix. The direct approach outlined above takes $O(n^4)$. In Section 7.1 we show that this can be reduced to $O(n^3)$. Thus the complexity of the algorithm is reduced to $O(n^8)$ arithimetic operations in total. Below is a result that further reduces the problem to $O(n^6)$; this is also proved in [5].

Theorem 1.3 *Let A and X be $V_r \times V_c$ matrices where A is of real entries and $X = (x_{ij})$ is homogeneous. Let $\tilde{X}$ be an evaluation of X, Y_r denote the set of avoidable rows of $\tilde{X} + A$, and Y_c denote the set of avoidable columns of $(\tilde{X} + A)[Y_r, V_c]$. Then:*

i) If $(V_r \backslash Y_r) \cup (V_c \backslash Y_c)$ is a cover of X, then $rank(\tilde{X} + A) = rank(X + A)$.

ii) If there exists $x_{ij} \neq 0$ where $i \in Y_r$ and $j \in Y_c$ then there exists $\alpha \in \{1, 2, \ldots, |V_r| + 1\}$ such that $\tilde{X}(x_{ij} \to \alpha) + A \succ \tilde{X} + A$.

Let $|V_r \cup V_c| = n$. For each evaluation, we have to find the rank and determine the sets Y_r and Y_c. This requires $O(n^3)$ arithmetic operations. As we have a total of $|V_r| + 1$ different entries to check, this increases each perturbation to $O(n^4)$. Up to $O(n^2)$ perturbations required produces a total complexity time of $O(n^6)$.

Later on, we will derive techniques to solve the problem in $O(n^4)$ calculations. Let an evaluation be an iteration. After each iteration either the rank increases or the set of avoidable rows increases. Without a more global strategy, we cannot avoid the fact that in the worst-case scenario any perturbation algorithm can require up to $O(n^2)$ perturbations. Since our matrices have $O(n^2)$ entries, each iteration is likely to require at least $O(n^2)$ arithmetic operations. Therefore, the $O(n^4)$ algorithm described is as efficient an algorithm one could expect from Theorem 1.3.

One of the major applications of the maximum rank matrix completion problem is determining the cardinality of a maximum matching in a bipartite graph. Let $G = (V_r \cup V_c, E)$ be a bipartite graph. The *bipartite matching matrix* $X = (x_{ij})$ of G is the $V_r \times V_c$ matrix where

$$x_{ij} = \begin{cases} z_e & \text{if } e = ij \in E \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Note that any two distinct indeterminates in the set $\{z_e : e \in E\}$ are independent of one another.

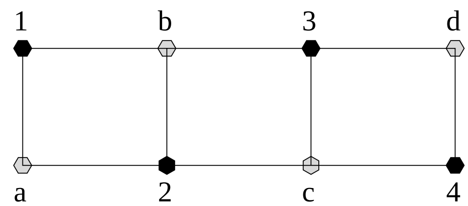

Figure 1: A bipartite graph.

Below is the bipartite matching matrix for the graph in figure 1.

$$X = \begin{pmatrix} z_{1a} & z_{1b} & 0 & 0 \\ z_{2a} & z_{2b} & z_{2c} & 0 \\ 0 & z_{3b} & z_{3c} & z_{3d} \\ 0 & 0 & z_{4c} & z_{4d} \end{pmatrix} \tag{3}$$

Let $\nu(G)$ denote the size of a maximum matching for a graph G. Then we have the following result.

Theorem 1.4 *Let $G = (V_r \cup V_c, E)$ be a graph and let X be its corresponding matching matrix. Then $\nu(G) = rank(X)$.* ■

Theorem 1.4 implies that finding the cardinality of a maximum matching for a bipartite graph is equivalent to solving the maximum rank matrix completion problem for a given bipartite matching matrix. But the size of a maximum matching using augmenting paths can be calculated in $O(n^3)$-time, and can be improved to $O(n^{5/2})$; see [9]. The algorithms based on Theorem 1.3 are not competitive with the algorithms mentioned above but are relatively simpler to implement. In addition, results have been proven that show if the indeterminates from some bipartite matching matrix X are assigned values from some arbitrary large field then the probability is high that $rank(\tilde{X}) = rank(X)$; see [3,10].

Let $G = (V_r \cup V_c, E)$ be a graph and X be its bipartite matching matrix. Notice if $(V_r \backslash Y_r) \cup (V_c \backslash Y_c)$ is a cover for X, then the vertices indexed by $(V_r \backslash Y_r) \cup (V_c \backslash Y_c)$ form a vertex cover for G. Theorem 1.1 implies that $rank(X) \leq |V_r \backslash Y_r| + |V_c \backslash Y_c|$ and that there is some cover such that the relationship is met with equality. Thus Theorems 1.1 and 1.4 together imply Konig's Theorem. For other similar results and applications for the maximum rank matrix completion problem see [6]. Although a bipartite matching matrix might be useful when working with a bipartite graph, what if we were to work with any graph in general? A matrix A is said to be *skew-symmetric* if $a_{ij} = -a_{ji}$ for all ij, and $a_{ii} = 0$ for all i. Let $G = (V, E)$ be a graph. Then we define the *Tutte matrix* $T = (t_{ij})$ of G as the following $V \times V$ skew-symmetric matrix of indeterminate and zero entries.

$$t_{ij} = \begin{cases} \pm z_e & \text{if } e = ij \in E \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Again, any two distinct indeterminates from the set $\{z_e : e \in E\}$ are independent of one another.

$$T = \begin{pmatrix} 0 & z_{12} & -z_{13} & 0 \\ -z_{12} & 0 & z_{23} & z_{24} \\ z_{13} & -z_{23} & 0 & -z_{34} \\ 0 & -z_{24} & z_{34} & 0 \end{pmatrix} \tag{5}$$

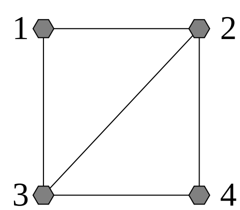

Figure 2: A graph and its corresponding Tutte matrix

Figure 2 above shows an example. Tutte matrices can be used to determine the size of a maximum matching for any graph in general. We have the following result regarding graphs in general and the size of their maximum matchings; see Tutte [11].

Theorem 1.5 *Let $G = (V, E)$ be a graph, and T be its corresponding Tutte matrix. Then $2\nu(G) = rank(T)$.* ■

Define a *skew-perturbation* on an evaluation of a Tutte matrix to be the process of changing both t_{ij} and t_{ji} such that the perturbed matrix remains skew-symmetric. A skew-perturbation is denoted by $T(z_e \to \alpha)$ where t_{ij} is replaced with α and t_{ji} is replaced with $-\alpha$ for $e = ij$. We get the following perburbation result; see [5].

Theorem 1.6 *Let $G = (V, E)$ be a graph, T be its corresponding Tutte matrix, and let $\tilde{T}$ be an evaluation of T. Then either $rank(\tilde{T}) = rank(T)$ or there exists some edge $e \in E$ and $\alpha \in \{1, 2, \ldots, 2|V| + 3\}$ such that $\tilde{T}(z_e \to \alpha) \succ \tilde{T}$.*

There is another stronger result regarding any submatrix of the Tutte matrix. The following is an extension of Theorem 1.6.

Theorem 1.7 *Let $G = (V, E)$ be a graph, and let T be its Tutte matrix. If S is an $R \times C$ submatrix of T and $\tilde{S}$ is an evaluation of S, then either $rank(\tilde{S}) = rank(S)$ or there exists some edge $e \in E$ and $\alpha \in \{1, 2, \ldots, 2|R| + 3\}$ such that $\tilde{S}(z_e \to \alpha) \succ \tilde{S}$.*

Note that the complexity of our algorithms are given in terms of arithmetic operations rather than time. We can avoid numbers growing out of control during the algorithm by working over a finite field. For example, if we are applying Theorem 1.7 we could work over Z_p where $p \geq 2|R| + 3$. The following result allows us to choose this field and keep a minimal storage space required.

Theorem 1.8 (Bertrand's Postulate) *For each positive integer $n > 1$, there exists a prime number p such that $n < p < 2n$.* ■

Joseph Bertrand verified this theorem for all numbers less than three million and it was proved by Chebyshev in 1850-1851. For a proof see [7]. Therefore we can work over matrices with entries in Z_p. This keeps our matrix operations relatively easy and keeps the storage space required for the matrix entries at $O(n \lg n)$. Choosing a finite field to work over instead of the real numbers also avoids the problem of numerical instability; see [1].

Throughout the remainder of this paper, we will focus on the proofs of the theorems we have stated and develop an algorithm that will solve the maximum rank matrix completion problem in $O(n^4)$ arithmetic operations. First we must introduce some general matroid concepts.

2 Matroid Theory

We begin this section by defining a matroid. The results below and other properties regarding matroids can be found in [5].

Let E be a finite set and let $\mathcal{I}$ be a collection of subsets, called independent sets, of elements taken from the set E. Let $\mathcal{B}$ be the collection of maximal independent subsets of E. Then $M = (E, \mathcal{I})$ is called a *matroid* if the following properties are satisifed.

P_1 : The empty set is independent.

P_2 : Any subset of an independent set is independent.

P_3 : For $S \subseteq E$, all maximal independent subsets contained in S have equal cardinality.

An element $B \in \mathcal{B}$ is called a *basis* of M. The *rank* of a set $S \subseteq E$, denoted $r_M(S)$ is the maximum size of an independent subset of S.

Consider a matrix A whose columns form the set E. Let $\mathcal{B}$ be the set of column-bases of A. Then $M_c(A) = (E, \mathcal{I})$ is clearly a matroid and is called the *column-matroid* of A. Notice any unavoidable column of A is contained in every basis of $\mathcal{B}$. An element contained in every basis of a matroid is called a *coloop.* Analogously we can define the *row-matroid,* $M_r(A)$ of A.

If e is an element of E then the matroid $M \backslash e = (E \backslash e, \mathcal{I}')$ is obtained from M by deleting e. The new basis set $\mathcal{B}'$ is the set of bases of $E \backslash e$. As any independent set remains independent by deleting an element, then any coloop $e' \neq e$ of M will also be a coloop of $M \backslash e$.

We get the following result.

Lemma 2.1 *Let B_r be a row-basis and B_c be a column-basis of a $V_r \times V_c$ matrix A. Then $A[B_r, B_c]$ is non-singular.*

Proof: Deleting an avoidable row of A does not affect the column dependencies of A. Thus $M_c(A) = M_c(A[B_r, V_c])$. Hence, B_c is a column-basis of

$A[B_r, V_c]$. A symmetric arguement can be used to show B_r is a row-basis of $A[B_r, B_c]$ by deleting an avoidable column of $A[B_r, V_c]$. Thus $A[B_r, B_c]$ is non-singular. ∎

Let x and y be any two distinct elements of a matroid $M = (E, \mathcal{I})$. If both x and y are not coloops of $M = (E, \mathcal{I})$ but $r_M(E \backslash \{x, y\}) < r_M(E)$, then (x, y) is called a *series-pair* of M. The following result implies series-pairs are transitive.

Lemma 2.2 *Let x, y, and z be distinct elements of a matroid. If (x, y) and (y, z) are series-pairs, then so is (x, z).*

Proof: Suppose (x, z) is not a series-pair. Then by definition there exists a basis B_0 that avoids both x and z. But (x, y) and $(y,)$ are series-pairs, hence $y \in B_0$. Since y is not a coloop, $r_M(E \backslash y) = r_M(E)$. But $r_M(E) = |B_0|$. Therefore there exists basis B'_0 of matroid $M \backslash y$ extended from $B_0 \backslash y$ such that $|B'_0| = |B_0 \backslash y| + 1$. As $r_M(E) = r_M(E \backslash y)$, B'_0 is also a basis of M. But B'_0 cannot contain both x and z and B_0 contains neither. Therefore at least one of (x, y) or (y, z) is not a series pair which is a contradiction. ∎

In addition we need the following result regarding the rank of a special type of matroid.

Lemma 2.3 *Let $M = (E, \mathcal{I})$ be a matroid. Then each pair of elements of M is a series-pair if and only if $r_M(E) = |E| - 1$.*

Proof: Assume for any two elements $i, j \in E$, (i, j) is a series-pair. Then $r_M(E) < |E|$. By definition, there does not exist a basis of M that avoids both i and j. As series-pairs are transitive then any basis avoids at most one element. Hence $r_M(E) \geq |E| - 1$ and the result follows.

Now assume $r_M(E) = |E| - 1$. Then any basis of M avoids exactly one element. Let $i, j \in E$ for $i \neq j$. Then no basis exists that avoids both i and j. By definition then (i, j) is a series-pair. Therefore any two elements of E form a series-pair. ∎

3 Matching Matrices and Path Matchings

We are now ready to derive Theorems 1.4 and 1.5. We will prove results regarding graphs containing perfect matchings and then apply the results to non-singular submatrices of given matching matrices; see [4,11].

Let $G = (V_r \cup V_c, E)$ be a bipartite graph and let X be its matching matrix. If $\mathcal{M}$ is the set of all perfect matchings of G, then by considering the permutation expansion of $det(X)$ we have

$$det(X) = \sum_{M \in \mathcal{M}} \pm \prod_{e \in M} z_e. \tag{6}$$

This implies that if a bipartite graph has a perfect matching then its corresponding matching matrix will be non-singular. In fact, by (6):

Theorem 3.1 *Let $G = (V_r \cup V_c, E)$ be a bipartite graph and let X be its bipartite matching matrix. Then G has a perfect matching if and only if X is non-singular.* ■

Let A be any $V_r \times V_c$ matrix. If B_r is a row-basis of A and B_c is a column-basis of A, then by Theorem 2.1 $A[B_r, B_c]$ is non-singular. As vertex-induced subgraphs of G have bipartite matching matrices which are submatrices of X, Theorem 1.4 now follows.

Before we prove an analogue of Theorem 3.1 for Tutte matrices, a well known property of skew-symmetric matrices must be introduced.

Lemma 3.2 *Let A be a $V \times V$ skew-symmetric matrix. Then A has even rank.* ■

Let $G = (V, E)$ be a graph and let T be its corresponding Tutte matrix. If $\mathcal{M}$ is the set of all perfect matchings of G, then it is not difficult to prove that

$$det(T) = (\sum_{M \in \mathcal{M}} \pm \prod_{e \in M} z_e)^2. \tag{7}$$

For a proof see [11]. The next result is an easy consequence of (7) but to be self-contained, we give a direct proof.

Theorem 3.3 *Let $G = (V, E)$ be a graph and let T be its Tutte matrix. Then G has a perfect matching if and only if T is non-singular.*

Proof: Assume G has a perfect matching M. Let $\tilde{T}$ be the matrix obtained by setting $z_e = 1$ for each $e \in M$ and setting $z_e = 0$ otherwise. Then each row/column contains exactly one ± 1 entry which implies $det(\tilde{T}) \neq 0$. Clearly $rank(\tilde{T}) \leq rank(T)$. Thus T is non-singular.

Assume T is non-singular. From Lemma 3.2 we can assume $|V| = 2n$. The proof is by induction on n. If T is 2×2, G is either K_2 or $\bar{K}_2$ and the result is trivial. Assume the result is true for all skew-symmatric matrices on less than $2n$ rows/columns. Consider the row i expansion of the determinant of $T_{2n \times 2n}$:

$$det(T) = \sum_k \pm t_{ik} det(T[V \backslash i, V \backslash k]) \tag{8}$$

Then there exist non-zero entry t_{ij} such that $T[V \backslash i, V \backslash j]$ is non-singular. Now $rank(T[V \backslash \{i,j\}, V \backslash \{i,j\}]) \geq rank(T[V \backslash i, V \backslash j]) - 2 = (2n-1) - 2 = 2n - 3$. Therefore $rank(T[V \backslash \{i,j\}, V \backslash \{i,j\}]) = 2n - 2$ as skew-symmetric matrices have even rank. By the inductive hypothesis $G' = (V \backslash \{i,j\}, E')$ has a perfect matching M. Since $t_{ij} \neq 0$ then $ij \in E$. Hence $M \cup \{ij\}$ is a perfect matching for G. ■

If B_r is a row-basis of T then B_r is also a column-basis as T is skew-symmetric. From this observation and by Lemma 2.1, $T[B_r, B_r]$ is skew-symmetric and non-singular. Notice for some $A \subseteq V$, $T[A, A]$ denotes the Tutte matrix of the induced subgraph of G spanned by the vertices in the set A. Combining this arguement with Theorem 3.3, we get a direct proof of Theorem 1.5.

Given a graph $G = (V, E)$, we say that a set of vertices is *stable* if no two are adjacent to one another. Let $T_1, T_2 \subseteq V$ and $W = V \backslash (T_1 \cup T_2)$. Denote this partition of G by $G = (T_1, T_2, W, E)$.

A *path-matching* is a set of vertex-disjoint paths $P_1, P_2, \ldots, P_r$ where each path has all its internal vertices in W together with a matching of the vertices in $W \backslash (V(P_1) \cup V(P_2) \cup \cdots \cup V(P_r))$ such that no P_i has both ends in either T_1 or T_2. Figure 3 provides an example.

Suppose $|T_1| = |T_2| = r$. Then we define a *perfect path-matching* as a path-matching of r vertex disjoint paths $P_1, P_2, \ldots, P_r$, where each vertex in T_1 and T_2 is an endpoint on exactly one of these paths and the vertex set of $W \backslash (P_1 \cup P_2 \cup \cdots \cup P_r)$ contains a perfect matching. In fact the path-matching

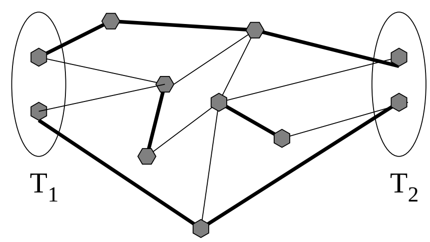

Figure 3: Bolded arcs are contained in the perfect path-matching shown.

of figure 3 is a perfect path-matching. We get the following result regarding perfect path-matchings; see [4]. The proof is similar to that of Theorem 3.3.

Theorem 3.4 *Let $G = (V, E)$ be a graph and let T be its corresponding Tutte matrix. Let S be an $R \times C$ submatrix of T where $R, C \subseteq V$. Then S is non-singular if and only if $G' = (R \backslash C, C \backslash R, R \cap C, E')$ contains a perfect path-matching.* ■

From this result we can conclude that a graph $G = (T_1, T_2, W, E)$ has a perfect path-matching with respect to the partition given if and only if $T[T_1 \cup W, T_2 \cup W]$ is non-singular. If $P = P_1 \cup P_2 \cup \cdots \cup P_r \cup M$ is a path matching then we define the *value* of P by $|P| = (\sum |P_i|) + 2|M|$. The value of the largest path-matching with respect to $G = (T_1, T_2, W, E)$ is denoted by ψ^G. Notice from this definition, the values of any distinct perfect path-matchings are equal regardless of the number of edges contained in each of the paths. From this definition and Theorem 3.4 we get the following result.

Corollary 3.5 *Let $G = (T_1, T_2, W, E)$ be a graph. Then $\psi^G = rank(T[T_1 \cup W, T_2 \cup W])$ with respect to the partition given.* ■

4 The Maximum Rank Matrix Completion Problem

Now we are ready to start proving Theorems 1.1, 1.2 and 1.3. We require some preliminary results which can be found in [5]. Recall that $A(a_{ij} \to \alpha)$ represents a perturbation on A by replacing entry a_{ij} with α.

4.1 Required Results

Theorem 4.1 *Let A be a $V_r \times V_c$ matrix. If $\alpha \neq a_{ij}$ then $rank(A(a_{ij} \rightarrow \alpha)) > rank(A)$ if and only if $i \in D_r(A)$ and $j \in D_c(A)$.*

Proof: Suppose $i \in D_r(A)$ and $j \in D_c(A)$. Then there exist row-basis B_r avoiding row i and column-basis B_c avoiding column j. Theorem 2.1 implies $det(A[B_r, B_c]) \neq 0$. Consider the row expansion on row i of the determinant on the singular matrix $A[B_r \cup i, B_c \cup j]$.

$$det(A[B_r \cup i, B_c \cup j]) = \pm a_{ij} det(A[B_r, B_c]) + K = 0 \tag{9}$$

In the equation, K is some constant representing the summation of the remaining terms on the row expansion. Since the equation is linear, a_{ij} is the unique value such that $A[V_r \cup i, V_c \cup j]$ is singular. We can then conclude for any perturbation on entry a_{ij}, $rank(A(a_{ij} \rightarrow \alpha)) > rank(A)$.

Now suppose $i \notin D_r(A)$. The case $j \notin D_c(A)$ is equivalent under symmetry. If we delete any row from a matrix either the rank stays the same or decreases by one. Thus for any row i, $rank(A(a_{ij} \rightarrow \alpha)) \leq rank(A(a_{ij} \rightarrow \alpha)[V_r \backslash i, V_c]) + 1$. Since i is unavoidable, $rank(A[V_r \backslash i, V_c]) = rank(A) - 1$. Since $A[V_r \backslash i, V_c]$ and $A(a_{ij} \rightarrow \alpha)[V_r \backslash i, V_c]$ have equal rank, $rank(A(a_{ij} \rightarrow \alpha)) \leq rank(A)$ to complete the proof. ∎

Lemma 4.2 *Let A be an $R \times C$ matrix and let k be the number of occurrences of indeterminate z in A. Then there exists $\alpha \in \{1, 2, \ldots, k(|R|+1)+1\}$ such that $A(z \rightarrow \alpha) \approx A$.*

Proof: Let B_r be a row-basis of A and B_c be a column-basis. Since the indeterminate z appears k times in A, the determinant of $A[B_r, B_c]$ will be a polynomial of degree at most k, say $p(z)$. Thus there exist at most k roots $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $p(\alpha_i) = 0$.

Now $A \approx A(z \rightarrow \alpha)$ if and only if their ranks are equal initially and their ranks are equal when the same row is deleted from each. For a given row i, let B_r (resp. B_c) be a row-basis (resp. column-basis) of $A[R \backslash i, C]$. By the same arguement above there exists at most k roots $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $det(A(z \rightarrow \alpha_i)[B_r, B_c]) = 0$. Thus there exists $\alpha \in \{1, 2, \ldots, k(|R| + 1) + 1\}$ such that $A(z \rightarrow \alpha) \approx A$. ∎

The following is a direct corollary of Lemma 4.2.

Corollary 4.3 *Let A be a $V_r \times V_c$ matrix. For any a_{ij} and indeterminate z there exists $\alpha \in \{1, 2, \ldots, |V_r| + 1\}$ such that $A(a_{ij} \to z) \approx A(a_{ij} \to \alpha)$.* ■

Lemma 4.4 *Let A be a $V_r \times V_c$ matrix and let $Y_r = D_r(A)$. Then the following hold:*
i) $rank(A) = rank(A[Y_r, V_c]) + |V_r \backslash Y_r|$
ii) Each row of $A[Y_r, V_c]$ is avoidable.
iii) $D_c(A) \subseteq D_c(A[Y_r, V_c])$

Proof: The first two results follow immediately by deleting the set of coloops from the row matroid $M_r(A)$. Notice each basis of $M_c(A)$ contains a basis of $M_c(A[Y_r, V_c])$. Therefore any avoidable column of A is also avoidable in $A[Y_r, V_c]$. The third result now follows. ■

Lemma 4.5 *Let A be a $V_r \times V_c$ matrix and let $Y_r = D_r(A)$. For an indeterminate z, if $rank(A(a_{ij} \to z)[Y_r, V_c]) > rank(A[Y_r, V_c])$ then $A(a_{ij} \to z) \succ A$.*

Proof: Trivially $A(a_{ij} \to z) \succeq A$. So assume $A(a_{ij} \to z) \approx A$. Lemma 4.4 part *i*) implies $rank(A[Y_r, V_c]) = rank(A) - |V_r \backslash Y_r|$ and $rank(A(a_{ij} \to z)[Y_r, V_c]) = rank(A(a_{ij} \to z)) - |V_r \backslash Y_r|$. Thus $rank(A[Y_r, V_c]) = rank(A(a_{ij} \to z)[Y_r, V_c]$ which is a contradiction. ■

The following is a direct consequence of Corollary 4.3 and Lemma 4.5

Corollary 4.6 *Let A be a $V_r \times V_c$ matrix and let $Y_r = D_r(A)$. If there exists some indeterminate z such that $rank(A(a_{ij} \to z)[Y_r, V_c]) > rank(A[Y_r, V_c])$, then there exists $\alpha \in \{1, 2, \ldots, |V_r| + 1\}$ such that $A(a_{ij} \to \alpha) \succ A$.* ■

4.2 Returning to the Key Thoerems

Now we are ready to begin proving the theorems found in the introduction section. Let A and X be $V_r \times V_c$ matrices where A is of real entries and X is homogeneous. Let $Q = A + X$. For $X_r \subseteq V_r$ and $X_c \subseteq V_c$, call $X_r \cup X_c$ a *stable set* if $X[X_r, X_c] = 0$. Notice if $X_r \cup X_c$ is a stable set for mixed matrix Q then $(V_r \backslash X_r) \cup (V_c \backslash X_c)$ is a cover of Q. The following result is required.

Lemma 4.7 *Let A and X be $V_r \times V_c$ matrices where A is of real entries and X is homogeneous. If $Y_r = D_r(X + A)$ and $Y_c = D_c(X + A)[Y_r, V_c]$, then $Y_r \cup Y_c$ is a stable set.*

Proof: Assume there exists $x_{ij} \neq 0$ such that $i \in Y_r$ and $j \in Y_c$. Let $x_{ij} = z_e$. Lemma 4.4 part ii) implies i is an avoidable row of $(X + A)[Y_r, V_c]$. Let B_r be a basis of $M_r(X + A)[Y_r, V_c]$ such that $i \notin B_r$. Symmetrically, let B_c be a basis of $M_c(X + A)[Y_r, V_c]$ such that $j \notin B_c$. Theorem 2.1 implies $(X + A)[B_r, B_c]$ is non-singular. But $det(X + A)[B_r \cup i, B_c \cup j] = \pm z_e det(X + A)[B_r, B_c] + p(\bar{z})$ where $p(\bar{z})$ is some polynomial, and $\bar{z}$ is a set of indeterminates. Thus $det(X + A)[B_r \cup i, B_c \cup j]$ is non-singular which is a contradiction. Hence $x_{ij} = 0$ and $Y_r \cup Y_c$ is a cover. ■

Proof of Theorem 1.1: For any matrix Q if we delete any row or column, the rank can decrease by at most one. Thus for any $X_r \subseteq V_r$ and $X_c \subseteq V_c$, $rank(Q) \leq rank(Q)[X_r, X_c] + |V_r \backslash X_r| + |V_c \backslash X_c|$. Moreover, if $Y_r = D_r(Q)$ and $Y_c = D_r(Q)[Y_r, V_c]$ then by applying Lemma 4.4 part i) twice, $rank(Q) = rank(Q)[Y_r, Y_c] + |V_r \backslash Y_r| + |V_c \backslash Y_c|$. By Lemma 4.7, $Y_r \cup Y_c$ is stable set. Thus $(V_r \backslash Y_r) \cup (V_c \backslash Y_c)$ is a cover. ■

Now we are ready to prove Theorem 1.3. There is one final lemma that is needed.

Lemma 4.8 *Let A and X be $V_r \times V_c$ matrices where A is of real entries and X is homogeneous. Let $\tilde{X}$ be an evaluation of X, $Y_r = D_r(A + \tilde{X})$, and let $Y_c = D_c(A + \tilde{X})[Y_r, V_c]$. If $Y_r \cup Y_c$ is a stable set then $rank(A + X) = rank(A + \tilde{X})$.*

Proof: Trivially $rank(A + \tilde{X}) \leq rank(A + X)$. Let us prove the result by showing the reverse inequality. Applying Lemma 4.4 part i) twice and using the fact that $\tilde{X}[Y_r, Y_c] = 0$, we obtain $rank(A + \tilde{X}) = rank(A[Y_r, Y_c]) + |V_r \backslash Y_r| + |V_c \backslash Y_c|$. Since $Y_r \cup Y_c$ is stable then $(V_r \backslash Y_r) \cup (V_c \backslash Y_c)$ is a cover. Thus by Theorem 1.1, $rank(A + X) \leq rank(A[Y_r, Y_c]) + |V_r \backslash Y_r| + |V_c \backslash Y_c|$. Therefore $rank(A + X) \leq rank(A + \tilde{X})$ to complete the proof. ■

The arguements for Theorems 1.2 and 1.3 are quite similar. Now we are ready for the proof of Theorem 1.3. Theorem 1.2 follows as a direct corollary.

Proof of Theorem 1.3: If $Y_r \cup Y_c$ is a stable set then $(V_r \backslash Y_r) \cup (V_c \backslash Y_c)$ is a cover and $i)$ follows immediately from Lemma 4.8.

Now assume $Y_r \cup Y_c$ is not stable. Then there exists $i \in Y_r$ and $j \in Y_c$ such that $x_{ij} \neq 0$. In particular Lemma 4.4 part $ii)$ then implies i is an avoidable row and j is an avoidable column of $(\tilde{X}+A)[Y_r, V_c]$. Then Theorem 4.1 implies for indeterminate z, $rank(\tilde{X}(x_{ij} \to z) + A)[Y_r, V_c] > rank(\tilde{X} + A)[Y_r, V_c]$. Corollary 4.6 then implies there exists $\alpha \in \{1, 2, \ldots, |V_r|+1\}$ such that $\tilde{X}(x_{ij} \to \alpha) + A \succ \tilde{X} + A$ to complete the proof of $ii)$. ■

Now we have proven a polynomial-time deterministic algorithm exists for solving the maximum rank matrix completion problem. Let us now extend the results we have obtained to solve the problem of calculating the rank of the Tutte matrix and any corresponding submatrix of the Tutte matrix.

5 Finding the Rank of the Tutte Matrix

Since the main application of Tutte matrices is to find specific properties of graphs, most of the theorems will be stated with respect to graphs. Although the general concept is similar to that of simple mixed matrices, the techniques for applying evaluations to determine the rank are more complex. In the next section we will extend the results found in this section and introduce other ideas that can be used on any given submatrix of a Tutte matrix. All the results of this section can be found in [5].

5.1 Properties of Tutte Matrices

We begin with an immediate corollary that follows directly from Lemma 4.2.

Corollary 5.1 *Let $G = (V, E)$ be a graph and let T be its corresponding Tutte matrix. For any edge $e \in E$ and indeterminate z there exists $\alpha \in \{1, 2, \ldots, 2|V| + 3\}$ such that $T(z_e \to z) \approx T(z_e \to \alpha)$.* ■

As a direct consequence of Lemmas 5.1 and 4.5 we have the following corollary.

Corollary 5.2 *Let $G = (V, E)$ be a graph and let T be its corresponding Tutte matrix. Let $Y_r = D_r(T)$. If there exists some edge $e \in E$ and indeterminate z such that $rank(T(z_e \to z)[Y_r, V]) > rank(T[Y_r, V])$ then there exists $\alpha \in \{1, 2, \ldots, 2|V| + 3\}$ such that $T(z_e \to \alpha) \succ T$.* ■

5.2 Blocks, Series-Classes, and the Main Result

We cannot apply Theorem 1.3 directly to skew-perturbations of skew-symmetric matrices since each indeterminate appears twice. Any perturbation on a skew-symmetric matrix requires the changing of two entries. Before we derive an analogue of Theorem 1.3 some definitions must be introduced.

Let A be a $V_r \times V_c$ matrix. Then $A[X_r, X_c]$ for $X_r \subseteq V_r$ and $X_c \subseteq V_c$ is called a *block* of A if $A[X_r, V_c \backslash X_c] = 0$, $A[V_r \backslash X_r, X_c = 0$, and $A[X_r, X_c]$ contains no zero rows/columns. Hence if A contains no zero rows or columns, the block formation of A is as follows.

$$A = \begin{pmatrix} B_1 & 0 & \cdots & \cdots & 0 \\ 0 & B_2 & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & B_k \end{pmatrix} \tag{10}$$

Note that blocks of a skew-symmetric matrix must be skew-symmetric. Define a block B to be *even* (resp. *odd*) if B contains an even (resp. odd) number of rows/columns.

Recall from Section 2, columns i and j of a matrix A are called a series-pair if there does not exist a column basis that avoids both i and j and neither columns are coloops of $M_c(A)$. Define a *series-class* for element $i \in E$ of a matroid as $\Lambda(i) = i \cup \{j \in E : (i, j) \text{ is a series-pair}\}$. The following is direct corollary of Lemma 2.3 by taking the column matroid of A.

Corollary 5.3 *Let A be an $n \times n$ skew-symmetric matrix such that every column is avoidable. Then A has only one series-class if and only if $rank(A) = n - 1$.* ■

We are almost ready to prove Theorem 1.6. First we need new results to determine when the rank of our matrix has reached its maximum value possible and our algorithm terminates.

Lemma 5.4 *Let T be a $V \times V$ Tutte matrix and $\tilde{T}$ be an evaluation of T. Let $Y_r = D_r(\tilde{T})$ and $Y_c = D_c(\tilde{T}[Y_r, V])$. If the following axioms hold then $rank(\tilde{T}) = rank(T)$.*

i) $T[Y_r, Y_c \backslash Y_r] = 0$.

ii) For all $i, j \in Y_r$ such that $t_{ij} \neq 0$, i and j are a series-pair of $M_c(\tilde{T}[Y_r, V])$.

Proof: Trivially $rank(\tilde{T}) \leq rank(T)$. Let us then show the reverse inequality. Assume the two axioms hold. Since axiom i) holds then $rank(\tilde{T}[Y_r, Y_c]) = rank(\tilde{T}[Y_r, Y_r])$. The same can be said for $T[Y_r, Y_c]$.

Due to skew-symmetry, $Y_r \subseteq Y_c$. Thus $rank(T) \leq rank(T[Y_r, Y_r]) + |V \backslash Y_r| + |V \backslash Y_c|$. Let k be the number of odd blocks in $T[Y_r, Y_r]$. Then $rank(T[Y_r, Y_r]) \leq |Y_r| - k$. Therefore $rank(T) \leq |Y_r| - k + |V \backslash Y_r| + |V \backslash Y_c|$.

By applying Lemma 4.4 part i) twice, $rank(\tilde{T}) = rank(\tilde{T}[Y_r, Y_c]) + |V \backslash Y_r| + |V \backslash Y_c|$. Consider non-zero entry t_{ij} of $\tilde{T}[Y_r, Y_r]$. Then by axiom ii), (i, j) is a series-pair of $M_c(\tilde{T}[Y_r, V])$. By definition of the column matroid, (i, j) is also a series-pair of $M_c(\tilde{T}[Y_r, Y_c])$. Thus by axiom i), (i, j) is a series-pair of $M_c(\tilde{T}[Y_r, Y_r])$.

Let B be any block of $\tilde{T}[Y_r, Y_r]$. By axiom ii), any pair of row and column indices i and j that lie in B form a series-pair. Therefore each block contains exactly one series-class and thus by Corollary 5.3, $rank(\tilde{T}[B, B]) = |B| - 1$. Hence each block contains an odd number of rows and columns and is skew-symmetric. Thus $rank(\tilde{T}[Y_r, Y_r]) = |Y_r| - k$. Thus $rank(\tilde{T}) = |Y_r| - k + |V \backslash Y_r| + |V \backslash Y_c| \geq rank(T)$ to complete the proof. ■

Proof of Theorem 1.6: Let $Y_r = D_r(\tilde{T})$ and let $Y_c = D_c(\tilde{T}[Y_r, V])$. By skew-symmetry $Y_r \subseteq Y_c$. If $rank(\tilde{T}) \neq rank(T)$, then one of the two axioms must fail in Lemma 5.4.

Case I: There exists $e = ij \in E$ such that $i \in Y_r$ and $j \in Y_c \backslash Y_r$. Then by Theorem 4.1, for indeterminate z, $rank(\tilde{T}(z_e \to z)[Y_r, V]) > rank(\tilde{T}[Y_r, V])$. Thus by Corollary 5.2, there exists $\alpha \in \{1, 2, \ldots, 2|V| + 3\}$ such that $\tilde{T}(z_e \to \alpha) \succ \tilde{T}$.

Case II: There exists $e = ij \in E$ such that $i, j \in Y_r$ and (i, j) is not a series-pair of $M_c(\tilde{T}[Y_r, V])$. The set of avoidable rows remain avoidable when we delete an avoidable column of $\tilde{T}$. Now by Lemma 4.4 part ii), every row is avoidable in $\tilde{T}[Y_r, V]$. Thus every row is avoidable in $\tilde{T}[Y_r, V\backslash i]$ as i is an avoidable column. Now by definition of a series-pair, j is an avoidable column of $\tilde{T}[Y_r, V\backslash i]$. Thus by Theorem 4.1, $rank(\tilde{T}(z_e \rightarrow z)[Y_r, V\backslash i]) > rank(\tilde{T}[Y_r, V\backslash i])$ for indeterminate z. But $rank(\tilde{T}[Y_r, V]) = rank(\tilde{T}[Y_r, V\backslash i])$ since i is an avoidable column of $\tilde{T}[Y_r, V]$. Thus $rank(\tilde{T}(z_e \rightarrow z)[Y_r, V]) > rank(\tilde{T}[Y_r, V])$. Hence by Corollary 5.2, there exists some $\alpha \in \{1, 2, \ldots, 2|V|+3\}$ such that $\tilde{T}(z_e \rightarrow \alpha) \succ \tilde{T}$. ■

The immediate consequence of Lemma 5.4 and the proof of Theorem 1.6 is the following main result. It is an analogue of Theorem 1.3 regarding Tutte matrices.

Theorem 5.5 *Let $G = (V, E)$ be a graph, T be its Tutte matrix, and $\tilde{T}$ be an evaluation of T. Let $Y_r = D_r(\tilde{T})$ and let $Y_c = D_c(\tilde{T}[Y_r, V])$.*

i) If there is some $e = ij \in E$ where $i \in Y_r$ and $j \in Y_c \backslash Y_r$, then there exists $\alpha \in \{1, 2, \ldots, 2|V| + 3\}$ such that $\tilde{T}(z_e \rightarrow \alpha) \succ \tilde{T}$.

ii) If there is some $e = ij \in E$ where $i, j \in Y_r$ and (i, j) is not a series-pair of $M_c(\tilde{T}[Y_r, V])$, then there exists $\alpha \in \{1, 2, \ldots, 2|V| + 3\}$ such that $\tilde{T}(z_e \rightarrow \alpha) \succ \tilde{T}$.

iii) Otherwise if the hypotheses in i) and ii) both do not hold then $rank(\tilde{T}) = rank(T)$. ■

We now can repeatedly apply Theorem 5.5 on a Tutte matrix to calculate its given rank and thus the cardinality of a maximum matching of a given graph. But in order to find the next perturbation, we may have to identify the series-class of a given element of $M_c(\tilde{T}[Y_r, V])$. An efficient method is shown in Section 7.

6 A Submatrix of the Tutte Matrix

Now we are ready to analyze the case when we take a submatrix of an evaluation of the Tutte matrix. Most of the material found in this section follows immediately from arguements found in the previous two sections. Theorems 6.2 and 6.5 are new results.

Let T be a given Tutte matrix and let S be an $R \times C$ submatrix of T where $R, C \subseteq V$. Let $\tilde{S}$ be an evaluation of S. Note that the results derived regarding the Tutte matrix cannot be used here as S may not be skew-symmetric. Let Y_r denote the set of avoidable rows of S and Y_c denote the set of avoidable columns of $S[Y_r, C]$.

Notice $S[Y_r \cap Y_c, Y_r \cap Y_c]$ is skew-symmetric. Thus all indeterminates entries of $S[Y_r \cap Y_c, Y_r \cap Y_c]$ occur twice. Then Theorem 5.5 immediately implies the following corollary.

Corollary 6.1 *Let $G = (V, E)$ be a graph and let T be its Tutte matrix. Let S be an $R \times C$ submatrix of T and let $\tilde{S}$ be an evaluation of S. Then for $e = ij \in E$ where $i, j \in Y_r \cap Y_c$, there exists $\alpha \in \{1, 2, \ldots, 2|R| + 3\}$ such that $\tilde{S}(z_e \to \alpha) \succ \tilde{S}$ if and only if (i, j) is not a series-pair of $\tilde{S}[Y_r, C]$.* ■

The result above does not provide all possible perturbations that can be used to find a more independent evaluation of a given S. There is still the case where indeterminate z occurs only once in $S[Y_r, Y_c]$. We cannot use previous results to efficiently find the rank in this case. We have the following theorem.

Theorem 6.2 *Let $G = (V, E)$ be a graph and let T be its Tutte matrix. Let $\tilde{S}$ be an evaluated submatrix of T. Assume there exists edge $e = ij \in E$ such that $i \in D_r(\tilde{S})$, $j \in D_c(\tilde{S})$, but $i \notin D_c(\tilde{S})$ and/or $j \notin D_r(\tilde{S})$. Then there exists indeterminate z such that $rank(\tilde{S}(z_e \to z)) > rank(\tilde{S})$.*

Proof: Assume $j \notin D_r(\tilde{S})$. A symmetric arguement can be given if $i \notin D_c(\tilde{S})$. Let B_r be a row-basis of $\tilde{S}$ avoiding row i but containing row j and let B_c be a column-basis avoiding column j. Then $\tilde{S}[B_r, B_c]$ is non-singular. Assume $s_{ij} = \alpha$. Let $\tilde{S}'$ be the matrix obtained by replacing entry s_{ij} with $\alpha + z$ and entry s_{ji} with $-\alpha - y$ for indeterminates z and y. Let A and A' denote $\tilde{S}[B_r \cup i, B_c \cup j]$ and $\tilde{S}'[B_r \cup i, B_c \cup j]$ respectively. The determinant of A' can be written in the form below.

$$det(A') = c_1 + c_2 z + c_3 y + c_4 zy \tag{11}$$

Now at $z = y = 0$, $det(A) = det(A') = c_1$. Since A is singular, $c_1 = 0$. If $i \notin B_c$ then $det(A') = c_1 + c_2 z = c_2 z$. But $c_2 = det(A[B_r, B_c])$ which is non-singular. Thus $c_2 \neq 0$ and A' is non-singular. Therefore $rank(\tilde{S}(z_e \rightarrow z)) > rank(\tilde{S})$.

Otherwise $i \in B_c$. If we substitute $z = 0$ in the equation for $det(A')$ above we get $det(A'[z = 0]) = c_1 + c_3 y$. Since $c_1 = 0$, $det(A'[z = 0]) = c_3 y$. Notice $c_3 = det(A[(B_r \backslash j) \cup i, (B_c \backslash i) \cup j])$. But $j \notin D_r(\tilde{S})$. Thus $(B_r \backslash j) \cup i$ is not a row-basis implying the determinant is zero. Therefore $c_3 = 0$. Setting $y = z$ we obtain $det(A(s_e \rightarrow z)) = c_2 z + c_4 z^2$. Thus $A(z_e \rightarrow z)$ is non-singular. Therefore $rank(\tilde{S}(z_e \rightarrow z)) > rank(\tilde{S})$. ∎

Corollary 6.3 *Let $G = (V, E)$ be a graph, T be its Tutte matrix and let S be an $R \times C$ submatrix of T. For edge $e = ij \in E$ and indeterminate z, there exists $\alpha \in \{1, 2, \ldots, 2|R| + 3\}$ such that $S(z_e \rightarrow z) \approx S(z_e \rightarrow \alpha)$.* ∎

Corollary 6.3 follows directly from Lemma 4.2. The following corollary follows from Lemma 4.5 and Corollary 6.3.

Corollary 6.4 *Let $G = (V, E)$ be a graph and T be its Tutte matrix. Let S be an $R \times C$ submatrix of T and $Y_r = D_r(S)$ If there is some edge $e = ij \in E$ such that for indeterminate z, $rank(S(z_e \rightarrow z)[Y_r, C]) > rank(S[Y_r, C])$ then there exists $\alpha \in \{1, 2, \ldots, 2|R| + 3\}$ such that $S(z_e \rightarrow \alpha) \succ S$.* ∎

We are now ready to state and prove the key result of this section. It is a more detailed version of Theorem 1.7.

Theorem 6.5 *Let $G = (V, E)$ be a graph, T be its Tutte matrix and let S be an $R \times C$ submatrix of T. Let $\tilde{S}$ be an evaluation of S, $Y_r = D_r(\tilde{S})$, and $Y_c = D_c(\tilde{S}[Y_r, C])$.*

i) If there is some $e = ij \in E$ where $i \in Y_r$ (resp. $Y_r \backslash Y_c$) and $j \in Y_c \backslash Y_r$ (resp. Y_c), then there exists $\alpha \in \{1, 2, \ldots, 2|R| + 3\}$ such that $\tilde{S}(z_e \rightarrow \alpha) \succ \tilde{S}$.

ii) If there is some $e = ij \in E$ where $i, j \in Y_r \cap Y_c$ and (i, j) is not a series-pair of $M_c(\tilde{S}[Y_r, C])$, then there exists $\alpha \in \{1, 2, \ldots, 2|R| + 3\}$ such that $\tilde{S}(z_e \rightarrow \alpha) \succ \tilde{S}$.

iii) Otherwise if the hypotheses in i) and ii) both do not hold then $rank(\tilde{S}) = rank(S)$.

Proof: Assume the hypothesis of i) occurs. Then Theorem 6.2 implies $rank(\tilde{S}(z_e \to z)[Y_r, C]) > rank(\tilde{S}[Y_c, C])$ for indeterminate z. Corollary 6.4 then implies there exists $\alpha \in \{1, 2, \ldots, 2|R| + 3\}$ such that $\tilde{S}(z_e \to \alpha) \succ \tilde{S}$

If the hypothesis of ii) occurs, the result immediately follows from Corollary 6.1. Assume neither hypotheses of i) or ii) occurs. Trivially $rank(\tilde{S}) \leq rank(S)$. Let us then show the reverse inequality.

Since i) fails, $\tilde{S}[Y_r \backslash Y_c, Y_c \backslash Y_r] = 0$. Thus $\tilde{S}[Y_r, Y_c] = \tilde{S}[Y_r \cap Y_c, Y_r \cap Y_c]$ and is skew-symmetric. Therefore $rank(\tilde{S}) = rank(\tilde{S}[Y_r \cap Y_c, Y_r \cap Y_c]) + |R \backslash Y_r| + |C \backslash Y_c|$.

Let k be the number of blocks of $\tilde{S}[Y_r \cap Y_c, Y_r \cap Y_c]$. Consider any block B of $\tilde{S}[Y_r \cap Y_c, Y_r \cap Y_c]$. By axiom ii), any pair of row and column indices i and j that lie in B form a series-pair. Therefore each block contains exactly one series-class and thus by Corollary 5.3, $rank(\tilde{S}[B, B]) = |B| - 1$. Hence each block contains an odd number of rows and columns and is skew-symmetric. Thus $rank(\tilde{S}[Y_r \cap Y_c, Y_r \cap Y_c]) = |Y_r \cap Y_c| - k$. Therefore $rank(\tilde{S}) = |Y_r \cap Y_c| - k + |R \backslash Y_r| + |C \backslash Y_c|$.

But $rank(S) \leq rank(S[Y_r, Y_c]) + |R \backslash Y_r| + |C \backslash Y_c|$. Thus $rank(S) \leq rank(S[Y_r \cap Y_c, Y_r \cap Y_c]) + |R \backslash Y_r| + |C \backslash Y_c|$. Since $S[Y_r \cap Y_c, Y_r \cap Y_c]$ has k odd blocks, $rank(S)[Y_r \cap Y_c] \leq |Y_r \cap Y_c| - k$. Hence $rank(S) \leq |Y_r \cap Y_c| - k + |R \backslash Y_r| + |C \backslash Y_c|$. We can now conclude $rank(S) \leq rank(\tilde{S})$ to complete the proof. ∎

We now have a polynomial-time algorithm that given a Tutte matrix T, we can calculate the rank of any submatrix S. We still need a process for determining what entry s_{ij} should be set equal to such that all avoidable rows remain avoidable if we cannot increase the rank. A procedure will be carried out to solve this in the next section.

7 Efficient Perturbation Techniques

So far we have taken a matrix A which contains indeterminate and possibly real entries, and found an entry a_{ij} to perturb if one exists. But we have not shown what new value a_{ij} should take on in order to get a more independent matrix A'. Also for each evaluation on A, we must calculate its rank and $D_r(A)$. Finally if we are dealing with any submatrix of a Tutte matrix, we may have to identify a series-class. All these issues are investigated here.

In this section we introduce techniques that can be used to improve on the complexity of simple matrix multiplication. For other applications on matrix multiplication and complexity improvements see [1,2].

7.1 Calculating the Rank and Finding the Set of Avoidable Rows

Let A be a $V_r \times V_c$ matrix. Assume that we want to find its rank and its set of avoidable rows. Let B_r be a row-basis of A. Then the *column-reduced form* of A with respect to B_r, denoted A^*, satisfies the equation $A^* = AQ$ for some $V_c \times V_c$ invertible matrix Q where A^* is of the following form (after possible row/column permutations).

$$A^* = \left(\begin{array}{c|c} I & 0 \\ \hline W & 0 \end{array} \right) \tag{12}$$

Analogously we can define the *row-reduced form* of A such that $(A^T)^* = A^T Q$ for some non-singular matrix Q. The rows of the identity matrix contained in A^* form a row-basis B_r of the row matroid $M_r(A^*)$. But since A and A^* have the same rank then we know $rank(A) = |B_r|$.

We can also find $D_r(A)$ from A^* since $D_r(A) = D_r(A^*)$. Let us then find the avoidable rows of A by identifying the coloops of $M_r(A^*)$. Let i be a row such that $i \notin B_r$. If $a^*_{ij} \neq 0$ for some $j \in B_r$, then row i can be replaced with row j in B_r to obtain $B'_r = (B_r \backslash j) \cup i$. But B'_r is also a basis of $M_r(A^*)$. Thus the coloops of $M_r(A^*)$ consist of rows $j \in B_r$ such that $a^*_{ij} = 0$ for all $i \notin B_r$.

7.2 Rank-One Updates

Let A and D be $V_r \times V_c$ matrices such that $rank(D) = 1$. D is called a *rank-one matrix.* Consider the matrix $A' = A + D$. A *rank-one update* for A' is the process of determining the column-reduced form of A', given the column-reduced form of A. Before we analyze the efficiency of a rank-one update some other results and definitions must be introduced. We have the following important result for rank-one matrices.

Theorem 7.1 *Let D be an $n \times m$ matrix. Assume $n \geq m$. Then there exist $u \in \Re^n$, $v \in \Re^m$ such that $D = uv^T$ if and only if D is a rank-one matrix. Furthermore, if D is rank-one then u and v can be found in $O(n^2)$-operations.*

Proof: If $D = uv^T$ then the result is trivial. Assume D is rank-one. Let d_{ij} be a non-zero entry and let $D' = 1/d_{ij}D$. If r_i denotes the i^{th} row and c_j denotes the j^{th} column of D' then $D' = c_j r_i^T$. Furthermore, we have found the two desired vectors that form D in $O(n^2)$ operations. ■

Let A be an $n \times m$ matrix and let c_j be any column of A. Then a *column-pivot* on A with respect to $a_{ij} \neq 0$ for some i is the process of replacing a column in A say c_k with $\alpha_1 c_j + \alpha_2 c_k$ where $\alpha_1, \alpha_2 \in \Re$ such that $a_{ik} = 0$. Let this new matrix be A'. An example on entry $a_{11} = \alpha$ is given below.

$$A = \left(\begin{array}{c|c} \alpha & x^T \\ \hline y & \hat{A} \end{array}\right) \tag{13}$$

$$A' = \left(\begin{array}{c|c} 1 & 0 \\ \hline \alpha^{-1}y & \hat{A} - \alpha^{-1}yx^T \end{array}\right) \tag{14}$$

Notice a pivot requires $O(n^2)$ operations. For any given pivot, there exists a *pivot matrix* P associated with it such that $A' = AP$. The pivot matrix associated with the previous pivot is given below.

$$P = \left(\begin{array}{c|c} \alpha^{-1} & \alpha^{-1}x^T \\ \hline 0 & I \end{array}\right) \tag{15}$$

The prime advantage to working with pivot matrices is the fact that multiplying any matrix by a given pivot matrix requires only $O(n^2)$ operations. Notice for any rank-one matrix D, its column-reduced form D^* can be found in a single pivot on any entry $d_{ij} \neq 0$ such that j forms a column-basis of D. Thus there exists pivot matrix P such that $D^* = DP$.

Now let us return to $A' = A + D$ where A and D are $V_r \times V_c$ and D is rank-one. Let $|V_r \times V_c| = n$. We know there exists a non-singular $V_c \times V_c$ matrix Q such that $A^* = AQ$. Let us multiply A' on the righthand side by Q to obtain the following equation.

$$A'Q = A^* + DQ. \tag{16}$$

Let $D = uv^T$. Then $DQ = u(v^TQ)$. But v^TQ is just a vector in $\Re^{|V_c|}$, say w, and thus can be calculated in $O(|V_c|^2)$ arithmetic operations. Now $DQ = uw^T$. Theorem 7.1 then implies DQ is a rank-one matrix and therefore it can be found in $O(n^2)$ operations.

Let DQ=U, j be a coloop of $M_c(U)$, and P_1 be the pivot matrix associated with column-pivot on non-zero entry u_{ij} for some i. Then $U^* = UP_1$. Let us multiply (16) on the righthand sides by P_1 to obtain the following.

$$A'QP_1 = A^*P_1 + U^* \tag{17}$$

Without loss of generality assume A^* is in the form given in (12) and $rank(A) = k$. Let $u_1, u_2, \ldots, u_{|V_c|}$ denote the columns of U. We have two cases that must be considered.

Case I: There exists $i > k$ such that $u_i \neq 0$. Without loss of generality, assume $u_{k+1} \neq 0$ and we do a column-pivot on $u_{i(k+1)} \neq 0$ for some i. Then if P_1 is our pivot matrix, $A^*P_1 = A^*$. Thus $A'QP_1$ is of the following form.

$$A'QP_1 = \left(\begin{array}{c|c|c} I_{k\times k} & & \\ \hline W & ? & 0 \end{array}\right) \tag{18}$$

Now for non-zero entries in positions $i, (k+1)$ where $i \leq k$, we can perform a pivot to set all these entries to zero by taking various multiples of the first k columns and subtracting them from the $(k+1)^{st}$. Thus there exists pivot matrix P_2 such that

$$A'QP_1P_2 = \left(\begin{array}{c|c|c} I_{k\times k} & 0 & \\ \hline W & ? & 0 \end{array}\right). \tag{19}$$

If $? = 0$ we have found A'^*. Otherwise there exists some non-zero entry in position $t, (k+1)$ for $t \geq k+1$. Hence we can do a pivot with respect to the entry in this position such that all other entries in row t become zero.

Therefore we have $A'^* = A'QP_1P_2P_3$. Thus we can obtain A'^* if we are given $A^* = AQ$ in $O(n^2)$ arithmetic operations.

Case II: $u_i = 0$ for all $i > k$. Without loss of generality, assume $u_k \neq 0$. Then after we pivot on some entry in column k we get

$$A^*P_1 = \left(\begin{array}{c|c|c} I_{(k-1)\times(k-1)} & 0 & \\ \cline{1-2} ? & 1 & 0 \\ \hline \multicolumn{2}{c|}{W'} & 0 \end{array}\right). \tag{20}$$

In particular column a_k remains unchanged after A^* is multiplied by P_1. But adding U^* to A^*P_1 then we get

$$A'QP_1 = \left(\begin{array}{c|c|c} I_{(k-1)\times(k-1)} & & \\ \cline{1-1} & ? & 0 \\ W' & & \end{array}\right). \tag{21}$$

Notice this matrix has the same structure as that given in (18). But we showed we can find the column-reduced form of the matrix in (18) by at most two more pivots. Thus we can find the column-reduced form of the matrix above in at most two more pivots. We have now shown that given $A' = A + D$ and $A^* = AQ$, we can find A'^* in $O(n^2)$ calculations.

7.3 Identifying a Series-Class

Let A be a $V_r \times V_c$ matrix. Consider we want to identify the series-class associated with some non-coloop element $e \in M_c(A)$. Then for $j \neq e$, $j \in \Lambda(e)$ if and only if j is not a coloop of $M_c(A)$ but is a coloop of $M_r(A[V_r, V_c \backslash e])$.

Assume we are given the row-reduced form of A, say A^*, and Q such that $A^* = QA$. We can immediately identify the coloops of $M_c(A)$ given A^*. Let U be the $V_r \times V_c$ matrix such that $u_{ij} = a_{ij}$ if $j = e$ and $u_{ij} = 0$ otherwise. Now let

$$A' = A - U \tag{22}$$

$$QA' = A^* - QU \tag{23}$$

Notice the coloops of $M_c(A')$ are equivalent to those of $M_c(A[V_r, V_c \backslash e])$. But since $rank(U) = 1$, $A' = A - U$ is just a rank-one update. Applying the technique from Section 7.2, we can find the row-reduced form of A' in $O(n^2)$ operations given Q and A^*. Hence we can find $\Lambda(e)$ in $O(n^2)$.

7.4 Finding a Perturbation

Let A be a $V_r \times V_c$ matrix. Assume we have found some entry a_{ij} that we can perturb such that the independence of A will increase. Let us assume that other entries e may be perturbed as well. We will now develop a technique to determine α such that $A(a_e \to a_e + \alpha) \succ A$.

Let A and D be $V_r \times V_c$ matrices where $d_{ij} = 1$ if ij is to be perturbed and $d_{ij} = 0$ otherwise. Assume k entries must be perturbed. Notice $rank(D) \leq k$. Let A' be the matrix obtained from the following equation where z is an indeterminate,

$$A' = A + zD. \tag{24}$$

We must solve the above equation for z such that $A' \succ A$. Let A^* be the column-reduced form of A. From Section 7.1 we know there exists a non-singular matrix Q such that $A^* = AQ$. Let us multiply the previous equation by Q on the righthand sides. Then we get the following.

$$A'Q = A^* + zDQ \tag{25}$$

We can rewrite DQ as $(D_1 + \cdots + D_k)Q$ where each D_i is of rank-one. Thus D_iQ is a rank-one matrix for all i. Let us apply a rank-one update to $A'_1 = A^* + zD_1$ given Q. The column-reduced form of A'_1 is given below.

$$A_1'^* = \left(\begin{array}{c|c} I & 0 \\ \hline W(z) & 0 \end{array} \right) \tag{26}$$

Each non-constant entry in the submatrix $W(z)$ is a linear rational function in z. This follows from the definition of the column-pivot described in Section 7.2.

We can repeat this process for each D_i with the final rank-one update being $A'_kQ = A'^*_{k-1} + zD_kQ$. Then $A'^* = A'^*_k$ will be of the same form as in (26) but now each non-constant entry in $W(z)$ will be a rational function $p_{ij}(z)/q_{ij}(z)$ of degree at most k. Therefore using the technique of Section

7.2 for each D_i, we can find the column-reduced form of A' given that of A in $O(kn^2)$ operations by applying k rank-one updates.

Now returning to the process of finding α such that $A'(z \rightarrow \alpha) \succ A$. If the rank of A'^* has increased then we can assign z any value such that $q_{ij}(z) \neq 0$ for all rational functions in $W(z)$. Otherwise for each avoidable row i, we must choose α such that $p_{ij}(\alpha) \neq 0$ for all avoidable rows j and $q_{ij}(z) \neq 0$ for all rational functions in $W(z)$.

Now each $p_{ij}(z)$ and $q_{ij}(z)$ has at most k unique roots. In either case, since we have $O(n^2)$ rational functions of degree k, we can find this value α in $O(kn^2)$ operations. In particular, when $k \leq 2$, we can find α in $O(n^2)$ arithmetic operations by finding the roots of the desired functions individually. We use this result in the next section.

8 A More Efficient Maximum Rank Matrix Completion Algorithm

Theorems 1.3 and 6.5 immediately imply there exist polynomial-time algorithms that solve the maximum rank matrix completion problem and find the rank of any submatrix of a given Tutte matrix in $O(n^6)$ arithmetic operations. Throughout the remainder of this section we use the techniques developed in the previous section to solve both problems in $O(n^4)$ calculations.

8.1 An $O(n^4)$ Algorithm For Solving the Maximum Rank Completion Problem

Let A and X be $V_r \times V_c$ matrices where A is of real entries and X is homogeneous. Assume $n = |V_r \cup V_c|$. Let $\tilde{X}$ be an evaluation of X, and let $\tilde{Q} = A + \tilde{X}$. Let $Y_r = D_r(\tilde{Q})$ and let $Y_c = D_c(\tilde{Q}[Y_r, V_c])$. The following is one complete iteration of an $O(n^4)$ perturbation algorithm.

Assume we currently know the column-reduced form for $\tilde{Q}$ and the row-reduced form of $\tilde{Q}[Y_r, V_c]$. Therefore we also know Y_r and Y_c. Hence we can apply Theorem 1.3. Either $X[Y_r, Y_c] = 0$ or there exists $i \in Y_r$ and $j \in Y_c$ such that $x_{ij} \neq 0$. If $X[Y_r, Y_c] = 0$ then $rank(\tilde{Q}) = rank(Q)$ and our rank is maximal.

Assume there exists $x_{ij} \neq 0$ where $i \in Y_r$ and $j \in Y_c$. Then by Theorem 1.3, for indeterminate z, $A + \tilde{X}(z_{ij} \rightarrow z) \succ A + \tilde{X}$. Using the methods of

Section 7.4, we can find $\alpha \in \{1, 2, \ldots, |V_r| + 1\}$ such that $A + \tilde{X}(z_{ij} \rightarrow \alpha) \approx A + \tilde{X}(z_{ij} \rightarrow z)$ in $O(n^2)$ operations.

Now let $\tilde{Q}' = A + \tilde{X}(z_{ij} \rightarrow \alpha)$. Then $\tilde{Q}' = \tilde{Q} + D$ where D is an all-zero matrix with the exception of entry d_{ij}. Therefore we can determine the column-reduced form of $\tilde{Q}'$, say $\tilde{Q}'^*$, given the column-reduced form of $\tilde{Q}$ in $O(n^2)$ arithmetic operations using a rank-one update.

Let $Y_r' = D_r(\tilde{Q}')$. Then we can identify $rank(\tilde{Q}')$ and Y_r' immediately from $\tilde{Q}'^*$. Now we need to determine the row-reduced form of $\tilde{Q}'[Y_r', V_c]$.

Note that the rank only increases $O(n)$ times during the algorithm. Thus, if $rank(\tilde{Q}') > rank(\tilde{Q})$ then we can find the row-reduced form of $\tilde{Q}'[Y_r', V_c]$ in $O(n^3)$ calculations by pivoting.

Let us assume $rank(\tilde{Q}') = rank(\tilde{Q})$. In this case, $Y_r \subset Y_r'$. Since $rank(\tilde{Q}'[Y_r, V_c] - \tilde{Q}[Y_r, V_c]) \leq 1$, we can determine the row-reduced form for $\tilde{Q}'[Y_r, V_c]$ from that of $\tilde{Q}[Y_r, V_c]$ in $O(n^2)$ using a rank-one update.

Assume we have t new avoidable rows. Let $Y_r' \backslash Y_r = \{y_1, y_2, \ldots, y_t\}$, $W_o = Y_r$, and let $W_i = Y_r \cup \{y_1, \ldots, y_i\}$ for $1 \leq i \leq t$. Then we can determine the row-reduced form for $\tilde{Q}'[W_i, V_c]$ from that of $\tilde{Q}'[W_{i-1}, V_c]$ using a rank-one update. Hence it takes $O(tn^2)$ operations to compute the row-reduced form for $\tilde{Q}'[Y_r', V_c]$ given the one for $\tilde{Q}[Y_r, V_c]$.

Each iteration of the algorithm such that the rank remains constant requires $O(tn^2)$ arithmetic operations where t is the number of new avoidable rows found. Assume the rank has just increased and it does not increase again until k perturbations later. Let $t_1, t_2, \ldots, t_k$ be the number of new avoidable rows during each of these iterations. Since $t_1 + t_2 + \cdots + t_k \leq n$, the rank increases in at most $O(n^3)$ calculations. Therefore the algorithm terminates in $O(n^4)$. We have the following result.

Theorem 8.1 *The maximum rank matrix completion problem can be solved in $O(n^4)$ arithmetic operations.* ■

8.2 An $O(n^4)$ Algorithm For Calculating the Rank of any Submatrix of a Given Tutte Matrix

Let $G = (V, E)$ be a graph and let T be its corresponding Tutte matrix. Let $\tilde{T}$ be an evaluation of T and let $\tilde{S}$ be an $R \times C$ submatrix of $\tilde{T}$. We now will give an algorithm that calculates the rank of $\tilde{S}$.

The procedure is almost identical to that of the algorithm in Section 8.1. There are a few differences we must look at. First we are choosing α from the set $\{1, 2, \ldots, 2|R| + 3\}$. But α can still be found in $O(n^2)$ operations using the technique of Section 7.4.

Second there is the additional aspect that we could be perturbing two entries at a time for a given $\tilde{S}$. But we can replace all rank-one updates in Section 8.1 with two consecutive rank-one updates $\tilde{S}' = \tilde{S} + D_1 + D_2$ if necessary. This will not increase the complexity of the algorithm.

Also there is the additional aspect of identifying a specific series-class. Assume $\tilde{S}[Y_r \backslash Y_c, Y_c] = 0$ and $\tilde{S}[Y_r, Y_c \backslash Y_r] = 0$ otherwise axiom $i)$ of Theorem 6.5 implies there is some edge $e = ij$ we can perturb. Then we want to find the series-classes of $\tilde{S}[Y_r \cap Y_c, Y_r \cap Y_c]$.

Let $B_1, B_2, \ldots, B_k$ be the blocks of $\tilde{S}[Y_r \cap Y_c, Y_r \cap Y_c]$. By Lemma 4.4 part $ii)$, for any block B_i every row of $\tilde{S}[B_i, B_i]$ is avoidable. Since B_i is skew-symmetric then every column of $\tilde{S}[B_i, B_i]$ is also avoidable. Let us now try to find a block that contains more than one series-class.

First consider there exists some even block B_i. Now every column of B_i is avoidable. Therefore if j is any column of B_i then $rank(B_i) = rank[B_i, B_i \backslash j]$. But $rank(B_i)$ is even and $[B_i, B_i \backslash j]$ has an odd number of columns. Thus $rank(B_i) < |B_i| - 1$ and by Corollary 5.3, B_i must contain more than one series-class. Therefore, if B_i is even then identify the series class associated with some element j of the block. We know there exists $e = ij$ where $i, j \in Y_r \cap Y_c$ and (i, j) is not a series-pair. Thus the hypothesis of axiom $ii)$ of Theorem 6.5 always holds for even blocks.

Now assume all blocks are odd. Consider the row-reduced form of $\tilde{S}[Y_r, C]$. Note that since every row is avoidable then there will be at least one all-zero row for each block.

$$\tilde{S}^*[Y_r, C] = \left(\begin{array}{c|c} I & Z \\ \hline 0 & 0 \end{array}\right) \tag{27}$$

Choose any all-zero row of $\tilde{S}^*[Y_r, C]$ and identify the corresponding block associated with that row. Note if any two zero rows are in the same block, say B_i, then $rank(B_i) < |B_i| - 1$. Corollary 5.3 then implies B_i has more than one series-class.

Therefore, if each block has exactly one all-zero row then $rank(B) = |B| - 1$ for each block B. We can then conclude that $rank(\tilde{S})$ is the optimal solution. All even blocks of a skew-symmetric matrix have more than one series-class. The number of series-classes of an odd block can be identified immediately by looking at the row-reduced form of $\tilde{S}[Y_r, C]$. We can restate Theorem 6.5 as follows.

Theorem 8.2 *Let $G = (V, E)$ be a graph, T be its Tutte matrix and let S be an $R \times C$ submatrix of T. Let $\tilde{S}$ be an evaluation of S, $Y_r = D_r(\tilde{S})$, and $Y_c = D_c(\tilde{S}[Y_r, C])$.*

i) If there is some $e = ij \in E$ where $i \in Y_r$ (resp. $Y_r \backslash Y_c$) and $j \in Y_c \backslash Y_r$ (resp. Y_c), then there exists $\alpha \in \{1, 2, \ldots, 2|R| + 3\}$ such that $\tilde{S}(z_e \rightarrow \alpha) \succ \tilde{S}$.

ii) If there exists some block B of $\tilde{S}[Y_r \cap Y_c, Y_r \cap Y_c]$ such that $rank(B) < |B| - 1$, then there exists $e = ij$ where $i, j \in B$ and $\alpha \in \{1, 2, \ldots, 2|R| + 3\}$ such that $\tilde{S}(z_e \rightarrow \alpha) \succ \tilde{S}$.

iii) Otherwise if the hypotheses of i) and ii) both do not hold then $rank(\tilde{S}) = rank(S)$. ■

The algorithm for calculating the rank of $\tilde{S}$ is identical to the algorithm from Section 8.1 with the exception that during each iteration we apply Theorem 8.2 instead of Theorem 1.3. We can still test the hypothesis of axiom *i*) of Theorem 8.2 in constant time since we know Y_r and Y_c at the beginning of an iteration. We can also test the hypothesis of axiom *ii*) in constant time since we calculated the row-reduced form of $\tilde{S}[Y_r, C]$ in the previous iteration. Assume there is a block B such that we can apply axiom *ii*). Then we can apply the technique from Section 7.3 to identify the series-class associated with a given element of B. Since this is just a rank-one update, it can be done in $O(n^2)$ calculations. Therefore the efficiency of each iteration has not increased from the algorithm in Section 8.1. Thus we can calculate the rank of any submatrix of a given Tutte matrix in $O(n^4)$ operations. The paper is concluded with the following result.

Theorem 8.3 *Let T be a Tutte matrix and let S be a submatrix of T. Then $rank(S)$ can be found in $O(n^4)$ arithmetic operations.* ■

References

[1] D. Bini and V. Pan, *Polynomial and Matrix Computations Volume 1: Fundamental Algorithms*, Birkhauser, Boston, 1994.

[2] P. Burgisser, M. Clausen, and M. Amin Shokrollahi, *Algebraic Complexity Theory*, Springer-Verlag, Berlin, 1997.

[3] J. Cheriyan, "Randomized $\tilde{O}(M(|V|))$ Algorithms for Problems in Matching Theory", SIAM J. Comput. **26**(1997), 1635-1655.

[4] W.H. Cunningham and J.F. Geelen, "The Optimal Path-Matching Problem", Combinatorica **17** (3) (1997), 315-337.

[5] J.F. Geelen, "Matching Theory", Euler Institute for Discrete Mathematics and its Applications, 2000.

[6] J.F. Geelen, "Maximum rank matrix completion", Linear Algebra Applications, **288**(1999), 211-217.

[7] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 1960.

[8] D.J. Hartfiel and R. Loewy, "A determinental version of the Frobenius-Konig theorem", Linear Multilinear Algebra **16**(1984), 155-165.

[9] J.E. Hopcroft and R.M. Karp, "An $n^{5/2}$ algorithm for maximum matching in bipartite graphs", SIAM Journal on Computing 2 (1973), 225-231.

[10] L. Lovasz, "On determinants, matchings and random algorithms", in *Fundamentals of Computation Theory*, Akademie-Verlag, Berlin, 1979, 565-574.

[11] W.T. Tutte, "The factorization of linear graphs", J. London Math. Soc., **22**(1947), 107-111.