Cutting Planes for Mixed Integer Programming

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Abstract

The purpose of this paper is to present an overview of families of cutting planes for mixed integer programming problems. We examine the families of disjunctive inequalities, split cuts, mixed integer rounding inequalities, mixed integer Gomory cuts, intersection cuts, lift-and-project cuts, and reduce-and-split cuts. In practice, mixed integer Gomory cuts are very useful in obtaining solutions to mixed integer programming problems. Hence, we also examine how to use intersection cuts, lift-and-project cuts, and reduce-and-split cuts to obtain cuts which are stronger than a mixed integer Gomory cut.

Contents

1 Introduction

Mixed integer programming is a generalization of integer programming wherein only some variables are constrained to take integer values; for example, a mixed integer programming minimization problem can be denoted as $\min\{cx : Ax \leq b,\, x \geq 0\,, x_j \in \mathbb{Z} \text{ for } j \in \{1, \ldots, p\}\}$ where $A \in \mathbb{R}^{m\times(p+n)}$, $b \in \mathbb{R}^m$. As mixed integer programming is a generalization of pure integer programming, it remains NP-hard to find optimal solutions to such problems; however, there are a significant number of real-world problems that one can model as a mixed integer programming problem. As a result, there is a practical importance to having the capability to solve mixed integer programming problems of a relatively large size.

A general solution technique in integer programming is to use the linear programming relaxation along with various types of cutting planes to assist the solver in finding an optimal integer solution (a cutting plane is a linear inequality that all integer solutions of the linear programming relaxation satisfy). Mixed integer linear programming follows this same idea, using linear programming and cutting planes to arrive at solutions which have integral components where specified. An additional intrigue, however, of mixed integer programming is the fact that we cannot form Gomory cutting planes as in pure integer programming problems (this is due to the fact that the argument used to derive Gomory cuts does not apply in the mixed integer case). We thus need to develop families of cutting planes which will either generalize in this context or apply solely to this situation.

This paper will present several families of cutting planes that could be used for mixed integer programming problems. We assume that the reader has a good background in linear programming as well as some knowledge of integer programming. The goal is to provide the reader with a short introduction to many of the families of cutting planes used to solve mixed integer programming problems in current codes.

We first introduce two families of cutting planes (disjunctive inequalities and split cuts) which generalize from integer programming to the mixed integer case as well as one family of cuts formed by a technique called mixed integer rounding. Next, we introduce the family of cuts Gomory proposed for mixed integer problems to be used in place of the pure integer cutting planes. A particular focus will be made on mixed integer Gomory cuts since they have a high level of practical importance (they are among the most generally useful cuts available for use in current mixed integer solver codes). This paper will demonstrate links between the other families of cutting planes and the mixed integer Gomory cuts.

The remainder of this essay will introduce several additional families of cuts with the aim of finding a family of cuts that are stronger than mixed integer Gomory cuts, as such a family will potentially be of practical importance in improving the quality of mixed integer programming codes. The paper proceeds as follows:

- The remainder of Section 1 introduces the notational conventions used throughout this paper.
- Section 2 introduces three types of mixed integer cutting planes and demonstrates the equivalence between these families.
- Section 3 introduces the mixed integer Gomory cut (a cut with links to the pure integer Gomory cuts) and intersection cuts. This section will also establish the mixed integer Gomory cut as a member of several of the families of cutting planes.
- Section 4 introduces a family of cutting planes called lift-and-project cuts. The section also covers the links between these cuts and mixed integer Gomory cuts.
- Section 5 presents techniques for improving mixed integer Gomory cuts.

1.1 Notational conventions

Consider a mixed integer problem with p integer-constrained and n non-integer constrained variables. We denote the set of feasible points to this problem as $\{x \in \mathbb{R}^{p+n} \mid Ax \leq b,\, x \geq 0,\, x_j \in \mathbb{Z} \text{ for } j \in I\}$ where $A \in \mathbb{R}^{m\times(p+n)}$, $b \in \mathbb{R}^m$, and I is the set of indices of the integer-constrained variables (without loss of generality, we consider this to be $\{1, \ldots, p\}$). We denote this set of feasible points by P_I and let $P = \{x \in \mathbb{R}^{p+n} : Ax \leq b,\, x \geq 0\}$ be the feasible set of the LP relaxation obtained from relaxing the integral constraints on the variables x_j for $j \in I$. We denote by J the set of indices of the remaining, non-negative real variables.

Where necessary, we denote an optimal solution to the LP relaxation of this problem (namely $\min\{cx : x \in P\}$ for some cost function $c^T \in \mathbb{R}^{p+n}$) by $\bar{x}$. Without loss of generality, we will assume that $\bar{x}$ is a basic solution and denote by B and N the set of indices of variables that are basic and nonbasic, respectively, in $\bar{x}$. Let $B_I = B \cap I$, $B_J = B \cap J$, $N_I = N \cap I$, and $N_J = N \cap J$ denote the integer-constrained and continuous basic and nonbasic variables, respectively.

We denote a row k of the simplex tableau as follows:

$$x_k + \sum_{i \in N_I} \bar{a}_{kj} x_j + \sum_{j \in N_J} \bar{a}_{kj} x_j = \bar{b}_k \ .$$

In many cases, we will be interested in rows of the simplex tableau for the LP relaxation of (MIP) having $k \in I$ but $\bar{b}_k \notin \mathbb{Z}$ (since then $\bar{x}$ is not a feasible solution to the mixed integer problem and we will then want to form cutting planes that will cut off this point). Under these circumstances, we define $f = \bar{b}_k - \lfloor \bar{b}_k \rfloor$, $f_j = \bar{a}_{kj} - \lfloor \bar{a}_{kj} \rfloor$, and $I' = I \backslash \{k\}$.

1.2 Basic concepts of valid inequalities

Proposition 1.1. *If $x \geq 0$ and both $\sum_{j=1}^{n} c_j^1 x_j \leq c_0^1$ and $\sum_{j=1}^{n} c_j^2 x_j \leq c_0^2$ are valid inequalities for S_1 and S_2, respectively, then $\sum_{j=1}^{n} \min\{c_j^1, c_j^2\} x_j \leq \max\{c_0^1, c_0^2\}$ is a valid inequality for $S_1 \cup S_2$.*

Proof. Let $x \in S_1 \cup S_2$. Then, $\sum_{j=1}^{n} \min\{c_j^1, c_j^2\} x_j \leq c_0^1$ and $\sum_{j=1}^{n} \min\{c_j^1, c_j^2\} x_j \leq c_0^2$. Thus, $\sum_{j=1}^{n} \min\{c_j^1, c_j^2\} x_j \leq \max\{c_0^1, c_0^2\}$. □

2 Disjunctive inequalities, split cuts, and mixed integer rounding

In the study of valid inequalities, we see that there are two ideas reused many times among the various families of cutting planes. The first technique, rounding, is the basis of the Gomory cutting plane algorithm for pure integer programming problems. The idea is that given an inequality of the form $cx \leq c_0$ where c is an integral vector and all of the variables of x are integer-constrained, we must have that $cx \leq \lfloor c_0 \rfloor$ is satisfied by every integral x.

The second technique, disjunction, relies on a disjunction of the form $\pi^1 x \leq \pi_0^1 \vee \pi^2 x \geq \pi_0^2$ where points must satisfy one of the two conditions $\pi^1 x \leq \pi_0^1$ or $\pi^2 x \geq \pi_0^2$. A special type of disjunction called a split disjunction is of particular interest. Consider P as above and let $(\pi, \pi_0) \in \mathbb{Z}^{p+n+1}$ with $\pi_j = 0$ for $j \in J$. Then all points in P_I, that is, all points $x \in P$ such that $x_j \in \mathbb{Z}$, $j \in \{1, \ldots, p\}$, satisfy either $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$. A simple example of a split disjunction is as follows: Suppose that $\bar{x}$ is a solution for a linear programming relaxation with feasible region P, $\bar{x}_k \notin \mathbb{Z}$ for some $k \in I$, and let π be such that $\pi_j = 1$ if $j = k$ and 0 otherwise, and let $\pi_0 = \lfloor \bar{x}_i \rfloor$. For any solution to a mixed integer programming problem P_I, we must have either $x_k \leq \pi_0$ or $x_k \geq \pi_0 + 1$.

These two techniques, rounding and disjunction, play vital roles in the validity and construction of many families of cuts. As we shall see, we often need both techniques to derive valid cutting planes for mixed integer programming problems.

2.1 Disjunctive inequalities and split cuts

Definition 2.1.

- We denote by $D(\pi, \pi_0^1, \pi_0^2)$ a disjunction of the form $\pi x \leq \pi_0^1 \vee \pi x \geq \pi_0^2$. Furthermore, let F_D be the set of points x satisfying $D(\pi, \pi_0^1, \pi_0^2)$, where $F_D = \{x \mid \pi x \leq \pi_0^1 \text{ or } \pi x \geq \pi_0^2\}$.

- We call a disjunction a *split disjunction* if $(\pi, \pi_0) \in \mathbb{Z}^{p+n+1}$, $\pi_j = 0$ for $j \in J$, and $\pi_0^2 = \pi_0^1 + 1$. We denote such a disjunction by $D(\pi, \pi_0)$. If the disjunction is not a split disjunction, we call it a *general disjunction*.
- The inequality $cx \le c_0$ is a *disjunctive inequality* (or *disjunctive cut*) if there exist α, $\beta \in \mathbb{R}^+$ and $(\pi, \pi_0) \in \Pi$ such that

$$\begin{aligned} cx - \alpha(\pi x - \pi_0) &\le c_0 \\ cx + \beta(\pi x - (\pi_0 + 1)) &\le c_0 \end{aligned}$$

  are valid inequalities for P.
- An inequality $cx \le c_0$ is a *split cut* with respect to I if there exists $(\pi, \pi_0) \in \mathbb{Z}^{p+n+1}$ such that $cx \le c_0$ is valid for both $\{x \in P \mid \pi x \le \pi_0\}$ and $\{x \in P \mid \pi x \ge \pi_0 + 1\}$ (i.e. $cx \le c_0$ is valid for $P \cap F_{D(\pi,\pi_0)}$).

Remark 2.2.

- We note that some general disjunctions are valid for P_I whereas others are not. By contrast, every split disjunction is valid for P_I.
- If $cx \le c_0$ is a disjunctive inequality then we see that it is valid for P_I: let $x \in P_I$.
  - If $\pi x \le \pi_0$, then $-\alpha(\pi x - \pi_0) \ge 0$. So $cx \le c_0$.
  - If $\pi x \ge \pi_0 + 1$, then $\beta(\pi x - (\pi_0 + 1)) \ge 0$. Therefore $cx \le c_0$.
- If $cx \le c_0$ is a split cut, the we see that it is valid for P_I since $\{x \in P \mid \pi_0 < \pi x < \pi_0 + 1\}$ contains no integral points.

We now show a result which indicates that using only a relatively small set of disjunctions allows us to generate all valid disjunctive inequalities.

Proposition 2.3. *Let P be bounded and suppose $cx \le c_0$ is a valid inequality for $(P \cap \{x \mid x_k \le \pi_0\}) \cup (P \cap \{x \mid x_k \ge \pi_0 + 1\})$. Then there exist $\alpha \ge 0$ and $\beta \ge 0$ such that*

$$cx - \alpha(x_k - \pi_0) \le c_0 \tag{1}$$

$$cx + \beta(x_k - (\pi_0 + 1)) \le c_0 \tag{2}$$

are valid inequalities for P.

Before beginning the proof of this proposition, we need an auxiliary lemma:

Lemma 2.4. *Let $cx \le c_0$ be a valid inequality for $P = \{x \mid Ax \le b,\ x \ge 0\}$. If $A = \begin{pmatrix} A' \\ I \end{pmatrix}$ for some matrix A', then there exists $u \ge 0$ such that $uA \ge c$ and $ub \le c_0$.*

Proof. Suppose that $A = \begin{pmatrix} A' \\ I \end{pmatrix}$ where A' is an $m \times n$ matrix and I is an $n \times n$ matrix.

- Case 1: $P \ne \emptyset$. Then $\max\{cx \mid x \in P\}$ has a feasible solution bounded above by c_0; hence the dual of $\max\{cx \mid x \in P\}$ is also feasible, so there exists a real-valued, non-negative $1 \times m$ matrix u such that $uA \ge c$ and $ub < c_0$.
- Case 2: $P = \emptyset$. Consider the dual problem to $\max\{cx \mid x \in P\}$, namely $\min\{ub \mid uA \ge c,\ u \ge 0\}$, where u is a real-valued, non-negative $1 \times (m + n)$ matrix. Consider a particular u where

$$u_j = \begin{cases} 0 & \text{if } j \in \{1, \ldots, m\} \\ c_i & \text{if } j = m + i,\ i \in \{1, \ldots, n\}. \end{cases}$$

  We see that $uA = c$ so u is a feasible solution to the dual, so by the duality theorem $\min\{ub \mid uA \ge c\}$ is unbounded. Hence, there exists $u \ge 0$ such that $uA \ge c$ and $ub \le c_0$.

In both cases, we see that there exists $u \geq 0$ such that $uA \geq c$ and $ub \leq c_0$. $\square$

Proof of Proposition 2.3. Suppose that $cx \leq c_0$ is valid for $(P \cap \{x \mid x_k \leq \pi_0\}) \cup (P \cap \{x \mid x_k \geq \pi_0 + 1\})$. Then $cx \leq c_0$ is valid for $P \cap \{x \mid x_k \leq \pi_0\}$ and $P \cap \{x \mid x_k \geq \pi_0 + 1\}$. Since P is bounded, either P is of the form necessary for Lemma 2.4 or there exists $K \in \mathbb{R}^{p+n}$ such that $x \leq K$ and we can construct an equivalent constraint matrix for P of the correct form. Hence, without loss of generality we can conclude by Lemma 2.4 there exists a real-valued, non-negative $1 \times m$ matrix u_1 such that $u_1 A + \alpha e_k \geq c$, $u_1 b + \alpha \pi_0 \leq c_0$ and there exists a real-valued, non-negative $1 \times m$ matrix u_2 such that $u_2 A - \beta e_k \geq c$, $u_2 b - \beta(\pi_0 + 1) \leq c_0$. Now, let $x \in P$ and we see:

$$\begin{aligned} cx &\leq (u_1 A + \alpha e_k)x \\ &= u_1 Ax + \alpha e_k x \\ &\leq u_1 b + \alpha x_k \\ &\leq c_0 - \alpha \pi_0 + \alpha x_k. \end{aligned}$$

So, $cx - \alpha(x_k - \pi_0) \leq c_0$ is valid for P. Similarly,

$$\begin{aligned} cx &\leq (u_2 A - \beta e_k)x \\ &= u_2 Ax - \beta e_k x \\ &\leq u_2 b - \beta x_k \\ &\leq c_0 + \beta(\pi_0 + 1) - \beta x_k. \end{aligned}$$

So, $cx + \beta(x_k - (\pi_0 + 1)) \leq c_0$ is valid for P. $\square$

Proposition 2.5. *Let $P_1(\pi, \pi_0) = P \cap \{x \mid \pi x \leq \pi_0\}$ and $P_2(\pi, \pi_0) = P \cap \{x \mid \pi x \geq \pi_0 + 1\}$. Then an inequality $cx \leq c_0$ is valid for $P_1(\pi, \pi_0) \cup P_2(\pi, \pi_0)$ if and only if it is valid for $\operatorname{conv}(P_1(\pi, \pi_0) \cup P_2(\pi, \pi_0))$.*

Proof. "$\Rightarrow$" Suppose $x \in \operatorname{conv}(P_1(\pi, \pi_0) \cup P_2(\pi, \pi_0))$, then $x = \lambda x^1 + (1 - \lambda)x^2$ for some $\lambda \in [0, 1]$, $x^1 \in P_1$, and $x^2 \in P_2$. Since $cx^1 \leq c_0$ and $cx^2 \leq c_0$, we thus have $cx \leq c_0$.
"$\Leftarrow$" Let $x \in P_1(\pi, \pi_0) \cup P_2(\pi, \pi_0)$ Then either $x \in P_1(\pi, \pi_0)$ or $x \in P_2(\pi, \pi_0)$. Since $cx \leq c_0$ is valid for $\operatorname{conv}(P_1(\pi, \pi_0) \cup P_2(\pi, \pi_0))$, we thus have that it is certainly valid in the above cases. $\square$

2.2 Mixed integer rounding inequalities

The concept of mixed integer rounding was motivated by the work of Gomory [10]. In this section, we define the mixed integer rounding cuts and verify the validity of these inequalities.

Definition 2.6 (Nemhauser and Wolsey, [13])**.** Let $c^1, c^2 \in \mathbb{R}^{p+n}$ be such that $c_j := c_j^1 = c_j^2$ for all $j \in J$, let c_0^1, $c_0^2 \in \mathbb{R}$ and suppose that $c^1 x \leq c_0^1$ and $c^2 x \leq c_0^2$ are valid for P. Let $f = (c_0^2 - c_0^1) - \lfloor c_0^2 - c_0^1 \rfloor$, and define the *mixed integer rounding inequality* as

$$\sum_{j \in I} \lfloor c_j^2 - c_j^1 \rfloor x_j + \tfrac{1}{1-f} \left(\sum_{j \in I} c_j^1 x_j + \sum_{j \in J} c_j x_j - c_0^1 \right) \leq \lfloor c_0^2 - c_0^1 \rfloor.$$

Remark 2.7. There is another definition (due to Wolsey, [15]) that is more compact but less general than the above. If $cx \leq c_0$ is valid for P, then the following is valid for P_I:

$$\sum_{\substack{j \in I \\ f_j \leq f}} \lfloor c_j \rfloor x_j + \sum_{\substack{j \in I \\ f_j > f}} (\lfloor c_j \rfloor + \tfrac{f_j - f}{1-f}) x_j + \tfrac{1}{1-f} \sum_{\substack{j \in J \\ c_j < 0}} c_j x_j \leq \lfloor c_0 \rfloor.$$

We will see a proof of the validity of the inequality in this form in Proposition 2.12 which will demonstrate why it is less general than Definition 2.6. Articles in the literature refer to mixed integer rounding as one of the above inequalities, though not always the exact form the author had in mind. Care must be taken to use the correct definition for the application at hand.

We begin the process of demonstrating the validity of the mixed integer rounding inequality with a preliminary lemma:

Lemma 2.8 (Nemhauser and Wolsey, [12]). *Given an inequality $cx \leq c_0$ that is valid for P, then the following inequality is valid for P_I:*

$$\sum_{j \in I} \lfloor c_j \rfloor x_j + \frac{1}{1-f} \sum_{\substack{j \in J \\ c_j < 0}} c_j x_j \leq \lfloor c_0 \rfloor.$$

Proof. Let $x \in P_I$ and recall that $f = c_0 - \lfloor c_0 \rfloor$.

- Case 1: $\sum_{j \in J} c_j x_j > f - 1$. Since $x_j \geq 0$, we see that

$$\begin{aligned}
\sum_{j \in I} \lfloor c_j \rfloor x_j &\leq \sum_{j \in I} c_j x_j \\
&\leq c_0 - \sum_{j \in J} c_j x_j, \quad \text{since } cx \leq c_0 \text{ is valid for } P \\
&< c_0 - (f-1) \\
&= c_0 - (c_0 - \lfloor c_0 \rfloor - 1) \\
&= \lfloor c_0 \rfloor + 1.
\end{aligned}$$

  So, by the integrality of the left-hand side of the inequality, $\sum_{j \in I} \lfloor c_j \rfloor x_j \leq \lfloor c_0 \rfloor$. Moreover,

$$\frac{1}{1-f} \sum_{\substack{j \in J \\ c_j < 0}} c_j x_j \leq 0$$

  so we have

$$\sum_{j \in I} \lfloor c_j \rfloor x_j + \frac{1}{1-f} \sum_{\substack{j \in J \\ c_j < 0}} c_j x_j \leq \lfloor c_0 \rfloor.$$

- Case 2: $\sum_{j \in J} c_j x_j \leq f - 1$. Since $x_j \geq 0$ we see that

$$\begin{aligned}
\sum_{j \in I} \lfloor c_j \rfloor x_j + \tfrac{1}{1-f} \Bigg(\sum_{\substack{j \in J \\ c_j < 0}} c_j x_j \Bigg) &\leq \sum_{j \in I} c_j x_j + \tfrac{1}{1-f} \Bigg(\sum_{\substack{j \in J \\ c_j < 0}} c_j x_j \Bigg) \\
&\leq c_0 - \sum_{j \in J} c_j x_j + \tfrac{1}{1-f} \Bigg(\sum_{\substack{j \in J \\ c_j < 0}} c_j x_j \Bigg) \\
&= c_0 - \sum_{\substack{j \in J \\ c_j \geq 0}} c_j x_j - \sum_{\substack{j \in J \\ c_j < 0}} c_j x_j + \tfrac{1}{1-f} \Bigg(\sum_{\substack{j \in J \\ c_j < 0}} c_j x_j \Bigg) \\
&\leq c_0 - \sum_{\substack{j \in J \\ c_j < 0}} c_j x_j + \tfrac{1}{1-f} \Bigg(\sum_{\substack{j \in J \\ c_j < 0}} c_j x_j \Bigg) \\
&\leq c_0 + (\tfrac{1}{1-f} - 1) \Bigg(\sum_{\substack{j \in J \\ c_j < 0}} c_j x_j \Bigg) \\
&\leq c_0 + \tfrac{f}{1-f}(f-1), \quad \text{since } \sum_{\substack{j \in J \\ c_j < 0}} c_j x_j \leq \sum_{j \in J} c_j x_j \leq f - 1 \\
&\leq c_0 - f \\
&= \lfloor c_0 \rfloor.
\end{aligned}$$

So, the inequality is valid for P_I. $\square$

Theorem 2.9 (Nemhauser and Wolsey, [12])**.** *Given two inequalities $c^1x \leq c_0^1$ and $c^2x \leq c_0^2$ that are valid for P, define $f = (c_0^2 - c_0^1) - \lfloor c_0^2 - c_0^1 \rfloor$. Then*

$$\sum_{j\in I} \lfloor c_j^2 - c_j^1 \rfloor x_j + \tfrac{1}{1-f}\left(\sum_{j\in I} c_j^1 x_j + \sum_{j\in J} \min\{c_j^1, c_j^2\} x_j - c_0^1\right) \leq \lfloor c_0^2 - c_0^1 \rfloor$$

is a valid inequality for P_I.

Proof. Let $x \in P_I$. Since $x_j \geq 0$ for all $j \in J$, $c^1x \leq c_0^1$, and $c^2x \leq c_0^2$, we see that $\sum_{i\in I} c_j^1 x_j + \sum_{j\in J} \min\{c_j^1, c_j^2\} x_j \leq c_0^1$ and $\sum_{i\in I} c_j^2 x_j + \sum_{j\in J} \min\{c_j^1, c_j^2\} x_j \leq c_0^2$. So, we have:

$$\sum_{j\in I} c_j^2 x_j + \sum_{j\in J} \min\{c_j^1, c_j^2\} x_j - \sum_{j\in I} c_j^1 x_j + \sum_{j\in I} c_j^1 x_j - c_0^1 \leq c_0^2 - c_0^1$$
$$\sum_{j\in I} (c_j^2 - c_j^1) x_j - (c_0^1 - \sum_{j\in I} c_j^1 x_j - \sum_{j\in J} \min\{c_j^1, c_j^2\} x_j) \leq c_0^2 - c_0^1.$$

Let $s = c_0^1 - \sum_{j\in I} c_j^1 x_j - \sum_{j\in J} \min\{c_j^1, c_j^2\} x_j$. Since $\sum_{j\in I} c_j^1 x_j + \sum_{j\in J} \min\{c_j^1, c_j^2\} x_j \leq c_0^1$, we see that $s \geq 0$ and so $-s \leq 0$. Consider $\sum_{j\in I} (c_j^2 - c_j^1) x_j + (-s) \leq c_0^2 - c_0^1$ and apply Lemma 2.8 to obtain $\sum_{j\in I} \lfloor c_j^2 - c_j^1 \rfloor x_j - \frac{1}{1-f} s \leq \lfloor c_0^2 - c_0^1 \rfloor$. Replacing s, we get the desired result. $\square$

Corollary 2.10. *The mixed integer rounding inequality is a valid inequality for P_I.*

Proof. Take $c_j^1 = c_j^2$ for each $j \in J$ in Theorem 2.9 and we obtain this result. $\square$

Notation 2.11. For x, $y \in \mathbb{R}$, define $(x-y)^+ = \begin{cases} 0 & \text{if } x \leq y \\ x-y & \textit{otherwise} \end{cases}$.

Proposition 2.12. *If $cx \leq c_0$ is valid for P, then the inequality*

$$\sum_{i\in I} (\lfloor c_j \rfloor + \tfrac{(f_j - f)^+}{1-f}) x_j + \tfrac{1}{1-f} \sum_{\substack{j\in J \\ c_j < 0}} c_j x_j \leq \lfloor c_0 \rfloor$$

is valid for P_I.

Proof. We note that

$$0 \leq 0 \tag{1}$$

is valid for P. We have the following:

$$\sum_{\substack{j\in I \\ f_j \leq f}} c_j x_j + \sum_{\substack{i\in I \\ f_j > f}} c_j x_j + \sum_{j\in J} c_j x_j \leq c_0$$
$$\sum_{\substack{j\in I \\ f_j \leq f}} c_j x_j + \sum_{\substack{j\in I \\ f_j > f}} (\lceil c_j \rceil - (1 - f_j)) x_j + \sum_{j\in J} c_j x_j \leq c_0$$
$$\sum_{\substack{j\in I \\ f_j \leq f}} c_j x_j + \sum_{\substack{j\in I \\ f_j > f}} \lceil c_j \rceil x_j + \sum_{j\in J} c_j x_j - \sum_{\substack{j\in I \\ f_j > f}} (1 - f_j) x_j \leq c_0.$$

We note that $s = \sum_{\substack{j\in I \\ f_j > f}} (1 - f_j) x_j > 0$ so we have

$$\sum_{\substack{j\in I \\ f_j \leq f}} c_j x_j + \sum_{\substack{j\in I \\ f_j > f}} \lceil c_j \rceil x_j + \sum_{j\in J} c_j x_j - s \leq c_0. \tag{2}$$

We apply Theorem 2.9 to (1) and (2):

$$\sum_{\substack{j\in I\\ f_j\le f}} \lfloor c_j\rfloor x_j + \sum_{\substack{j\in I\\ f_j> f}} \lfloor\lceil c_j\rceil\rfloor x_j + \tfrac{1}{1-f}(\sum_{\substack{j\in J\\ c_j<0}} c_j x_j - s) \le \lfloor c_0\rfloor$$

$$\sum_{\substack{j\in I\\ f_j\le f}} \lfloor c_j\rfloor x_j + \sum_{\substack{j\in I\\ f_j> f}} \lceil c_j\rceil x_j + \tfrac{1}{1-f}(\sum_{\substack{j\in J\\ c_j<0}} c_j x_j - \sum_{\substack{j\in I\\ f_j> f}} (1-f_j)x_j) \le \lfloor c_0\rfloor$$

$$\sum_{\substack{j\in I\\ f_j\le f}} (\lfloor c_j\rfloor + \tfrac{(f_j-f)^+}{1-f})x_j + \sum_{\substack{j\in I\\ f_j> f}} (\lceil c_j\rceil - \tfrac{1-f_j}{1-f})x_j + \tfrac{1}{1-f}\sum_{\substack{j\in J\\ c_j<0}} c_j x_j \le \lfloor c_0\rfloor$$

$$\sum_{\substack{j\in I\\ f_j\le f}} (\lfloor c_j\rfloor + \tfrac{(f_j-f)^+}{1-f})x_j + \sum_{\substack{j\in I\\ f_j> f}} (\lfloor c_j\rfloor + 1 - \tfrac{1-f_j}{1-f})x_j + \tfrac{1}{1-f}\sum_{\substack{j\in J\\ c_j<0}} c_j x_j \le \lfloor c_0\rfloor$$

$$\sum_{\substack{j\in I\\ f_j\le f}} (\lfloor c_j\rfloor + \tfrac{(f_j-f)^+}{1-f})x_j + \sum_{\substack{j\in I\\ f_j> f}} (\lfloor c_j\rfloor + \tfrac{f_j-f}{1-f})x_j + \tfrac{1}{1-f}\sum_{\substack{j\in J\\ c_j<0}} c_j x_j \le \lfloor c_0\rfloor$$

$$\sum_{j\in I} (\lfloor c_j\rfloor + \tfrac{(f_j-f)^+}{1-f})x_j + \tfrac{1}{1-f}\sum_{\substack{j\in J\\ c_j<0}} c_j x_j \le \lfloor c_0\rfloor.$$

This is thus valid for P_I. □

2.3 The equivalence of disjunctive, split, and mixed integer rounding inequalities

Theorem 2.13 (Nemhauser and Wolsey, [13])**.** *The families of disjunctive inequalities, split cuts, and mixed integer rounding inequalities are the same.*

Proof.

1. We show that a split cut is a disjunctive inequality. Let $D(\pi, \pi_0)$ be a split disjunction and $cx \le c_0$ be a split cut for this disjunction. Now the inequality $cx \le c_0$ is valid for $P \cap \{x \mid \pi x \le \pi_0\}$ if and only if there exists $\alpha \ge 0$ such that $cx - \alpha(\pi x - \pi_0) \le c_0$ is valid for P. We can see this using LP duality:

$$\begin{aligned} & cx \le c_0 \text{ is valid for } \{x \mid Ax \le b,\ x \ge 0,\ \pi x \le \pi_0\} \\ \Leftrightarrow\quad & \max\{cx \mid Ax \le b,\ \pi x \le \pi_0,\ x \ge 0\} \le c_0 \\ \Leftrightarrow\quad & \exists y \ge 0,\ \alpha \ge 0 \text{ such that } yA + \alpha\pi \ge c \text{ and } yb + \alpha\pi_0 \le c_0. \end{aligned}$$

   Then for every $x \in P$,

$$\begin{aligned} (c - \alpha\pi)x &\le yAx \\ &\le yb \\ &\le c_0 - \alpha\pi_0. \end{aligned}$$

   Hence, for all $x \in P$, $cx - \alpha(\pi x - \pi_0) \le c_0$.

   Likewise, $cx \le c_0$ is valid for $P \cap \{x \mid \pi x \ge \pi_0 + 1\}$ if and only if there exists $\beta \ge 0$ such that $cx + \beta(\pi x - \pi_0 - 1) \le c_0$ is valid for P, which we can again see using LP duality:

$$\begin{aligned} & cx \le c_0 \text{ is valid for } \{x \mid Ax \le b,\ x \ge 0,\ \pi x \ge \pi_0 + 1\} \\ \Leftrightarrow\quad & \max\{cx \mid Ax \le b,\ \pi x \ge \pi_0 + 1,\ x \ge 0\} \le c_0 \\ \Leftrightarrow\quad & \exists y \ge 0,\ \beta \ge 0 \text{ such that } yA - \beta\pi \ge c \text{ and } yb - \beta(\pi_0 + 1) \le c_0. \end{aligned}$$

Then for every $x \in P$,

$$\begin{aligned}(c+\beta\pi)x &\leq yAx \\ &\leq yb \\ &\leq c_0 + \beta(\pi_0+1).\end{aligned}$$

Hence, for all $x \in P$, $cx + \beta(\pi x - (\pi_0 + 1)) \leq c_0$.

So by Definition 2.1 we have that $cx \leq c_0$ is a disjunctive inequality.

2. If $cx \leq c_0$ is a disjunctive inequality, there exist $\alpha \geq 0$ and $\beta \geq 0$ such that $cx - \alpha(\pi x - \pi_0) \leq c_0$ and $cx + \beta(\pi x - (\pi_0 + 1)) \leq c_0$ for all $x \in P$. We can use the above argument in the other direction to conclude that $cx \leq c_0$ is a split cut. Hence, disjunctive inequalities and split cuts are equivalent.

3. Next, we will show that the mixed integer rounding inequality is a disjunctive inequality. Let $x \in P$ and suppose that

$$c_I^1 x_I + c_J x_J \leq c_0^1 \tag{1}$$
$$c_I^2 x_I + c_J x_J \leq c_0^2. \tag{2}$$

Let $\pi_I = c_I^2 - c_I^1$, $\pi_J = 0$, $\pi_0 = \lfloor c_0^2 - c_0^1 \rfloor$, and $\gamma = c_0^2 - c_0^1 - \pi_0$. Note that $\pi x = \pi_I x_I$.

Take $\frac{1}{1-\gamma} \cdot (1)$ and let $\alpha = 1$. We notice that $\frac{1}{1-\gamma} \geq 0$ and we obtain:

$$\begin{aligned} c_I^1 x_I + c_J x_J - c_0^1 &\leq 0 \\ \tfrac{1}{1-\gamma}(c_I^1 x_I + c_J x_J - c_0^1) &\leq 0 \\ \pi x + \tfrac{1}{1-\gamma}(c_I^1 x_I + c_J x_J - c_0^1) - (\pi x - \pi_0) &\leq \pi_0 \\ \pi x + \tfrac{1}{1-\gamma}(c_I^1 x_I + c_J x_J - c_0^1) - \alpha(\pi x - \pi_0) &\leq \pi_0 \end{aligned} \tag{1*}$$

Similarly, we take $\frac{1}{1-\gamma} \cdot (2)$ and $\beta = \frac{\gamma}{1-\gamma}$ to obtain:

$$\begin{aligned} c_I^2 x_I + c_J x_J - c_0^2 &\leq 0 \\ c_I^2 x_I + c_J x_J - (\gamma + c_0^1 + \pi_0) &\leq 0 \\ (c_I^2 - c_I^1)x_I + (c_I^1 x_I + c_J x_J - c_0^1) - \gamma &\leq \pi_0 \\ \pi_I x_I + \tfrac{1}{1-\gamma}(c_I^1 x_I + c_J x_J - c_0^1) + \tfrac{1-\gamma-1}{1-\gamma}(c_I^1 x_I + c_J x_J - c_0^1) - \gamma &\leq \pi_0 \\ \pi x + \tfrac{1}{1-\gamma}(c_I^1 x_I + c_J x_J - c_0^1) + \tfrac{\gamma}{1-\gamma}(-c_I^1 x_I - c_J x_J + c_0^1 - 1 + \gamma) &\leq \pi_0 \\ \pi x + \tfrac{1}{1-\gamma}(c_I^1 x_I + c_J x_J - c_0^1) + \tfrac{\gamma}{1-\gamma}(-c_I^1 x_I - c_J x_J + (c_0^1 + \gamma) - 1) &\leq \pi_0 \\ \pi x + \tfrac{1}{1-\gamma}(c_I^1 x_I + c_J x_J - c_0^1) + \beta(-c_I^1 x_I - c_J x_J + (c_0^2 - \pi_0) - 1) &\leq \pi_0 \\ \pi x + \tfrac{1}{1-\gamma}(c_I^1 x_I + c_J x_J - c_0^1) + \beta(-c_I^1 x_I - c_J x_J + c_0^2 - \pi_0 - 1) &\leq \pi_0. \end{aligned}$$

We note that $-c_J x_J + c_0^2 \geq c_I^2 x_I$ and obtain:

$$\begin{aligned} \pi x + \tfrac{1}{1-\gamma}(c_I^1 x_I + c_J x_J - c_0^1) + \beta(-c_I^1 x_I + c_I^2 x_I - \pi_0 - 1) &\leq \pi_0 \\ \pi x + \tfrac{1}{1-\gamma}(c_I^1 x_I + c_J x_J - c_0^1) + \beta(\pi x - \pi_0 - 1) &\leq \pi_0 \end{aligned} \tag{2*}$$

Now since (1*) and (2*) are valid for P, we can apply Definition 2.1 to (1*) and (2*) to obtain the mixed integer rounding inequality, namely:

$$\pi x + \tfrac{1}{1-\gamma}(c_I^1 x_I + c_J x_J - c_0^1) \leq \pi_0.$$

4. Finally, we show that a disjunctive inequality is a mixed integer rounding inequality. Let $x \in P$ and let π be such that $\pi_j = 0$ for all $j \in J$ (hence we note that $\pi_I x_I = \pi x$). Suppose that for some $\alpha \geq 0$, $\beta \geq 0$ we have:

$$cx - \alpha(\pi x - \pi_0) \leq c_0 \tag{1}$$
$$cx + \beta(\pi x - \pi_0 - 1) \leq c_0. \tag{2}$$

Let $f = (c_0 + \beta(\pi_0+1) - c_0 + \alpha\pi) - \lfloor c_0 + \beta(\pi_0+1) - c_0 + \alpha\pi \rfloor = (\alpha\pi + \beta(\pi_0+1)) - \lfloor \alpha\pi + \beta(\pi_0+1) \rfloor$. If both $\alpha = 0$ and $\beta = 0$, apply Definition 2.6 to (1) and (2) to obtain $\frac{1}{1-f}(cx - c_0) \leq 0$, and hence $cx \leq c_0$.

If $\alpha = 0$ but $\beta \neq 0$, apply Definition 2.6 to $\frac{1}{\beta}(1)$ and $\frac{1}{\beta}(2)$, noting that $f = (\frac{\beta(\pi_0+1)}{\beta}) - \lfloor \frac{\beta(\pi_0+1)}{\beta} \rfloor = 0$:

$$\begin{aligned}
(\tfrac{c_I}{\beta} + \pi_I - \tfrac{c_I}{\beta})x_I + \tfrac{1}{1-f}\tfrac{1}{\beta}(cx - c_0) &\leq \lfloor \tfrac{1}{\beta}(c_0 + \beta(\pi_0+1) - c_0) \rfloor \\
\pi_I x_I + \tfrac{1}{1-0}\tfrac{1}{\beta}(cx - c_0) &\leq \lfloor \tfrac{\beta}{\beta}(\pi_0+1) \rfloor \\
\pi x + \tfrac{1}{\beta}(cx - c_0) &\leq \pi_0 + 1 \\
\tfrac{1}{\beta}(cx - c_0) &\leq \pi_0 + 1 - \pi x \qquad (*) \\
cx + \beta(\pi x - (\pi_0+1)) &\leq c_0.
\end{aligned}$$

So, if $\pi x \leq \pi_0 + 1$ then by line (*) we have $\frac{1}{\beta}(cx - c_0) \leq 0$, and hence $cx \leq c_0$. Otherwise, if $\pi x \geq \pi_0 + 1$, then $\beta(\pi x - (\pi_0+1)) \geq 0$ and thus $cx \leq c_0$.

If $\beta = 0$ but $\alpha \neq 0$, apply Definition 2.6 to $\frac{1}{\alpha}(1)$ and $\frac{1}{\alpha}(2)$:

$$\begin{aligned}
\tfrac{1}{\alpha}(\alpha\pi_I x_I) + \tfrac{1}{\alpha}(cx - \alpha\pi x + \alpha\pi_0 - c_0) &\leq \lfloor \pi_0 \rfloor \\
\pi_I x_I + \tfrac{1}{\alpha}(cx - c_0) - \pi x + \pi_0 &\leq \pi_0 \\
\pi x + \tfrac{1}{\alpha}(cx - c_0) - \pi x + \pi_0 &\leq \pi_0 \\
cx - c_0 &\leq 0.
\end{aligned}$$

Hence, $cx \leq c_0$.

If we are not in one of the special cases, take $\frac{1}{\alpha+\beta}\cdot(1)$ and $\frac{1}{\alpha+\beta}\cdot(2)$ and rearrange in terms of x_I and x_J.

$$\tfrac{1}{\alpha+\beta}c_I x_I + \tfrac{1}{\alpha+\beta}c_J x_J - \tfrac{\alpha}{\alpha+\beta}(\pi x - \pi_0) \leq \tfrac{1}{\alpha+\beta}c_0 \tag{1*}$$
$$\tfrac{1}{\alpha+\beta}c_I x_I + \tfrac{1}{\alpha+\beta}c_J x_J + \tfrac{\beta}{\alpha+\beta}(\pi x - \pi_0 - 1) \leq \tfrac{1}{\alpha+\beta}c_0. \tag{2*}$$

Now we compute $c_I^2 - c_I^1 = (\frac{1}{\alpha+\beta}(c_I + \beta\pi_I)) - (\frac{1}{\alpha+\beta}(c_I - \alpha\pi_I)) = \frac{\beta+\alpha}{\alpha+\beta}\pi_I = \pi_I$ and $c_0^2 - c_0^1 = \frac{1}{\alpha+\beta}(c_0 + \beta(\pi_0+1)) - \frac{1}{\alpha+\beta}(c_0 - \alpha\pi_0) = \pi_0 + \frac{\beta}{\alpha+\beta}$. Let $\gamma = \frac{\beta}{\alpha+\beta}$ and note that $0 < \gamma < 1$. Apply Definition 2.6 to (1*) and (2*) to obtain:

$$\begin{aligned}
\pi_I x_I + \tfrac{1}{1-\gamma}\left(\tfrac{1}{\alpha+\beta}\left((c_I - \alpha\pi_I)x_I + c_J x_J - c_0 + \alpha\pi_0\right)\right) &\leq \pi_0 \\
\pi x + \frac{1}{1-\frac{\beta}{\alpha+\beta}}\tfrac{1}{\alpha+\beta}\left((c_I - \alpha\pi_I)x_I + c_J x_J - c_0 + \alpha\pi_0)\right) &\leq \pi_0 \\
\pi x + \tfrac{1}{\alpha}(c_I x_I + c_J x_J - c_0) - \pi x + \pi_0 &\leq \pi_0 \\
\tfrac{1}{\alpha}(c_I x_I + c_J x_J - c_0) &\leq 0 \\
cx &\leq c_0.
\end{aligned}$$

Hence, we have that $cx \leq c_0$.

□

2.4 Are all mixed integer cuts disjunctive/split/mixed integer rounding cuts?

In the special case where $0 \le x_j \le 1$ for all $j \in I \cup J$, we have the following result:

Theorem 2.14 (Nemhauser and Wolsey, [13])**.** *Let $P = \{x \in \mathbb{R}^{p+n} \mid Ax \le b,\, 0 \le x \le 1\}$ and let $T = \{x \in P \mid x_j \in \{0,1\}$ for all $j \in I\}$. Then all valid inequalities for T are disjunctive inequalities for P.*

Using Theorem 2.13, we get the following corollary:

Corollary 2.15. *All valid inequalities for $T = \{x \in \mathbb{R}^{p+n} \mid Ax \le b,\, 0 \le x \le 1,\, x_j \in \{0,1\}$ for all $j \in I\}$ are disjunctive inequalities, mixed integer rounding inequalities, and split cuts.*

We naturally wonder whether or not the same idea holds in the general integer case; however, there is an example due to Schrijver (see [12] for more information) which establishes that there are valid inequalities to a general mixed integer problem which do not belong to any of the above families.

3 Mixed integer Gomory cuts and intersection cuts

As mentioned in the introduction, the Gomory pure integer cutting plane method does not convert into a method for mixed integer programming (the technique used to create that family of cutting planes is not valid in the mixed integer case). Realizing this, Gomory introduced a new family of cuts for the mixed integer problem in [10]; we call these cuts mixed integer Gomory cuts. These cuts rely on the use of a disjunction and rounding to cut off an extreme point of the polyhedron of the LP relaxation of a mixed integer problem that is fractional for integer-constrained variables. There are several ways to derive this cut due to the fact that this cut belongs to a number of families of cutting planes (as we shall demonstrate throughout the remainder of this paper). We begin our discussion of the mixed integer Gomory cut with an elementary derivation and definition of the cut.

3.1 Derivation of the mixed integer Gomory cut

Theorem 3.1. *Let $I' = I \backslash \{k\}$ and suppose we have a row from an optimal simplex tableau of the form $x_k + \sum_{i \in I'} \bar{a}_{kj} x_j + \sum_{j \in J} \bar{a}_{kj} x_j = \bar{b}_k$ where $k \in I$ and $\bar{b}_k \notin \mathbb{Z}$. Define $f_j = \bar{a}_{kj} - \lfloor \bar{a}_{kj} \rfloor$ and $f = \bar{b}_k - \lfloor \bar{b}_k \rfloor$. Then the following inequality is valid for P_I:*

$$\frac{1}{f} \sum_{\substack{j \in I' \\ f_j \le f}} f_j x_j + \frac{1}{1-f} \sum_{\substack{j \in I' \\ f_j > f}} (1 - f_j) x_j + \frac{1}{f} \sum_{\substack{j \in J \\ \bar{a}_{kj} \ge 0}} \bar{a}_{kj} x_j - \frac{1}{1-f} \sum_{\substack{j \in J \\ \bar{a}_{kj} < 0}} \bar{a}_{kj} x_j \ge 1. \tag{MIG}$$

Proof. Rewrite $x_k + \sum_{i \in I'} \bar{a}_{kj} x_j + \sum_{j \in J} \bar{a}_{kj} x_j = \bar{b}_k$ as

$$x_k + \sum_{i \in I'} \bar{a}_{kj} x_j + \sum_{\substack{j \in J \\ \bar{a}_{kj} \ge 0}} \bar{a}_{kj} x_j + \sum_{\substack{j \in J \\ \bar{a}_{kj} < 0}} \bar{a}_{kj} x_j = \bar{b}_k. \tag{1}$$

Let $x \in P_I$ and set t as follows (so that t is integral):

$$t = x_k + \sum_{\substack{i \in I' \\ f_j \le f}} \lfloor \bar{a}_{kj} \rfloor x_j + \sum_{\substack{i \in I' \\ f_j > f}} \lceil \bar{a}_{kj} \rceil x_j. \tag{2}$$

Take (1) $-$ (2) to obtain:

$$\sum_{\substack{i \in I' \\ f_j \le f}} (\bar{a}_{kj} - \lfloor \bar{a}_{kj} \rfloor) x_j + \sum_{\substack{i \in I' \\ f_j > f}} (\bar{a}_{kj} - \lceil \bar{a}_{kj} \rceil) x_j + \sum_{\substack{j \in J \\ \bar{a}_{kj} \ge 0}} \bar{a}_{kj} x_j + \sum_{\substack{j \in J \\ \bar{a}_{kj} < 0}} \bar{a}_{kj} x_j = \bar{b}_k - t$$

$$\sum_{\substack{i \in I' \\ f_j \le f}} f_j x_j + \sum_{\substack{i \in I' \\ f_j > f}} (f_j - 1) x_j + \sum_{\substack{j \in J \\ \bar{a}_{kj} \ge 0}} \bar{a}_{kj} x_j + \sum_{\substack{j \in J \\ \bar{a}_{kj} < 0}} \bar{a}_{kj} x_j = \bar{b}_k - t. \tag{*}$$

Case 1: $\lfloor \bar{b}_k \rfloor \geq t$. We note that $\sum\limits_{\substack{j \in I' \\ f_j > f}} (f_j - 1)x_j \leq 0$ and $\sum\limits_{\substack{j \in J \\ \bar{a}_{kj} < 0}} \bar{a}_{kj} x_j \leq 0$. Hence, by (*)

$$\sum_{\substack{j \in I' \\ f_j \leq f}} f_j x_j + \sum_{\substack{j \in J \\ \bar{a}_{kj} \geq 0}} \bar{a}_{kj} x_j \geq \bar{b}_k - t \geq \bar{b}_k - \lfloor \bar{b}_k \rfloor = f.$$

Since $\frac{1}{1-f} \geq 0$, we have:

$$\frac{1}{f} \sum_{\substack{j \in I' \\ f_j \leq f}} f_j x_j + \frac{1}{f} \sum_{\substack{j \in J \\ \bar{a}_{kj} \geq 0}} \bar{a}_{kj} x_j + \frac{-1}{1-f} \sum_{\substack{j \in I' \\ f_j > f}} (f_j - 1) x_j + \frac{-1}{1-f} \sum_{\substack{j \in J \\ \bar{a}_{kj} < 0}} \bar{a}_{kj} x_j \geq 1$$

$$\frac{1}{f} \sum_{\substack{j \in I' \\ f_j \leq f}} f_j x_j + \frac{1}{1-f} \sum_{\substack{j \in I' \\ f_j > f}} (1 - f_j) x_j + \frac{1}{f} \sum_{\substack{j \in J \\ \bar{a}_{kj} \geq 0}} \bar{a}_{kj} x_j - \frac{1}{1-f} \sum_{\substack{j \in J \\ \bar{a}_{kj} < 0}} \bar{a}_{kj} x_j \geq 1.$$

Case 2: $\lceil \bar{b}_k \rceil \leq t$. Then by (*)

$$\begin{aligned}
\sum_{\substack{j \in I' \\ f_j > f}} (1 - f_j) x_j - \sum_{\substack{j \in J \\ \bar{a}_{kj} < 0}} \bar{a}_{kj} x_j &= \underbrace{\sum_{\substack{j \in I' \\ f_j \leq f}} f_j x_j + \sum_{\substack{j \in J \\ \bar{a}_{kj} \geq 0}} \bar{a}_{kj} x_j}_{\geq 0} + t - \bar{b}_k \\
&\geq t - \bar{b}_k \\
&\geq \lceil \bar{b}_k \rceil - \bar{b}_k \\
&= \lfloor \bar{b}_k \rfloor + 1 - \bar{b}_k \\
&= 1 - f.
\end{aligned}$$

Since $\frac{1}{f} \sum\limits_{\substack{j \in I' \\ f_j \leq f}} f_j x_j \geq 0$ and $\frac{1}{f} \sum\limits_{\substack{j \in J \\ \bar{a}_{kj} \geq 0}} \bar{a}_{kj} x_j \geq 0$, we see that

$$\frac{1}{f} \sum_{\substack{j \in I' \\ f_j \leq f}} f_j x_j + \frac{1}{1-f} \sum_{\substack{j \in I' \\ f_j > f}} (1 - f_j) x_j + \frac{1}{f} \sum_{\substack{j \in J \\ \bar{a}_{kj} \geq 0}} \bar{a}_{kj} x_j - \frac{1}{1-f} \sum_{\substack{j \in J \\ \bar{a}_{kj} < 0}} \bar{a}_{kj} x_j \geq 1.$$

Since the inequality is valid under both terms of the disjunction $t \leq \lfloor \bar{b}_k \rfloor$ and $\lceil \bar{b}_k \rceil \leq t$, we thus have that the inequality is valid for P_I. □

Definition 3.2. The *mixed integer Gomory cut* (or *MIG cut*) is the inequality (MIG) derived as above.

3.2 Mixed integer Gomory cuts and pure integer Gomory cuts

We recall the derivation of the pure integer Gomory cuts, where $J = \emptyset$. Given a row of a simplex tableau of the form $x_k + \sum\limits_{j \in N} \bar{a}_{kj} x_j = \bar{b}_k$, we rewrite the row as

$$x_k + \sum_{j \in N} \lfloor \bar{a}_{kj} \rfloor x_j + \sum_{j \in N} f_j x_j = \lfloor \bar{b}_k \rfloor + f.$$

We rearrange the terms to obtain

$$\sum_{j \in N} f_j x_j - f = \lfloor \bar{b}_k \rfloor - x_k - \sum_{j \in N} \lfloor \bar{a}_{kj} \rfloor x_j.$$

Notice that when $x \in P_I$ the right hand side is integral, and also that $\sum\limits_{j \in N} f_j x_j \geq 0$. Since $\sum\limits_{j \in N} f_j x_j - f$ must be integral for a feasible solution x, we see that $\sum\limits_{j \in N} f_j x_j - f \geq 0$. We call the resulting valid inequality $\sum\limits_{j \in N} f_j x_j \geq f$ the *pure integer Gomory cut*.

We wish to compare this to the mixed integer Gomory cut so suppose now that $J = \emptyset$. Assuming that $\bar{b}_k \notin \mathbb{Z}$ we obtain the mixed integer Gomory cut:

$$\tfrac{1}{f} \sum_{\substack{j\in I' \\ f_j \le f}} f_j x_j + \tfrac{1}{1-f} \sum_{\substack{j\in I' \\ f_j > f}} (1-f_j)x_j \ge 1.$$

We thus only recover the Gomory cut from the mixed integer Gomory cut when $\{j \in I \mid f_j > f\} = \emptyset$. Moreover, the mixed integer Gomory cut will have a slack that is not integer constrained, whereas in the pure integer context this slack is always integer constrained. As a result, we see that the mixed integer Gomory cut and the pure integer Gomory cut are not the same in general.

3.3 Mixed integer Gomory cuts and mixed integer rounding

The mixed integer Gomory cut is closely related to the mixed integer rounding inequalities. In fact, if we expand the definition of a mixed integer rounding inequality to the inequality shown in Theorem 2.9, we can derive the mixed integer Gomory cut as an expanded mixed integer rounding inequality. We define this modified mixed integer rounding inequality as follows.

Definition 3.3. Let $c^1, c^2 \in \mathbb{R}^{p+n}$, c_0^1, $c_0^2 \in \mathbb{R}$ be such that $c^1 x \le c_0^1$ and $c^2 x \le c_0^2$ are valid for P. Then the *expanded mixed integer rounding inequality* is defined as

$$\sum_{j\in I} \lfloor c_j^2 - c_j^1 \rfloor x_j + \tfrac{1}{1-f}\left(\sum_{j\in I} c_j^1 x_j + \sum_{j\in J} \min\{c_j^1, c_j^2\} x_j - c_0^1\right) \le \lfloor c_0^2 - c_0^1 \rfloor.$$

The difference between this definition and Definition 2.6 lies in the relaxation of the requirement that $c_J^1 = c_J^2$. As mentioned in the motivation of this definition, the validity of the inequality for P_I was established in Theorem 2.9. We now use this definition to derive the mixed integer Gomory cut [12].

Lemma 3.4. *The mixed integer Gomory cut is an expanded mixed integer rounding inequality.*

Proof. Suppose $x_k + \sum_{j\in I'} \bar{a}_{kj} x_j + \sum_{j\in J} \bar{a}_{kj} x_j = \bar{b}_k$ with $k \in I$ and $\bar{b}_k \notin \mathbb{Z}$. We can apply Proposition 2.12 to the equation (this hides the application of Definition 3.3). So assume that $x \in P_I$ and then

$$x_k + \sum_{j\in I'} (\lfloor \bar{a}_{kj} \rfloor + \tfrac{(f_j - f)^+}{1-f}) x_j + \tfrac{1}{1-f} \sum_{\substack{j\in J \\ \bar{a}_{kj} < 0}} \bar{a}_{kj} x_j \le \lfloor b \rfloor.$$

We substitute for x_k and simplify to obtain the following:

$$\begin{aligned}
(b - \sum_{j\in I'} \bar{a}_{kj} x_j - \sum_{j\in J} \bar{a}_{kj} x_j) + \sum_{j\in I'} (\lfloor \bar{a}_{kj} \rfloor + \tfrac{(f_j-f)^+}{1-f}) x_j + \tfrac{1}{1-f} \sum_{\substack{j\in J \\ \bar{a}_{kj}<0}} \bar{a}_{kj} x_j &\le \lfloor b \rfloor \\
\sum_{j\in I'} (\lfloor \bar{a}_{kj} \rfloor - \bar{a}_{kj} + \tfrac{(f_j-f)^+}{1-f}) x_j - \sum_{\substack{j\in J \\ \bar{a}_{kj}\ge 0}} \bar{a}_{kj} x_j - (1 - \tfrac{1}{1-f}) \sum_{\substack{j\in J \\ \bar{a}_{kj}<0}} \bar{a}_{kj} x_j &\le -b + \lfloor b \rfloor \\
\sum_{j\in I'} (-f_j + \tfrac{(f_j-f)^+}{1-f}) x_j - \sum_{\substack{j\in J \\ \bar{a}_{kj}\ge 0}} \bar{a}_{kj} x_j + \tfrac{f}{1-f} \sum_{\substack{j\in J \\ \bar{a}_{kj}<0}} \bar{a}_{kj} x_j &\le -f \\
\tfrac{1}{f} \sum_{\substack{j\in I' \\ f_j\le f}} f_j x_j + \tfrac{1}{f} \sum_{\substack{j\in I' \\ f_j > f}} (f_j - \tfrac{f_j - f}{1-f}) x_j + \tfrac{1}{f} \sum_{\substack{j\in J \\ \bar{a}_{kj}\ge 0}} \bar{a}_{kj} x_j - \tfrac{1}{1-f} \sum_{\substack{j\in J \\ \bar{a}_{kj}<0}} \bar{a}_{kj} x_j &\ge 1 \\
\tfrac{1}{f} \sum_{\substack{j\in I' \\ f_j\le f}} f_j x_j + \tfrac{1}{f} \sum_{\substack{j\in I' \\ f_j > f}} \tfrac{f(-f_j+1)}{1-f} x_j + \tfrac{1}{f} \sum_{\substack{j\in J \\ \bar{a}_{kj}\ge 0}} \bar{a}_{kj} x_j - \tfrac{1}{1-f} \sum_{\substack{j\in J \\ \bar{a}_{kj}<0}} \bar{a}_{kj} x_j &\ge 1 \\
\tfrac{1}{f} \sum_{\substack{j\in I' \\ f_j\le f}} f_j x_j + \tfrac{1}{1-f} \sum_{\substack{j\in I' \\ f_j > f}} (1 - f_j) x_j + \tfrac{1}{f} \sum_{\substack{j\in J \\ \bar{a}_{kj}\ge 0}} \bar{a}_{kj} x_j - \tfrac{1}{1-f} \sum_{\substack{j\in J \\ \bar{a}_{kj}<0}} \bar{a}_{kj} x_j &\ge 1.
\end{aligned}$$

Hence, the mixed integer Gomory cut can be derived as an expanded mixed integer rounding inequality. □

3.4 Intersection cuts

We next consider a type of cut introduced by Balas [3] based on a split disjunction. Suppose that we have an optimal solution $\bar{x}$ to the LP relaxation of (MIP). Assume that $\bar{x}$ is fractional for some $j \in I$ and also that $\bar{x}$ is basic with basis B. Given a non-basic variable $j \in N$, define r^j as follows:

$$r^j_k = \begin{cases} -\bar{a}_{kj} & \text{if } k \in B \\ 1 & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}$$

Intuitively, r^j is the direction we move in if x_j enters on a simplex pivot. Let $D(\pi, \pi_0)$ be a split disjunction. Define $\epsilon(\pi, \pi_0) = \pi\bar{x} - \pi_0$ as the amount the basic solution $\bar{x}$ violates the split disjunction. Furthermore, for $\alpha \in \mathbb{R}$ let $x_j(\alpha) = \bar{x} + \alpha r^j$ be the line following r^j for nonbasic, integer constrained variable x_j. We set $\alpha_j(\pi, \pi_0)$ to be the smallest value α such that $x_j(\alpha)$ satisfies $D(\pi, \pi_0)$ (so $\alpha_j(\pi, \pi_0) = \min\{\alpha \mid x_j(\alpha) \in P \cap F_{D(\pi,\pi_0)}\}$).

Proposition 3.5. *If $j \in N$, then*

$$\alpha_j(\pi, \pi_0) = \begin{cases} -\frac{\epsilon(\pi,\pi_0)}{\pi r^j} & \textit{if } \pi r^j < 0 \\ \frac{1-\epsilon(\pi,\pi_0)}{\pi r^j} & \textit{if } \pi r^j > 0 \\ \infty & \textit{otherwise.} \end{cases}$$

Proof.

- Case 1: $\pi r^j < 0$. So we will approach $\pi x = \pi_0$ as we follow r^j toward $P \cap F_{D(\pi,\pi_0)}$.

$$\begin{aligned} \pi x_j(\alpha_j(\pi, \pi_0)) &= \pi_0 \\ \pi(\bar{x} + \alpha_j(\pi, \pi_0) r^j) &= \pi_0 \\ \pi\bar{x} + \alpha_j(\pi, \pi_0)\pi r^j &= \pi_0 \\ \alpha_j(\pi, \pi_0) &= \frac{\pi_0 - \pi\bar{x}}{\pi r^j} \\ &= -\frac{\epsilon(\pi, \pi_0)}{\pi r^j} \end{aligned}$$

- Case 2: $\pi r^j > 0$. So we will approach $\pi x = \pi_0 + 1$ as we follow r^j toward $P \cap F_{D(\pi,\pi_0)}$.

$$\begin{aligned} \pi x_j(\alpha_j(\pi, \pi_0)) &= \pi_0 + 1 \\ \pi(\bar{x} + \alpha_j(\pi, \pi_0) r^j) &= \pi_0 + 1 \\ \alpha_j(\pi, \pi_0) &= \frac{\pi_0 + 1 - \pi\bar{x}}{\pi r^j} \\ &= \frac{1 - \epsilon(\pi, \pi_0)}{\pi r^j} \end{aligned}$$

- Case 3: $\pi r^j = 0$. Since there is no intersection, set $\alpha_j(\pi, \pi_0) = \infty$.

□

Now, we consider the hyperplane passing through each of the points $x_j(\alpha_j(\pi, \pi_0))$, where $\alpha_j(\pi, \pi_0)$ is finite.

Definition 3.6. The *intersection cut* is defined as $\sum\limits_{j \in N} \frac{1}{\alpha_j(\pi,\pi_0)} x_j \geq 1$.

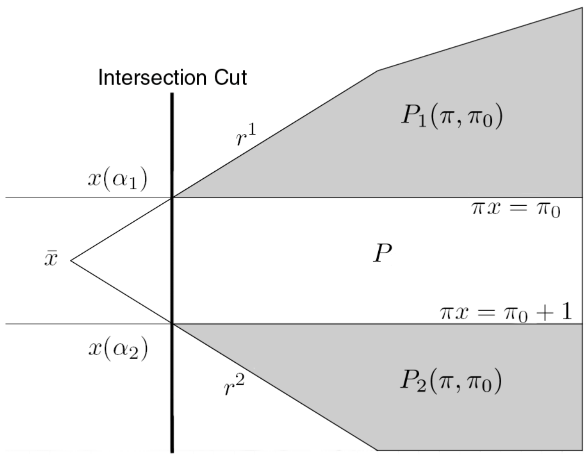

Figure 1: An example of an intersection cut

Example. Let's consider a polyhedron P in two dimensions. Let $P_1(\pi, \pi_0) = P \cap \{x \mid \pi x \leq \pi_0\}$ and $P_2(\pi, \pi_0) = P \cap \{x \mid \pi x \geq \pi_0 + 1\}$, then we can visualize the intersection cut geometrically as shown in Figure 1.

Theorem 3.7 ((Balas, [3]). *All points in $P \cap F_{D(\pi,\pi_0)}$ satisfy the intersection cut $\sum_{j \in N} \frac{1}{\alpha_j(\pi,\pi_0)} x_j \geq 1$, where $\alpha_j(\pi, \pi_0)$ is defined as above.*

Proof. Denote the intersection cut as $\gamma x \geq 1$; since we can assume that $\bar{x} \notin \{x \mid \gamma x \geq 1\}$ by our construction of the cut, we want to show that $P \cap \{x \mid \gamma x < 1\}$ contains no integral points.

Let C be the closure of $P \cap \{x \mid \gamma x < 1\}$ (i.e. the set $P \cap \{x \mid \gamma x < 1\}$ and all of its finite limit points). This is the convex hull of $\bar{x}$ and the points of intersection $x_j(\alpha_j(\pi, \pi_0)) = \bar{x} + \alpha_j(\pi, \pi_0) r^j$; each of these points $x_j(\alpha_j(\pi, \pi_0))$, however, lies on $\{x \mid \gamma x = 1\}$. Hence, $\{x \mid \gamma x < 1\} \cap F_{D(\pi,\pi_0)} = \emptyset$. So, the intersection cut $\gamma x \geq 1$ does not cut off any integral points. $\square$

Intersection cuts are inequalities that are valid for $P \cap \{x \mid \pi x \leq \pi_0\}$ and $P \cap \{x \mid \pi x \geq \pi_0 + 1\}$. We thus have that intersection cuts are split cuts; geometrically we note that intersection cuts appear to be relatively strong inequalities for $P \cap F_{D(\pi,\pi_0)}$ (where $F_{D(\pi,\pi_0)} = \{x \mid \pi x \leq \pi_0 \text{ or } \pi x \geq \pi_0 + 1\}$). We briefly examine this idea:

Definition 3.8.

- The set of all split disjunctions is denoted as $\Pi = \{D(\pi, \pi_0) \mid (\pi, \pi_0) \in \mathbb{Z}^{p+n+1},\ \pi_j = 0 \text{ for all } j \in J\}$
- The *split closure* of P is defined as

$$SC(P) = \bigcap_{D(\pi,\pi_0) \in \Pi} \text{conv}(P \cap F_{D(\pi,\pi_0)}).$$

In particular, if we denote by $P(B)$ the polyhedral cone $\{\bar{x} + \sum_{j \in J} \lambda_j r^j \mid \lambda_j \geq 0\}$, we have the following:

Theorem 3.9 (Anderson, Cornuéjols, and Li, [1]).

$$conv(P(B) \cap F_{D(\pi,\pi_0)}) = \{x \in P(B) \mid \sum_{j \in N} \frac{x_j}{\alpha_j(\pi, \pi_0)} \geq 1\}$$

A deeper result along these lines is also possible. Let $\mathcal{B}$ be the set of all bases of (MIP), and let Π be the set of all split disjunctions. Then we have the following:

Theorem 3.10 (Anderson, Cornuéjols, and Li, [1])**.**

$$SC = \bigcap_{B \in \mathcal{B}} \bigcap_{D(\pi,\pi_0) \in \Pi} conv(P(B) \cap F_{D(\pi,\pi_0)})$$

Thus we need only the family of intersection cuts to fully describe the split closure of P.

3.5 The mixed integer Gomory and intersection cuts

We now show that the mixed integer Gomory cut is in fact an intersection cut with a particular choice of π and π_0.

Lemma 3.11 (Anderson, Cornuéjols, and Li, [2])**.** *Let B be a basis of the LP relaxation of (MIP) and let $\bar{x}$ be its associated basic solution. Assume that $\bar{x}_k \notin \mathbb{Z}$ for some $k \in I$. The mixed integer Gomory cut obtained from the row of the simplex tableau corresponding to x_k is given by $\sum_{j \in N} \frac{1}{\alpha_j(\pi,\pi_0)} x_j \geq 1$, where $\pi_0 = \lfloor \bar{x}_k \rfloor$, $f = \bar{x}_k - \lfloor \bar{x}_k \rfloor$, and*

$$\pi_j = \begin{cases} \lfloor \bar{a}_{kj} \rfloor & \textit{if } j \in N_I \textit{ and } f_j \leq f \\ \lceil \bar{a}_{kj} \rceil & \textit{if } j \in N_I \textit{ and } f_j > f \\ 1 & \textit{if } j = k \\ 0 & \textit{otherwise} \end{cases}$$

Proof. We need to compute $\alpha_j(\pi,\pi_0)$:

- First, compute $\epsilon(\pi,\pi_0)$:
$$\epsilon(\pi,\pi_0) = \pi\bar{x} - \pi_0 = \bar{x}_k - \lfloor \bar{x}_k \rfloor = f.$$

- We now compute πr^j (noting that $\pi_i r_i^j = 0$ if $i \neq k, j$):

$$\begin{aligned} \pi r^j &= \pi_k r_k^j + \pi_j r_j^j + \sum_{i \neq k,j} \pi_i r_i^j \\ &= \pi_k r_k^j + \pi_j r_j^j + 0 \\ &= r_k^j + \pi_j \\ &= -\bar{a}_{kj} + \begin{cases} \lfloor \bar{a}_{kj} \rfloor & \text{if } j \in N_I \text{ and } f_j \leq f \\ \lceil \bar{a}_{kj} \rceil & \text{if } j \in N_I \text{ and } f_j > f \\ 0 & \text{otherwise} \end{cases} \\ &= -\bar{a}_{kj} + \begin{cases} \lfloor \bar{a}_{kj} \rfloor & \text{if } j \in N_I \text{ and } f_j \leq f \\ \lfloor \bar{a}_{kj} \rfloor + 1 & \text{if } j \in N_I \text{ and } f_j > f \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} -f_j & \text{if } j \in N_I, f_j \leq f \\ 1 - f_j & \text{if } j \in N_I, f_j > f \\ -\bar{a}_{kj} & \text{if } j \notin N_I \end{cases} \end{aligned}$$

- We compute $\alpha_j(\pi, \pi_0)$:

$$\alpha_j(\pi,\pi_0) = \begin{cases} -\frac{\epsilon(\pi,\pi_0)}{\pi r^j} & \text{if } \pi r^j < 0 \\ \frac{1-\epsilon(\pi,\pi_0)}{\pi r^j} & \text{if } \pi r^j > 0 \\ \infty & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{f}{f_j} & \text{if } \pi r^j < 0,\ j \in N_I,\ f_j \leq f \\ \frac{f}{\bar{a}_{kj}} & \text{if } \pi r^j < 0,\ \text{otherwise} \\ \frac{1-f}{1-f_j} & \text{if } \pi r^j > 0,\ j \in N_I,\ f_j > f \\ \frac{1-f}{-\bar{a}_{kj}} & \text{if } \pi r^j > 0,\ \text{otherwise} \\ \infty & \text{otherwise} \end{cases}$$

- Inserting $\alpha_j(\pi, \pi_0)$ into the intersection cut formula, we obtain the mixed integer Gomory cut:

$$\sum_{j\in N} \frac{x_j}{\alpha_j(\pi,\pi_0)} \geq 1$$

$$\sum_{\substack{j\in N_I \\ f_j \leq f}} \frac{f_j}{f} x_j + \sum_{\substack{j\in N_I \\ f_j > f}} \frac{1-f_j}{1-f} x_j + \sum_{\substack{j\in N_J \\ \pi r^j < 0}} \frac{\bar{a}_{kj}}{f} x_j + \sum_{\substack{j\in N_J \\ \pi r^j > 0}} \frac{-\bar{a}_{kj}}{1-f} x_j \geq 1$$

$$\frac{1}{f} \sum_{\substack{j\in N_I \\ f_j \leq f}} f_j x_j + \frac{1}{1-f} \sum_{\substack{j\in N_I \\ f_j > f}} (1-f_j) x_j + \frac{1}{f} \sum_{\substack{j\in N_J \\ \bar{a}_{kj} \geq 0}} \bar{a}_{kj} x_j - \frac{1}{1-f} \sum_{\substack{j\in N_J \\ \bar{a}_{kj} < 0}} \bar{a}_{kj} x_j \geq 1$$

$$\frac{1}{f} \sum_{\substack{j\in I \\ f_j \leq f}} f_j x_j + \frac{1}{1-f} \sum_{\substack{j\in I \\ f_j > f}} (1-f_j) x_j + \frac{1}{f} \sum_{\substack{j\in J \\ \bar{a}_{kj} \geq 0}} \bar{a}_{kj} x_j - \frac{1}{1-f} \sum_{\substack{j\in J \\ \bar{a}_{kj} < 0}} \bar{a}_{kj} x_j \geq 1.$$

We note that the change from N_I and N_J to I and J occurs because $\bar{a}_{kj} = 0$ for all $j \in B$. We now have the mixed integer Gomory cut, as desired.

□

4 Lift-and-project cuts

In the context of 0-1 programming problems, several authors have advocated the use of a lift-and-project procedure for generating cuts. This idea is quite different from the approaches mentioned previously, and there are several methods to generate such cutting planes. Among the main approaches are those of Lovasz and Schrijver [11], Sherali and Adams [14], and Balas, Ceria, and Cornuéjols [6]. In this paper, we will consider the approach taken in [6] to establish such cuts. We hold off on defining what we will consider to be a lift-and-project cut and first present the theory that leads to the family of cutting planes.

For the remainder of this section, let $K = \{x \in \mathbb{R}^n \mid Ax \geq b,\ x \geq 0,\ x_j \leq 1 \text{ for all } j \in I\} := \{x \in \mathbb{R}^n \mid \tilde{A}x \geq \tilde{b}\}$. We then define $K_I = \{x \in K \mid x_j \in \{0,1\} \text{ for all } j \in I\}$.

4.1 Sequential convexification procedure

4.1.1 One iteration of the procedure

1. Select $j \in \{1, \ldots, p\}$.
2. Multiply $\tilde{A}x \geq \tilde{b}$ by $(1 - x_j)$ and x_j:

$$(1-x_j)(\tilde{A}x - \tilde{b}) \geq 0$$
$$x_j(\tilde{A}x - \tilde{b}) \geq 0.$$

3. Linearize the convex body obtained in the previous step by setting $x_j = x_j^2$ and $y_i = x_i x_j$ for $i \neq j$. Let $M_j(K)$ be the polyhedron defined by the resulting set of inequalities.

4. Let $P_j(K) = \{x \mid (x, y) \in M_j(K)\}$.

Remark 4.1.

1. $K_I \subseteq P_j(K)$ since if $x \in K_I$ then $x_j = 0$ or $x_j = 1$. Thus, we have $x_j = 0 = 0^2 = x_j^2$ or $x_j = 1 = 1^2 = x_j^2$ and hence $x \in P_j(K)$.
2. $P_j(K) \subseteq K$ since if $x \in P_j(K)$ then there exists y such that $(x, y) \in M_j(K)$. Hence, we have that $x \in K$.
3. We recall from polyhedral theory that the set formed by the projection of a polyhedron into a lower-dimensional space is itself a polyhedron. Hence, $P_j(K)$ is a polyhedron since it is the projection of the polyhedron $M_j(K)$ onto the space of the x variables.
4. If $x \in K$ then x satisfies

$$\begin{aligned}(1 - x_j)(\tilde{A}x - \tilde{b}) &\geq 0\\ x_j(\tilde{A}x - \tilde{b}) &\geq 0.\end{aligned}$$

   Hence, points from K satisfy the constraints for the convex body created by step 2.
5. Replacing $x_i x_j$ with y_i does not tighten the constraints of K, either.
6. If $x_j \in (0, 1)$, then the point x is not in $P_j(K)$. Hence, the assignment $x_j = x_j^2$ accounts for the tightening of constraints in the convexification procedure.

4.1.2 Iterated procedure

To iterate the procedure from the previous section, we use the following procedure:

1. Initialize $K' = \{x \in \mathbb{R}^n \mid \tilde{A}x \geq \tilde{b}\}$.
2. Set $j' = 1$.
3. Run the procedure of the previous section, selecting variable j' in step 1.
4. Set $K' = P_j'(K')$ at the termination of each iteration.
5. If $j' = p$ then STOP (we have $K' = \text{conv}(K_I)$).
6. Otherwise, set $j' = j' + 1$ and go to step 3.

4.2 Correctness of convexification procedure

We now demonstrate that this algorithm does in fact find $\text{conv}(K_I)$ as claimed in step 5. To do so, we present two theorems and two corollaries that together show that iterating the sequential convexification procedure for $j' = 1$ to p is a valid means of obtaining the convex hull of K_I.

Theorem 4.2 (Balas, Ceria, and Cornuéjols, [6])**.**

$$P_j(K) = conv(K \cap \{x \in \mathbb{R}^n \mid x_j \in \{0, 1\}\})$$

Proof. First, we show that $\text{conv}(K \cap \{x \mid x_j \in \{0, 1\}\} \subseteq P_j(K)$. Let $\bar{x} \in \text{conv}(K \cap \{x \mid x_j \in \{0, 1\}\})$, let $y_i = \bar{x}_i \bar{x}_j$ for $i \neq j$. Now $(\bar{x}, y) \in M_j(K)$ as $x_i^2 = x_i$, so $\bar{x} \in P_j(K)$.

Next, we show that $P_j(K) \subseteq \text{conv}(K \cap \{x \mid x_j \in \{0, 1\}\})$. Let $(c, c_0) \in \mathbb{R}^{p+n+1}$ and assume that for all $x \in \text{conv}(K \cap \{x \mid x_j \in \{0, 1\}\})$, $cx \geq c_0$. We want to show that $cx \geq c_0$ is valid for $P_j(K)$, so we first deal with the exceptional cases when $\text{conv}(K \cup \{x \mid x_j = 0\}) = \emptyset$ and $\text{conv}(K \cup \{x \mid x_j = 1\}) = \emptyset$:

- If $\mathrm{conv}(K \cup \{x \mid x_j = 0\}) = \emptyset$, then there exists $\epsilon > 0$ such that for all $x \in \mathrm{conv}(K \cap \{x \mid x_j = 0\})$, $x_j \geq \epsilon$. Since $x_j - \epsilon \geq 0$ can be written in the form $uAx \geq ub$ for some non-negative $1 \times m$ matrix u, any x which satisfies

$$\begin{aligned} (1 - x_j)(\tilde{A}x - \tilde{b}) &\geq 0 \\ x_j(\tilde{A}x - \tilde{b}) &\geq 0 \end{aligned}$$

must also satisfy $(1 - x_j)(x_j - \epsilon) \geq 0$. Thus, if $x \in \mathrm{conv}(K \cap \{x \mid x_j = 0\})$,

$$(1 - x_j)(x_j - \epsilon) = x_j - \epsilon - x_j^2 + x_j\epsilon \geq 0.$$

Substituting $x_j^2 = x_j$, we obtain $\epsilon(x_j - 1) \geq 0$ so $x_j \geq 1$ and thus $x_j = 1$.

- If $\mathrm{conv}(K \cup \{x \mid x_j = 1\}) = \emptyset$, then there exists $\epsilon > 0$ such that for all $x \in \mathrm{conv}(K \cap \{x \mid x_j = 1\})$, $x_j \leq 1 - \epsilon$. Thus, $(1 - x_j) - \epsilon \geq 0$ and since this can be written in the form $uAx \geq ub$ for some non-negative $1 \times m$ matrix u, any x which satisfies

$$\begin{aligned} (1 - x_j)(\tilde{A}x - \tilde{b}) &\geq 0 \\ x_j(\tilde{A}x - \tilde{b}) &\geq 0 \end{aligned}$$

must also satisfy $x_j((1 - x_j) - \epsilon) \geq 0$. So, if $x \in \mathrm{conv}(K \cap \{x \mid x_j = 0\}$, we have $x_j - x_j^2 - x_j\epsilon \geq 0$ and thus $x_j \leq 0$. So $x_j = 0$.

In both cases, we see that $P_j(K) \subseteq \mathrm{conv}(K \cap \{x \mid x_j \in \{0, 1\}\})$. We now proceed to the general case. Let $x \in \mathrm{conv}(K \cap \{x \mid x_j = 0\}$. We notice that $\mathrm{conv}(K \cap \{x \mid x_j = 0\} = \mathrm{conv}(K \cap \{x_j \mid x_j \leq 0\})$. Since $cx \geq c_0$,

$$\begin{aligned} & cx \geq c_0 \text{ is valid for } \{x \mid \tilde{A}x \geq \tilde{b},\ x_j \leq 0\} \\ \Leftrightarrow\quad & \min\{cx \mid \tilde{A}x \geq \tilde{b},\ x_j \leq 0\} \geq c_0 \\ \Leftrightarrow\quad & \exists y,\ \lambda \geq 0 \text{ such that } y\tilde{A} - \lambda e_j = c \text{ and } y\tilde{b} \geq c_0, \end{aligned} \tag{*}$$

where the last line follows from the duality theorem. Now, for any $x \in \{x \mid \tilde{A}x \geq \tilde{b},\ x_j \leq 0\}$,

$$(c + \lambda e_j)x = y\tilde{A}x \geq y\tilde{b} \geq c_0. \tag{1}$$

So, $\exists \lambda \geq 0$ such that $cx + \lambda x_j \geq c_0$ for all $x \in K$.

Similarly, let $x \in \mathrm{conv}(K \cap \{x \mid x_j = 1\})$ and notice again that $\mathrm{conv}(K \cap \{x \mid x_j = 1\}) = \mathrm{conv}(K \cap \{x \mid x_j \geq 1\})$. Since $cx \geq c_0$, we have

$$\begin{aligned} & cx \geq c_0 \text{ is valid for } \{x \mid \tilde{A}x \geq \tilde{b},\ x_j \geq 1\} \\ \Leftrightarrow\quad & \min\{cx \mid \tilde{A}x \geq \tilde{b},\ x_j \geq 1\} \geq c_0 \\ \Leftrightarrow\quad & \exists y,\ \mu \geq 0 \text{ such that } y\tilde{A} + \mu e_j = c \text{ and } y\tilde{b} + \mu \geq c_0. \end{aligned} \tag{**}$$

Now, for any $x \in \{x \mid \tilde{A}x \geq \tilde{b},\ x_j \geq 1\}$,

$$(c - \mu e_j)x = y\tilde{A}x \geq y\tilde{b} \geq c_0 - \mu. \tag{2}$$

So, $\exists \mu \geq 0$ such that $cx + \mu(1 - x_j) \geq c_0$ for all $x \in K$ (We note that $(*)$ and $(**)$ follow only if the dual problem is bounded - the two exceptional cases listed earlier account for situations where either $(*)$ or $(**)$ do not hold). Now, since we can write both $cx + \lambda x_j \geq c_0$ and $cx + \mu(1 - x_j) \geq c_0$ in the form $u\tilde{A}x \geq u\tilde{b}$ for some row vector u (one for each inequality), any x which satisfies

$$\begin{aligned} (1 - x_j)(\tilde{A}x - \tilde{b}) &\geq 0 \\ x_j(\tilde{A}x - \tilde{b}) &\geq 0 \end{aligned}$$

must also satisfy (using (1) and (2))

$$\begin{aligned}(1-x_j)(cx+\lambda x_j-c_0)&\geq 0\\ x_j(cx+\mu(1-x_j)-c_0)&\geq 0.\end{aligned}$$

Adding these two inequalities together we get $cx+(\lambda+\mu)(x_j-x_j^2)-c_0\geq 0$. Setting $x_j=x_j^2$, we see that $cx\geq c_0$ is thus valid for $P_j(K)$. □

Theorem 4.3 (Balas, Ceria, and Cornuéjols, [6])**.** *For* $t\in\{1,\ldots,p\}$,

$$P_{i_1,\ldots,i_t}(K)=conv(K\cap\{x\in\mathbb{R}^n\mid x_j\in\{0,1\}\ \textit{for all}\ j\in\{i_1,\ldots,i_t\}\}).$$

Proof. Without loss of generality, let $\{i_1,\ldots,i_t\}=\{1,\ldots,t\}$ and define $F_q=\{x\mid x_j\in\{0,1\},\ j=1,\ldots,q\}$. Proceed by induction on t:

- Base case: $t=1$. Apply Theorem 4.2 to get the result.
- Suppose that for $t=q-1$, we have that $P_{1,\ldots,t}(K)=\text{conv}(K\cap\{x\in\mathbb{R}^n\mid x_j\in\{0,1\},j\in\{1,\ldots,t\}\}$.
- Let $t=q$, $2\leq q\leq p$. So,

$$\begin{aligned}P_{1,\ldots,q}(K)&=P_q(P_{1,\ldots,q-1}(K))\\ &=P_q(\text{conv}(K\cap F_{q-1}))\quad\text{(by the induction hypothesis)}\\ &=\text{conv}\,(\,\text{conv}(K\cap F_{q-1})\cap\{x\mid x_q\in\{0,1\}\})\quad\text{(by Theorem 4.2)}\\ &=\text{conv}\big(\,(\text{conv}(K\cap F_{q-1})\cap\{x\mid x_q=0\})\cup(\text{conv}(K\cap F_{q-1})\cap\{x\mid x_q=1\})\big).\end{aligned}$$

Claim. Let $S\subseteq\mathbb{R}^n$ and $H=\{x\in\mathbb{R}^n\mid cx=c_0\}$ be such that $\forall x\in S, cx\geq c_0$. Then $H\cap\text{conv}(S)=\text{conv}(S\cap H)$.

Proof. $x\in H\cap\text{conv}(S)\Rightarrow cx=c_0$ and $x=\sum_i\lambda_i s^i$, $\sum_i\lambda_i=1$, $\lambda_i>0$, $s^i\in\text{conv}(S)$, and $cs^i\geq c_0$. Now,

$$c_0=cx=\alpha\sum_i\lambda_i s^i=\sum_i\lambda_i cs^i\geq\sum_i\lambda_i c_0=c_0.$$

Hence, $cs^i=c_0$ for each s^i and we have $x\in\text{conv}(S)$.
Suppose now that $x\in\text{conv}(S\cap H)$. Then x satisfies $cx=c_0$, and hence $x\in H\cap\text{conv}(S)$. □

Applying the claim with $S=\text{conv}(K\cap F_{q-1})$, and H as each of $\{x\mid x_q=0\}$ and $\{x\mid x_q=1\}$ in succession, we have:

$$\begin{aligned}P_{1,\ldots,q}(k)&=\text{conv}(\text{conv}(K\cap F_{q-1}\cap\{x\mid x_q=0\})\cup\text{conv}(K\cap F_{q-1}\cap\{x\mid x_q=1\}))\\ &=\text{conv}(K\cap F_{q-1}\cap\{x\mid x_q\in\{0,1\}\})\\ &=\text{conv}(K\cap F_q)\end{aligned}$$

where the second last equality follows from the fact that for two sets A and B,

$$\text{conv}(\,\text{conv}(A)\cup\text{conv}(B))=\,\text{conv}(A\cup B).$$

□

Corollary 4.4 (Balas, Ceria, and Cornuéjols, [6])**.**

$$P_{1,\ldots,p}(K)=conv(K_I)$$

Proof. Set $t=p$ in Theorem 4.3 and the result follows. □

Corollary 4.5 (Balas, Ceria, and Cornuéjols, [6])**.**

$$P_i(P_j(K))=P_j(P_i(K))\ \textit{for}\ i,\,j\in I,\ i\neq j$$

Proof. Apply Theorem 4.3 for $\{i,j\}$ and $\{j,i\}$ and the result follows. □

4.3 Generating cutting planes using sequential convexification

To use the results from this convexification procedure in a cutting plane algorithm, we need to define a way to create cuts for K using the polyhedron $P_j(K)$. A result of Balas [4] aids in this procedure.

Theorem 4.6 (Balas, [4])**.**

$$P_j(K) = \{x \in \mathbb{R}^n \mid \alpha x \geq \beta \textit{ for all } (\alpha, \beta) \in P_j^*(K)\},$$

where $P_J^*(K)$ is the set of all (α, β) such that there exists $u, v \in \mathbb{R}^{m+n+p}, u_0, v_0 \in \mathbb{R}$ that satisfy the following equations:

$$\begin{array}{lllllllllll} \alpha & & & - & u\tilde{A} & + & u_0 e_j & & & & = 0 \\ \alpha & & & & & & & - & v\tilde{A} & - \; v_0 e_j & = 0 \\ & - & \beta & + & u\tilde{b} & & & & & & = 0 \\ & - & \beta & + & & & & + & v\tilde{b} & + \; v_0 & = 0 \\ & & & & u, & & & & v & & \geq 0. \end{array}$$

Definition 4.7. A *lift-and-project* cut $\alpha x \geq \beta$ is a cut such that $(\alpha, \beta) \in P_j^*(K)$.

Remark 4.8. A lift-and-project cut is a cut for $P_j(K)$. We can select a cut which maximizes the Euclidean distance from $\bar{x}$ to the cutting plane by taking (α, β) as follows:

$$\begin{array}{llllllllllll} \min & \alpha\bar{x} & - & \beta & & & & & & & & \\ \text{subject to} & \alpha & & & - & u\tilde{A} & + & u_0 e_j & & & & \geq 0 \\ & \alpha & & & & & & & - & v\tilde{A} & - \; v_0 e_j & \geq 0 \\ & & - & \beta & + & u\tilde{b} & & & & & & = 0 \\ & & - & \beta & + & & & & + & v\tilde{b} & + \; v_0 & = 0 \\ & & & & & u, & & & & v & & \geq 0. \end{array}$$

with the additional normalization constraint

$$\sum_{i=1}^{m+p} u_i + u_0 + \sum_{i=1}^{m+p} v_i + v_0 = 1.$$

We thus obtain a lift-and-project cut that would be valid for $P_j(K)$ without needing to compute $P_j(K)$. We now introduce the general cutting plane procedure using lift-and-project cuts.

4.3.1 General procedure

1. Let $S = \{(\alpha, \beta) \mid \|\alpha\|_1 \leq 1\}$
2. Set $K^1 = K = \{x \in \mathbb{R}^n \mid \tilde{A}x \geq \tilde{b}\}$.
3. for $t = 1,\ 2,\ \ldots\{$
4. Find $cx^t = \min\{cx \mid x \in K^t\}$
5. If $x_j^t \in \{0, 1\}$ for all $j \in I$ then STOP.
6. For all j such that $0 < x_j^t < 1$
7. Find $-x^t\alpha^j + \beta^j = \min\{\alpha x^t - \beta \mid (\alpha, \beta) \in P_j^*(K^t) \cap S\}$
8. $K^{t+1} = K^t \cap (\bigcap_j \{x \mid \alpha^j x \geq \beta^j\})$
9. $\}$

Remark 4.9.

1. In line 8, we could replace the coefficient $-x^t$ of α and 1 of β with any value of our choice. The selection of $-x^t$ and 1 was made to obtain what "deepest cut" [6], namely to maximize the Euclidean distance from $\bar{x}$ to the hyperplane $\alpha x = \beta$.
2. S is a normalization to truncate the cone $P_j^*(K)$. Other suggested choices from [6] for S include:
   (a) $\{(\alpha, \beta) \mid \beta = \pm 1\}$
   (b) $\{(\alpha, \beta) \mid \|\alpha\|_\infty \leq 1\}$
3. It can be shown that with the specification of some implementation details, a slight modification to line 6, and one of these normalizations that this cutting plane algorithm will obtain an optimal solution in a finite number of steps.

5 Improving mixed integer Gomory cuts

The use of mixed integer Gomory cuts appears to have a large impact on the speed of solving of real-world mixed integer programming problems [8]. Bixby has argued in the same paper [8] that their implementation is the most important innovation in recent years for obtaining solutions to ever-larger MIP problems. As a result of their practical importance, the idea of somehow improving mixed integer Gomory cuts appears capable of yielding results that translate into faster codes. In this section, we examine approaches made to generate cuts which are stronger than mixed integer Gomory cuts.

5.1 Reduce-and-split cuts

We recall from Section 3.5 that the mixed integer Gomory cut is an intersection cut with a specific choice of split disjunction. The first idea in obtaining a cut that improves on a mixed integer Gomory cut lies in strengthening a split disjunction of the form $D(\pi, \pi_0)$ in order to produce a new disjunction that remains valid for P_I but such that the intersection cut resulting from this strengthened disjunction cuts off more of P from an extreme point $\bar{x}$. Following the approach of Anderson, Cornuéjols, and Li [2], we assume that $D(\pi, \pi_0)$ is violated at $\bar{x} \in P$, set $\pi_0 = \lfloor \pi \bar{x} \rfloor$, and attempt to replace $D(\pi, \pi_0)$ with $D(\pi^j(\delta), \pi_0^j(\delta))$, where $j \in I$, $\delta \in \mathbb{Z}$, $\pi^j(\delta) = \pi + \delta e_j$, and $\pi_0^j(\delta) = \lfloor \pi^j(\delta)\bar{x} \rfloor$. We note that for $j \in N$, $\pi^j(\delta)\bar{x} = \pi\bar{x}$ and $\pi_0^j(\delta) = \pi_0$. Hence, $\epsilon(\pi, \pi_0) = \epsilon(\pi^j(\delta), \pi_0^j(\delta))$.

Lemma 5.1 (Anderson, Cornuéjols, and Li, [2])**.** *Let B be a basis of the LP relaxation of (MIP) and $\bar{x}$ its associated basic solution, let $D(\pi, \pi_0)$ be a split disjunction violated by $\bar{x}$, let $k \in N_I$, and let $D(\pi^k(\delta), \pi_0)$ be defined as above. Let δ^* be as follows:*

$$\delta^* = \begin{cases} -\lfloor \pi r^k \rfloor & \text{if } \lceil \pi r^k \rceil - \pi r^k > \epsilon(\pi, \pi_0) \\ -\lceil \pi r^k \rceil & \text{if } \lceil \pi r^k \rceil - \pi r^k \leq \epsilon(\pi, \pi_0) \end{cases}$$

Then the intersection cut derived from $D(\pi^k(\delta^*), \pi_0)$ and B has the largest coefficient of x_k of all the intersection cuts derived from $D(\pi^k(\delta), \pi_0)$ and B with $\delta \in \mathbb{Z}$.

Proof. If $k \in N$ and $j \in N \backslash \{k\}$, $\pi^k(\delta)r^j = \pi r^j$ for all $\delta \in \mathbb{Z}$. Recall that the cut derived from $D(\pi^k(\delta^*), \pi_0)$ differs from the cut derived from $D(\pi, \pi_0)$ only in the coefficient of x_k. So we need to show that δ^* maximizes $\alpha_k(\pi^k(\delta), \pi_0)$. Since $r_k^k = 1$, we have $\pi^k(\delta)r^k = (\pi + \delta e_k)r^k = \pi r^k + \delta$. Since $\epsilon(\pi^k(\delta), \pi_0^k(\delta)) = \epsilon(\pi, \pi_0)$, we have

$$\alpha_k(\pi^k(\delta), \pi_0) = \begin{cases} -\frac{\epsilon(\pi,\pi_0)}{\pi r^k + \delta} & \text{if } \pi r^k + \delta < 0 \\ \frac{1-\epsilon(\pi,\pi_0)}{\pi r^k + \delta} & \text{if } \pi r^k + \delta > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\alpha_k(\pi^k(\delta), \pi_0)$ attains its maximum as $\pi^k(\delta) = \pi r^k + \delta \to 0$; there are two potential values for δ in $\mathbb{Z}$, $\delta_f = -\lfloor \pi r^k \rfloor$ and $\delta_c = -\lceil \pi r^k \rceil$ (the values when the denominator is as close to 0 as possible). $\square$

Remark 5.2. We note that we can decide which of δ_f or δ_c is the actual value to choose through the following sequence of equivalences:

$$\begin{aligned}
\alpha_k(\pi^k(\delta_c), \pi_0) < \alpha_k(\pi^k(\delta_f), \pi_0) &\Leftrightarrow -\frac{\epsilon(\pi, \pi_0)}{\pi r^k + \delta_c} < \frac{1 - \epsilon(\pi, \pi_0)}{\pi r^k + \delta_f} \\
&\Leftrightarrow -\frac{\epsilon(\pi, \pi_0)}{\pi r^k - \lceil \pi r^k \rceil} < \frac{1 - \epsilon(\pi, \pi_0)}{\pi r^k - \lfloor \pi r^k \rfloor} \\
&\Leftrightarrow \frac{\epsilon(\pi, \pi_0)}{\lceil \pi r^k \rceil - \pi r^k} < \frac{1 - \epsilon(\pi, \pi_0)}{\pi r^k - (\lceil \pi r^k \rceil - 1)} \\
&\Leftrightarrow \frac{\lceil \pi r^k \rceil - \pi r^k}{\epsilon(\pi, \pi_0)} > \frac{1 - (\lceil \pi r^k \rceil - \pi r^k)}{1 - \epsilon(\pi, \pi_0)}.
\end{aligned}$$

We now consider the cases where $\lceil \pi r^k \rceil - \pi r^k > \epsilon(\pi, \pi_0)$ and $\lceil \pi r^k \rceil - \pi r^k \leq \epsilon(\pi, \pi_0)$:

- Suppose $\lceil \pi r^k \rceil - \pi r^k > \epsilon(\pi, \pi_0)$. Then
$$\frac{\lceil \pi r^k \rceil - \pi r^k}{\epsilon(\pi, \pi_0)} > \frac{\epsilon(\pi, \pi_0)}{\epsilon(\pi, \pi_0)} = 1$$
and
$$\frac{1 - (\lceil \pi r^k \rceil - \pi r^k)}{1 - \epsilon(\pi, \pi_0)} < \frac{1 - \epsilon(\pi, \pi_0)}{1 - \epsilon(\pi, \pi_0)} = 1.$$
So, we see the following:
$$\frac{\lceil \pi r^k \rceil - \pi r^k}{\epsilon(\pi, \pi_0)} > 1 > \frac{1 - (\lceil \pi r^k \rceil - \pi r^k)}{1 - \epsilon(\pi, \pi_0)}.$$
- Suppose $\lceil \pi r^k \rceil - \pi r^k \leq \epsilon(\pi, \pi_0)$. Then
$$\frac{\lceil \pi r^k \rceil - \pi r^k}{\epsilon(\pi, \pi_0)} \leq \frac{\epsilon(\pi, \pi_0)}{\epsilon(\pi, \pi_0)} = 1$$
and
$$\frac{1 - (\lceil \pi r^k \rceil - \pi r^k)}{1 - \epsilon(\pi, \pi_0)} \geq \frac{1 - \epsilon(\pi, \pi_0)}{1 - \epsilon(\pi, \pi_0)} = 1$$
So, we see the following:
$$\frac{\lceil \pi r^k \rceil - \pi r^k}{\epsilon(\pi, \pi_0)} \leq 1 \leq \frac{1 - (\lceil \pi r^k \rceil - \pi r^k)}{1 - \epsilon(\pi, \pi_0)}.$$

We thus see that $\delta^* = \delta_f$ if and only if $\lceil \pi r^k \rceil - \pi r^k > \epsilon(\pi, \pi_0)$.

The previous lemma can be applied repeatedly to a particular intersection cut to derive the mixed integer Gomory cut:

Lemma 5.3 (Anderson, Cornuéjols, and Li, [2])**.** *Let $\bar{x}$ be a basic feasible solution to the LP relaxation of (MIP). The mixed integer Gomory cut for $\bar{x}_i$ can be obtained from the disjunction $D(e_i, \lfloor \bar{x}_i \rfloor)$ by applying the strengthening procedure of Lemma 5.1 successively for each $j \in N_I$.*

We are thus able to construct mixed integer Gomory cuts via the strengthening procedure. This leads us to consider the idea of [2] on a family of cuts called *reduce-and-split cuts*: we start with a split cut and attempt to make it stronger. In a similar manner to lift-and-project cuts, we attempt to improve the Euclidean distance between the hyperplane of the cut and the current basic feasible solution $\bar{x}$.

Definition 5.4. Let $d(B, \pi, \pi_0)$ be the shortest distance from $\bar{x}$ to the hyperplane $\{x \mid \sum_{j \in N} \frac{x_j}{\alpha_j(\pi, \pi_0)} = 1\}$.

Lemma 5.5 (Anderson, Cornuéjols, and Li, [2])**.**

$$d(B, \pi, \pi_0)^2 = \frac{1}{\sum_{j \in N} (\frac{1}{\alpha_j(\pi, \pi_0)})^2}$$

Proof. Let $\gamma^T x \geq 1$ be the intersection cut derived from B and $D(\pi, \pi_0)$. So,

$$\gamma_j = \begin{cases} 0 & \text{if } j \in B \\ \frac{1}{\alpha_j(\pi,\pi_0)} & \text{if } j \in N. \end{cases}$$

We note that γ is normal to the hyperplane $\{x \mid \sum_{j \in N} \frac{x_j}{\alpha_j(\pi,\pi_0)} = 1\}$. So, $d(B, \pi, \pi_0)$ satisfies:

$$\begin{aligned} \gamma^T(\bar{x} + d(B,\pi,\pi_0)\tfrac{\gamma}{\|\gamma\|_2}) &= 1 \\ \gamma^T \tfrac{\gamma}{\|\gamma\|_2}\, d(B,\pi,\pi_0) &= 1 - \gamma^T \bar{x} \\ \|\gamma\|_2\, d(B,\pi,\pi_0) &= 1 \quad \text{since } \bar{x}_j = 0 \text{ for all } j \in N \Rightarrow \gamma^T \bar{x} = 0 \\ d(B,\pi,\pi_0) &= \frac{1}{\|\gamma\|_2} \\ d(B,\pi,\pi_0) &= \frac{1}{\sqrt{\sum_{j\in N} (\frac{1}{\alpha_j(\pi,\pi_0)})^2}} \\ d(B,\pi,\pi_0)^2 &= \frac{1}{\sum_{j\in N} (\frac{1}{\alpha_j(\pi,\pi_0)})^2}. \end{aligned}$$

□

From this formula, we see that a decrease in the magnitude of the coefficient $\frac{1}{\alpha_j(\pi,\pi_0)}$ of x_j in the intersection cut will increase $d(B, \pi, \pi_0)$. We explore further how this may be of use to improve intersection cuts (and hence mixed integer Gomory cuts due to the results of Section 3.5).

Proposition 5.6. *If $D(\pi, \pi_0)$ is a strengthened disjunction, then for all $j \in N_I$, $\frac{1}{\alpha_j(\pi,\pi_0)} \in [0, 1]$.*

Proof. Since $D(\pi, \pi_0)$ is a strengthened disjunction, then $D(\pi, \pi_0) = D(\pi(\delta^*), \pi_0)$. Again we note that $\epsilon(\pi, \pi_0) = \epsilon(\pi^j(\delta), \pi_0^j(\delta))$ and determine $\alpha_j(\pi^j(\delta), \pi_0)$:

$$\alpha_j(\pi^j(\delta), \pi_0) = \begin{cases} -\frac{\epsilon(\pi,\pi_0)}{\pi r^j + \delta} & \text{if } \pi r^j + \delta < 0 \\ \frac{1-\epsilon(\pi,\pi_0)}{\pi r^j + \delta} & \text{if } \pi r^j + \delta > 0 \\ \infty & \text{otherwise.} \end{cases}$$

Taking δ^* as in Lemma 5.1, we get

$$\delta^* = \begin{cases} -\lceil \pi r^j \rceil & \text{if } \pi r^j + \delta < 0 \\ -\lfloor \pi r^j \rfloor & \text{if } \pi r^j + \delta > 0. \end{cases}$$

So we have 3 cases for $\alpha_j(\pi^j(\delta), \pi_0)$:

1. $\alpha_j(\pi^j(\delta), \pi_0) = \infty$. Then $\frac{1}{\alpha_j(\pi^j(\delta),\pi_0)} = 0$
2. $\delta^* = -\lceil \pi r^j \rceil$. Then
$$\frac{1}{\alpha_j(\pi^j(\delta), \pi_0)} = \frac{\pi r^j + \delta^*}{-\epsilon(\pi, \pi_0)} = \frac{\lceil \pi r^j \rceil - \pi r^j}{\epsilon(\pi, \pi_0)} \leq \frac{\epsilon(\pi, \pi_0)}{\epsilon(\pi, \pi_0)} = 1.$$
3. $\delta^* = -\lfloor \pi r^j \rfloor$. Then
$$\frac{1}{\alpha_j(\pi^j(\delta), \pi_0)} = \frac{\pi r^j + \delta^*}{1 - \epsilon(\pi, \pi_0)} = \frac{\pi r^j - \lfloor \pi r^j \rfloor}{1 - \epsilon(\pi, \pi_0)} \leq \frac{1 - (\lceil \pi r^j \rceil - \pi r^j)}{1 - \epsilon(\pi, \pi_0)} < \frac{1 - \epsilon(\pi, \pi_0)}{1 - \epsilon(\pi, \pi_0)} = 1.$$

Since by definition $\alpha(\pi^j(\delta), \pi_0) > 0$, $\frac{1}{\alpha_j(\pi^j(\delta),\pi_0)} \geq 0$. Hence, $\frac{1}{\alpha(\pi^j(\delta),\pi_0)} \in [0, 1]$. □

The same argument does not apply for $j \in N_J$, so to strengthen the cut (and create a reduce-and-split cut) we look to reduce the size of $|\pi r^j|$ for each $j \in N_J$.

Remark 5.7. The quantity $\epsilon(\pi, \pi_0)$ could also be used to improve the cut but the authors of [2] indicate that this parameter is hard to control for the purposes of modifying $\alpha_j(\pi, \pi_0)$.

5.1.1 Reduce-and-split algorithm

We now specify the reduce-and-split algorithm of [2], whose goal is to produce disjunctions that will give deeper intersection cuts. Let $\Pi^s = \{D(e_i, \lfloor \bar{x}_i \rfloor) \mid i \in B_I\}$ be the starting set of disjunctions, and we note $D(e_i, \lfloor \bar{x}_i \rfloor)$ is violated for all $k \in I$ such that $\bar{x}_k$ is fractional.

1. Set $\bar{\Pi} = \Pi^s$.
2. Modify $\bar{\Pi}$ repeatedly to obtain better disjunctions in the following manner:

   Let $D(\pi, \pi_0)$ and $D(\pi', \pi'_0) \in \bar{\Pi}$; set $\pi(\delta) = \pi + \delta\pi'$ and $\pi_0(\delta) = \lfloor (\pi + \delta\pi')\bar{x} \rfloor$. Now consider $D(\pi(\delta), \pi_0(\delta))$ for $\delta \in \mathbb{Z}$.

   As a way to indicate when $D(\pi(\delta), \pi_0(\delta))$ is better than $D(\pi, \pi_0)$, we introduce a quadratic merit function, $f(\delta) = \sum_{j \in N_J} (\pi(\delta) r^j)^2 = \delta^2 g(\pi') + 2\delta h(\pi, \pi') + g(\pi)$ where $g(\pi) = \sum_{j \in N_J} (\pi r^j)^2$ and $h(\pi, \pi') = \sum_{j \in N_J} (\pi r^j)(\pi' r^j)$. This function is always non-negative and hence is minimized at $f(\delta) = 0$. Restricting ourselves to integer solutions δ, we need only to consider the two values $\delta^* \in \{\lfloor -\frac{h(\pi,\pi')}{g(\pi')} \rfloor, \lceil -\frac{h(\pi,\pi')}{g(\pi')} \rceil\}$.

   We now replace $D(\pi, \pi_0)$ by $D(\pi(\delta^*), \pi_0(\delta^*))$ in $\bar{\Pi}$ if $f(\delta^*) < f(0)$.
3. Strengthen the resulting set of disjunctions $\bar{\Pi}$ via the technique of Lemma 5.1.

A final set of disjunctions can be converted into a set of intersection cuts that are valid for P_I. We use these cuts with the hope that they are deeper than the mixed integer Gomory cuts. Experimental evidence ([2]) indicates that on average this procedure produces cuts which are stronger than mixed integer Gomory cuts; however, there are examples where the cuts produced through this procedure are not as strong as the corresponding mixed integer Gomory cuts. This looks promising as a technique for solving some problems, though it would be more satisfying to have a systematic way of generating cuts that are stronger than mixed integer Gomory cuts. We focus on a technique to do exactly this in the next section.

5.2 Strengthening lift-and-project cuts

We recall that we obtain a lift-and-project cut $\alpha x \geq \beta$ via a solution to the following linear program:

$$\begin{array}{rcccccccccccl}
\min & \alpha\bar{x} & - & \beta & & & & & & & & & \\
\text{subject to} & \alpha & & & - & u\tilde{A} & + & u_0 e_j & & & & & \geq 0 \\
& \alpha & & & & & & & - & v\tilde{A} & - & v_0 e_j & \geq 0 \\
& & - & \beta & + & u\tilde{b} & & & & & & & = 0 \\
& & - & \beta & + & & & & + & v\tilde{b} & + & v_0 & = 0 \\
& & & & & u, & & & & v & & & \geq 0
\end{array}$$

Lift-and-project cutting planes as generated from the above LP are relatively expensive to compute though they do show experimental evidence of generating solutions that are better than solutions found using mixed integer Gomory cutting planes [6]. As a result, we consider the idea of strengthening such a cut via the technique of Balas and Jeroslow [5] to obtain a cut that will be stronger than a mixed integer Gomory cut. To do this, let $\hat{A}$ be the constraint matrix of the constraints of the original problem that are not of the form $x \geq 0$, and let $\hat{b}$ represent the corresponding set of entries of b.

For $k = 1, \ldots, p$, $k \neq j$, let $\alpha_k^1 = \hat{u}\hat{A}_k$, $\alpha_k^2 = \hat{v}\hat{A}_k$, $\alpha_j^1 = \hat{u}\hat{A}_j + u_0$, $\alpha_j^2 = \hat{v}\hat{A}_j + v_0$, and $\beta = \hat{u}\hat{b} = \hat{v}\hat{b} + v_0$. Set $\alpha_k = \max\{\alpha_k^1, \alpha_k^2\}$ and define $m_k = \frac{\alpha_k^2 - \alpha_k^1}{u_0 + v_0}$. We strengthen $\alpha x \geq \beta$ to $\gamma x \geq \frac{\beta}{|\beta|}$, where γ_k is defined as follows:

$$\gamma_k = \begin{cases} \min\{\frac{1}{|\beta|}(\alpha_k^1 + u_0\lceil m_k \rceil\,,\ \frac{1}{|\beta|}(\alpha_k^2 - v_0\lfloor m_k \rfloor)\} & \text{for } k \in I \\ \min\{\frac{1}{|\beta|}\alpha_k^1\,,\ \frac{1}{|\beta|}\alpha_k^2\} & \text{for } k \in J \end{cases}$$

Theorem 5.8 (Balas, Ceria, and Cornuéjols, [6]). *Let* $x_k + \sum_{i \in I'} \bar{a}_{kj} x_j + \sum_{j \in J} \bar{a}_{kj} x_j = \bar{b}_k$ *be a row from a simplex tableau with* $k \in I$ *and assume* $\bar{b}_k \notin \mathbb{Z}$*. The mixed integer Gomory cut for the row is* $\gamma x \geq \frac{\beta}{|\beta|}$ *where* $u'_i = v'_i = 0$ *for* $i \in I$*,* $u_0 = \frac{1}{f}$*,* $v_0 = \frac{1}{1-f}$*, where* $f = \bar{b}_k - \lfloor \bar{b}_k \rfloor$*.*

Proof. Since $\bar{b}_k \in (0,1)$, $\lfloor \bar{b}_k \rfloor = 0$, and so $f = \bar{b}_k$. We compute $\alpha_k^1 = \frac{\bar{a}_{kj}}{f}$, $\alpha_k^2 = \frac{-\bar{a}_{kj}}{1-f}$, $\beta = 1$, $m_k = \frac{\frac{\bar{a}_{kj}}{1-f} - \frac{\bar{a}_{kj}}{f}}{\frac{1}{f} + \frac{1}{1-f}} = -\bar{a}_{kj}$, and

$$\gamma_k = \begin{cases} \min\left\{ \left(\frac{\bar{a}_{kj}}{f} + \frac{1}{f}\lceil -\bar{a}_{kj} \rceil\right), \left(\frac{-\bar{a}_{kj}}{1-f} - \frac{1}{1-f}\lfloor -\bar{a}_{kj} \rfloor\right) \right\} & \text{for } k \in I \\ \max\left\{ \frac{\bar{a}_{kj}}{f}, \frac{-\bar{a}_{kj}}{1-f} \right\} & \text{for } k \in J \end{cases}$$

Consider now the cut $\gamma x \geq \frac{\beta}{|\beta|}$:

$$\begin{aligned} 1 \leq \gamma x &= \sum_{k \in I} \left(\min\left\{ \frac{\bar{a}_{kj} + \lceil -\bar{a}_{kj} \rceil}{f}, \frac{-\bar{a}_{kj} - \lfloor -\bar{a}_{kj} \rfloor}{1-f} \right\} \right) x_k + \sum_{k \in J} \left(\max\left\{ \frac{\bar{a}_{kj}}{f}, \frac{-\bar{a}_{kj}}{1-f} \right\} \right) x_k \\ &= \sum_{k \in I} \left(\min\left\{ \frac{\bar{a}_{kj} - \lfloor -\bar{a}_{kj} \rfloor}{f}, \frac{-\bar{a}_{kj} + \lceil \bar{a}_{kj} \rceil}{1-f} \right\} \right) x_k + \frac{1}{f} \sum_{\substack{k \in J \\ \bar{a}_{kj} \geq 0}} \bar{a}_{kj} x_j - \frac{1}{1-f} \sum_{\substack{k \in J \\ \bar{a}_{kj} < 0}} \bar{a}_{kj} x_k \\ &= \sum_{k \in I} \left(\min\left\{ \frac{f_j}{f}, \frac{1-f_j}{1-f} \right\} \right) x_k + \frac{1}{f} \sum_{\substack{k \in J \\ \bar{a}_{kj} \geq 0}} \bar{a}_{kj} x_j - \frac{1}{1-f} \sum_{\substack{k \in J \\ \bar{a}_{kj} < 0}} \bar{a}_{kj} x_k \\ &= \frac{1}{f} \sum_{\substack{k \in I \\ f_j \leq f}} f_j x_j + \frac{1}{1-f} \sum_{\substack{k \in I \\ f_j > f}} (1 - f_j) x_j + \frac{1}{f} \sum_{\substack{k \in J \\ \bar{a}_{kj} \geq 0}} \bar{a}_{kj} x_j - \frac{1}{1-f} \sum_{\substack{k \in J \\ \bar{a}_{kj} < 0}} \bar{a}_{kj} x_k. \end{aligned}$$

This is the mixed integer Gomory cut. □

While Balas, Ceria, and Cornuéjols [6] give a method to compute their lift-and-project cuts, Balas and Perregaard [7] provide a method to compute cuts that, though slightly different from the cuts presented here, are more efficient to compute. They also characterize the exact relationships between their lift-and-project cuts, split cuts, and mixed integer Gomory cuts. This relationship ensures that these strengthened cuts (if they exist) are in fact stronger than mixed integer Gomory cuts at the cost of being more expensive to compute. Experimental evidence ([7]) indicates that the use of these strengthened cuts does seem to reduce the total computation time for solving many problems in MIPLIB 3.0 [9], a standard library of mixed integer programming problems.

6 Conclusion

This report presented an outline of many families of cutting planes, namely disjunctive cuts, split cuts, mixed integer rounding cuts, intersection cuts, mixed integer Gomory cuts, lift-and-project cuts, and reduce-and-split cuts. The theme of this report was the generation of cutting planes for mixed integer programming problems. In particular, we focused on the mixed integer Gomory cut and families of cuts which may improve upon this family. Two such families, lift-and-project cuts and reduce-and-split cuts, aim to systematically improve on the distance between a fractional solution and the mixed integer Gomory cutting plane at an iteration of a cut procedure. Lift-and-project cuts, in their strengthened form, will yield an improvement if such a cutting plane exists, whereas there is not yet a criterion for determining if or when reduce-and-split cutting planes can guarantee an improvement; however, the techniques presented in section 5 show promise against the MIPLIB 3.0 ([9]) test library of mixed integer programming problems, as indicated in [2], [6], and [7].

Research into cut generation algorithms for mixed integer programming is very active at this time, with many results appearing in the past 15 years. Families of cutting planes such as those presented in this paper will no doubt improve the quality of mixed integer programming solver codes in the years to come.

References

[1] K. Anderson, G. Cornuéjols, Y. Li, "Split closure and intersection cuts," *Mathematical Programming Series A*, Vol. 102, 457-493 (2005).

[2] K. Anderson, G. Cornuéjols, Y. Li, "Reduce-and-split cuts: improving the performance of mixed integer Gomory cuts," to appear in *Management Science*.

[3] E. Balas, "Intersection cuts - a new type of cutting planes for integer programming," *Operations Research*, Vol. 19 19-39 (1971).

[4] E. Balas "Disjunctive Programming," *Annals of Discrete Mathematics*, Vol. 5, 3-51 (1979).

[5] E. Balas and R. Jeroslow, "Strengthening cuts for mixed integer programs," *European Journal of Operations Research* 4(4) 224-234 (1980).

[6] E. Balas, S. Ceria, and G. Cornuéjols, "A lift-and-project cutting plane algorithm for mixed 0-1 programs," *Mathematical Programming*, Vol. 58, 295-324 (1993).

[7] E. Balas, and M. Perregaard, "A precise correspondence between lift-and-project cuts, simple disjunctive cuts, and mixed integer Gomory cuts for 0-1 programming," *Mathematical Programming Series B*, Vol. 94, 221-245 (2003).

[8] R. E. Bixby, M. Fenelon, Z. Guo, E. Rothberg, and R. Wunderling, "Mixed integer programming: a progress report," *The Sharpest Cut - The Impact of Manfred Padberg and His Work*, SIAM, 309-326 (2004).

[9] R. E. Bixby, S. Ceria, C. M. McZeal, M. W. P. Savelsberg, "An updated mixed integer programming library: MIPLIB 3.0," `http://www.caam.rice.edu/~bixby/miplib/miplib.html`.

[10] R. E. Gomory, "An algorithm for the mixed integer problem," RM-2597, The Rand Corporation, 1960.

[11] L. Lovasz and A. Schrijver, "Cones of matrices and set functions and 0-1 optimization," *SIAM Journal on Optimization* 1 (2) 166-190 (1991).

[12] G. L. Nemhauser and L. A. Wolsey, *Integer and Combinatorial Optimization*, Wiley-Interscience, 1988.

[13] G. L. Nemhauser and L. A. Wolsey, "A recursive procedure to generate all cuts for 0-1 mixed integer programs," *Mathematical Programming*, Vol. 46, 379-390 (1990).

[14] H. Sherali and W. Adams, "A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming," *SIAM Journal on Discrete Mathematics* 3 (3) 411-430 (1990).

[15] L. A. Wolsey, *Integer Programming*, Wiley-Interscience, 1998.