Models of generating random regular graphs and their short cycle distribution

by

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An essay
presented to the University of Waterloo
in fulfilment of the
essay requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, Dec 2006

Abstract

Different models of generating random regular graphs have been widely studied, such as the configuration model, the d-process and the d-star process. We will have a brief overview of properties of these models. We will also review properties of Markov chains, and the coupling method, which is usually used to prove the mixing rate of a Markov chain. In this essay, we study a new algorithm, called pegging, to generate random d-regular graphs. We prove that the number of triangles has a limiting Poisson distribution, and estimate the rate at which it approaches its limiting distribution. The method we used is similar to the coupling method, but we will first extend the coupling lemma, so that we can measure the total variation distance between two random processes, instead of two copies of a same Markov chain. Therefore the case we study here is quite different from that we usually see in Markov chains. We conjecture this result also holds for k-cycle for any fixed k, and more precisely, the number of k-cycles, where $k = 3, 4, 5, \ldots$ are asymptotically independent Poisson random variables.

Acknowledgements

I would like to express my deep thanks to my supervisor, Nick Wormald. Without his support and guidance, I could not finish this essay. I would also like to thank my reader, Bruce Richmond, for his comments and careful reading of the draft.

Thanks to the professors, colleagues, and friends in the Department of Combinatorics and Optimization at the University of Waterloo.

Thanks to my parents for their love and everlasting encouragement.

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Chapter 1

Introduction

Random graphs were first introduced by Erdős, in his proof of existence of graphs with arbitrarily large girth and chromatic number. Later on, random regular graphs were introduced and studied by Bender, Canfield, Bollobás, and Wormald. In other areas such as Computer Science, people especially show interests in generating random graphs, or random regular graphs, with a given distribution.

Let $\mathscr{G}_{n,d}$ denote the probability space of all random d-regular graphs on vertex set $[n]$, where nd is even and $[n] := \{1,2,3,\cdots,n-1,n\}$, with uniform distribution. A crucial question is how to generate regular graphs from $\mathscr{G}_{n,d}$ uniformly. Currently there is no known efficient algorithm to generate random d-regular graphs with uniform distribution when d is greater than $O(n^{1/3})$. If $d = O(n^{1/3})$, Mckay and Wormald showed an efficient algorithm, given in [4], which uses "switching" to eliminate loops and double edges. When d is a constant, we can apply the *configuration* model, also called the *pairing* model, which was first given by Bollobás [5].

But using the pairing model for generating graphs becomes slow even for constant d when d is large, because with high probability we will get a multigraph with loops and multiple edges, so we have to abandon this multigraph and start again. This is repeated until we get a simple final graph. It can be shown that the probability of getting a simple d-regular graph from the pairing model is asymptotically $\exp((1-d^2)/4)$. Thus the expected number of times we need to generate graphs until we get a simple final graph is $\exp((d^2-1)/4)$. It becomes slower and slower when d becomes large. Also, the algorithm in [4], though efficient in some senses, is very complicated and has never been implemented. So it is worthwhile to turn to other near-uniform models which generate graphs faster. Quick approaches such as the *d-process* [6], and the *d-star-process* [7] do not generate d-regular graphs uniformly, but a.a.s. end up with a final graph that is d-regular.

One of the direct applications of generating random d-regular graphs is to model random networks. Bourassa and Holt [1] introduced a peer-to-peer network called the SWAN network, in which clients arrive and leave randomly. They suggested to keep the underlying topology a random d-regular graph, which is achieved by using *pegging*, which they call "clothespinning" (for arriving clients), and its reverse (for clients leaving). We will define this pegging operation in Chapter 2. They found that this type of random network has good expected connectivity and other properties. Cooper, Dyer and Greenhill [2, 3] defined a Markov chain on d-regular graphs with randomized size to model the SWAN network. Each move of the Markov chain is by a pegging operation or its reverse. They showed that conditional on the size of the network, the stationary distribution is uniform, and they gave a good estimate for the mixing time of the chain.

The reason that the short cycle distribution attract special interest is that in $\mathscr{G}_{n,d}$, the only possible local structures to appear asymptotically almost surely (a.a.s.) are trees and cycles. For any $S \subset [n]$, such that $\mid S \mid$ is a constant, $G[S]$, the subgraph induced by the vertex set S, is a tree a.a.s. The distribution of the number of k-cycles in $\mathscr{G}_{n,d}$, where $k = 3, 4, 5, \cdots$, is asymptotically independent Poisson. So a question that arises naturally is what the short cycle distribution is in the near-uniform models of generating d-regular random graphs. This essay studies the short cycle distribution in the pegging algorithm.

In Chapter 2, we introduce the different models mentioned above to generate random regular graphs, and we describe the known results concerning the connectivity and the short cycle distribution. We will define the pegging operation, and the pegging algorithm, which is just repeating the pegging operation in each step. We will also introduce Markov chains, and some important parameters and theorems of Markov chains, such as stationary distribution, convergence theorem, and mixing time. We will introduce the coupling lemma, which is a simple but powerful tool to estimate the mixing time. Then we will give some examples of applying Markov chains to sample random structures. In Chapter 3, we study in particular the limiting distribution of the number of short cycles in random d-regular graphs generated by the pegging algorithm. In order to do this, we will first extend the coupling lemma to estimate the rate that a random process goes to its limiting distribution. Results and proofs will also be given in Chapter 3.

Chapter 2

Models of generating random regular graphs

In this chapter, we will introduce different models of generating random graphs, and compare the advantages and disadvantages of each model. We will discuss several graph properties in each model. The one we pay most attention to is the short cycle distribution.

2.1 Uniform models for random graphs

2.1.1 Pairing Model

Let $\mathscr{G}_{n,d}$ denote the probability space of all random d-regular graphs on n vertices with uniform distribution, where nd is even. Suppose we want to generate graphs from $\mathscr{G}_{n,d}$. If d is a constant, we can use the pairing model to sample graphs from $\mathscr{G}_{n,d}$ u.a.r. The pairing model is described as follows.

Consider a set of nd points, where nd is even, and partition them into n buckets with d points in each of them. We take a random matching of the nd points, and then contract the d points in each bucket into a single vertex. The resulting graph is a d-regular multigraph on n vertices.

Let $\mathscr{P}_{n,d}$ denote the probability space of random d-regular multigraphs generated by the pairing model as described above. Every graph in $\mathscr{G}_{n,d}$ corresponds to $(d!)^n$ copies in $\mathscr{P}_{n,d}$, So all simple graphs, namely, graphs without loops and multiple edges, in $\mathscr{P}_{n,d}$ occur u.a.r.

The following result gives the probability of a multigraph from $\mathscr{P}_{n,d}$ being simple.

Theorem 2.1 (Bender and Canfield [12])

$$\mathbf{P}(\mathscr{P}_{n,d} \textit{ is simple}) \sim \exp\left(\frac{1-d^2}{4}\right).$$

When d is large, namely, tending to infinity as n goes to infinity, then $\mathbf{P}$($\mathscr{P}_{n,d}$ is simple) tends to 0. Nevertheless, we can still apply the pairing model in such a case. The generated graph is a.a.s. a multigraph with loops and multiple edges. Then McKay and Wormald [4] used *switching* method to eliminate the loops and multiple edges. The switching method is described as follows. Consider a pair of non-adjacent edges $\{a,b\}$ and $\{c,d\}$ in a d-regular graph, then replacing $\{a,b\}$ and $\{c,d\}$ by $\{a,c\}$ and $\{b,d\}$, or by $\{a,d\}$ and $\{b,c\}$, we get another d-regular graph. This operation is called "switching". They showed that the property that the simple d-regular graphs are distributed u.a.r. is preserved after the switchings. So they gave a polynomial time algorithm of generating random d-regular graphs uniformly by applying pairing model and switching method. They also used the switching method to prove the following theorem.

Theorem 2.2 (McKay [11]). *If $k = o(n^{1/3})$, the number of labelled k-regular graphs on n vertices is*

$$\frac{(nk)!\exp((1-k^2)/4)}{(nk/2)!2^{nk/2}}(1+o(1)).$$

uniformly as $n \to \infty$ with kn even.

McKay and Wormald then used a new type of switching to find the formula for $d = o(n^{1/2})$.

Corollary 2.1 (McKay and Wormald [13]). *For $d = o(n^{1/2})$,*

$$\mathbf{P}(\mathscr{P}_{n,d} \textit{ is simple}) = \exp\left(\frac{1-d^2}{4} - \frac{d^3}{12n} + O\left(\frac{d^2}{n}\right)\right).$$

We can generalize $\mathscr{G}_{n,d}$ to $\mathscr{G}_{n,\mathbf{d}}$, where $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ is an n-vector denoting the degree sequence. Then $\mathscr{G}_{n,\mathbf{d}}$ denotes the uniform probability space of all random graphs with given degree sequence. We can still apply the pairing model to generate graphs from $\mathscr{G}_{n,\mathbf{d}}$. Partition $m = \sum_i^n d_i$ points into n buckets, with d_i points in each bucket, and take a random matching of the m points. Let $d = \max\{d_1, d_2, \ldots, d_n\}$, then similarly we can show that $\mathbf{P}(\mathscr{P}_{n,d} \text{ is simple}) \geq \exp\left(\frac{1-d^2}{4}\right)$. So the pairing model still works efficiently for constant d, especially when d is not large. Since the expected time of generating a simple graph is exponential to d^2, the algorithm will get slow when d gets large. The pairing model is also called the *configuration model*.

The following two theorems give the short cycle distribution in the uniform probability space $\mathscr{G}_{n,d}$, and the pairing model.

Theorem 2.3 (Bollobás [5]).*For d fixed, let $X_i = X_{i,n} (i \geq 1)$ be the number of cycles of length i in the random multigraph coming from $\mathscr{P}_{n,d}$. For fixed $k \geq 1$, $X_1, \ldots, X_k$ are asymptotically independent Poisson random variables with means $\lambda_i = \frac{(d-1)^i}{2i}$.*

Theorem 2.3 explains the following theorem for the short cycle distribution in $\mathscr{G}_{n,d}$.

Theorem 2.4 (Bollobás [5], Wormald [18]). *For d fixed, let $X_i = X_{i,n} (i \geq 3)$ be the number of cycles of length i in a graph in $\mathscr{G}_{n,d}$. For fixed $k \geq 3$, $X_3, \ldots, X_k$ are asymptotically independent Poisson random variables with means $\lambda_i = \frac{(d-1)^i}{2i}$.*

To illustrate the use of switching, here we prove a weaker theorem showing that X_k is asymptotically a Poisson random variable for any fixed $k \geq 3$.

Theorem 2.5 *For any constants d, k fixed, let $X_k = X_{k,n} (k \geq 3)$ be the number of cycles of length k in a graph in $\mathscr{G}_{n,d}$. Then X_k is asymptotically a Poisson random variable with mean $\lambda_k = \frac{(d-1)^k}{2k}$.*

Proof

We first prove that for any fixed integer k, the expected number of k-cycles in $\mathscr{G}_{n,d}$ is bounded. We only need to prove this in $\mathscr{P}_{n,d}$, then it is also true for $\mathscr{G}_{n,d}$, according to Theorem 2.1. Assume S

is a k-vertex set, such that the k vertices are placed in a cyclic order, namely, $v_0, v_1, \ldots, v_{k-1}, v_0$, we regard each vertex v_i as a bucket which contains d points. An edge v_i, v_{i+1} corresponds to a matching from a point in v_i to a point to v_{i+1}. We call this a pair. Let $\{p_1, p_2, \ldots, p_k\}$ be a k-pair, such that the i-th pair matches a point in $v_{i \mod k}$ to $v_{i+1 \mod k}$, and every two pairs are disjoint, namely, no two pairs will share the same point. Clearly, after we contract the points in each bucket, a k-pair in $\mathscr{P}_{n,d}$ corresponds to a k-cycle in a multiple d-regular graph. Let $X_{p_1,p_2,\ldots,p_k}$ be the indicator random variable that $\{p_1, p_2, \ldots, p_k\}$ occurs in the pairing model.So For any given k pairs in the pairing model, the probability for it to occur is $(dn-1-2k)!!/(dn-1)!!$, which is asymptotically $(nd)^{-k}$. So

$$\mathbf{P}(X_{p_1,p_2,\ldots,p_k} = 1) \sim (nd)^{-k}.$$

Let $Q(n,k)$ be the set of all k-pairs in $\mathscr{P}_{n,d}$ We need to count $\mid Q(n,k) \mid$. We will first count the number of k-pairs for any given S. First choose one point in each bucket; there are d^k ways to do that, then for each chosen point in $v_{i \mod k}$, there are $(d-1)^k$ ways to choose its partner in $v_{i+1 \mod k}$. So the number of k-pairs in total is $d^k(d-1)^k$. There are $\binom{n}{k}$ ways to choose a k-vertex set, and for each k -vertex set, there are $(k-1)!$ ways to put it into a cyclic order with an orientation. Thus, $\mid Q_{n,k} \mid = \binom{n}{k}(k-1)!d^k(d-1)^k$. Let

$$N_k = \sum_{\{p_1,p_2,\ldots,p_k\} \in Q(n,k)} X_{p_1,p_2,\ldots,p_k}.$$

Then N_k is the number of k-cycles in the multiple random d-regular graph produced by the pairing model. So

$$\begin{aligned} \mathbf{E}(N_k) &= \sum_{\{p_1,p_2,\ldots,p_k\} \in Q(n,k)} X_{p_1,p_2,\ldots,p_k} \\ &\sim \binom{n}{k}(nd)^{-k}d^k(d-1)^k(k-1)!/2 \\ &\sim (d-1)^k/2k. \end{aligned}$$

So $\mathbf{E}(N_k) < \infty$. Thus the expected number of k-cycles in $\mathscr{G}_{n,d}$ is also bounded, for any fixed k. Let X_k be the number of k-cycles in $\mathscr{G}_{n,d}$, and choose $w(n) = o(n^{1/2})$ to be some function of n that goes to infinity slowly as n goes to infinity, then

$$\mathbf{P}(X_i > w(n)) = o(1) \qquad \text{for any fixed } i. \tag{2.1.1}$$

Let $\mathscr{G}_c$ be the set of graphs in $\mathscr{G}_{n,d}$ with c k-cycles, where $c \geq 0$ is any constant. For any $G \in \mathscr{G}_{c+1}$, the $c+1$ k-cycles are disjoint with probability $1-o(1)$. Choose an edge contained in a k-cycle, then choose another edge which is neither adjacent to the chosen edge, nor adjacent to any other edge that is contained in the same cycle as the chosen edge, nor contained in any k-cycle. Switching these two chosen edges, we create a $(k+1)$-path and a graph $G' \in \mathscr{G}_c$. Then for all $c \leq w(n)$, the number of ways to choose an edge contained in a k-cycle is $k(c+1)$, and the number of ways to choose another edge is $\frac{dn}{2} - O(w(n))$. There are two ways to switch these two chosen edges, so the number of ways to do a switching is $2k(c+1)(\frac{dn}{2} - O(w(n)))$. On the other hand, this procedure is reversible. For any $G \in \mathscr{G}_c$, choose a $(k+1)$-path, such that the end edges are not contained in any k-cycles, nor both contained in a $2k$-cycle, and then switch the end edges of the path. We get a graph $G' \in \mathscr{G}_{c+1}$. The number of such $(k+1)$-paths we can find in G is $\frac{d(d-1)^k n}{2} - O(w(n))$, a.a.s. since only $O(w(n))$ cycles with length at most $2k$ exist in $\mathscr{G}_c$. So

$$\begin{aligned}\frac{|\mathscr{G}_c|}{|\mathscr{G}_{c+1}|} &= \frac{2k(c+1)(dn/2) - O(w(n))}{d(d-1)^k n/2 - O(w(n))} \\ &= \frac{2k(c+1)}{(d-1)^k}\left(1 + O\left(\frac{w(n)}{n}\right)\right).\end{aligned}$$

So

$$\begin{aligned}\frac{|\mathscr{G}_c|}{|\mathscr{G}_0|} &= \prod_{j=0}^{c-1}\left(\frac{(d-1)^k}{2k(j+1)}\left(1+O\left(\frac{w(n)}{n}\right)\right)\right) \\ &= \left(\left(\frac{(d-1)^k}{2k}\right)^c /c!\right)\left(1+O\left(\frac{w(n)}{n}\right)\right)^c \\ &\leq \left(\left(\frac{(d-1)^k}{2k}\right)^c /c!\right)\left(1+O\left(\frac{w(n)}{n}\right)\right)^{w(n)} \\ &= \left(\left(\frac{(d-1)^k}{2k}\right)^c /c!\right)\exp\left(O\left(\frac{w(n)^2}{n}\right)\right) \\ &= \left(\left(\frac{(d-1)^k}{2k}\right)^c /c!\right)(1+o(1)).\end{aligned}$$

For $c > w(n)$, from (2.1.1), we know that

$$\frac{\sum_{c>w(n)} |\mathscr{G}_c|}{\sum_{c\geq 0} |\mathscr{G}_c|} = o(1).$$

Now

$$\begin{aligned}
\mathbf{P}(X_k = 0) &= \frac{|\mathscr{G}_0|}{\sum_{i=0}^{\infty} |\mathscr{G}_i|} \\
&= \frac{|\mathscr{G}_0|}{(1+o(1))\sum_{i=0}^{w(n)} |\mathscr{G}_i|} \\
&= \frac{|\mathscr{G}_0|}{(1+o(1))\sum_{i=0}^{w(n)} |\mathscr{G}_0| \left(\left(\frac{(d-1)^k}{2k}\right)^i /i!\right)(1+o(1))} \\
&= \frac{1}{(1+o(1))\sum_{i=0}^{\infty} \left(\frac{(d-1)^k}{2k}\right)^i /i!} \\
&= \exp\left(-\frac{(d-1)^k}{2k} + o(1)\right).
\end{aligned}$$

For constant $c > 0$,

$$\begin{aligned}
\mathbf{P}(X_k = c) &= \frac{|\mathscr{G}_c|}{\sum_{i=0}^{\infty} |\mathscr{G}_i|} \\
&= \mathbf{P}(X_k = 0)\frac{|\mathscr{G}_c|}{|\mathscr{G}_0|} \\
&= \exp\left(-\frac{(d-1)^k}{2k} + o(1)\right)\left(\frac{(d-1)^k}{2k} + o(1)\right)^c /c!.
\end{aligned}$$

This proves X_k is asymptotically Poisson random variable with mean $\frac{(d-1)^k}{2k}$, for any fixed integer k. ∎

2.1.2 Pseudograph Model

Sometimes people want to generate graphs from the probability space $\mathscr{G}_{n,m}$, in which each graph with n vertices and m edges appears with the same probability. Chvátal's random pseudograph model is useful to generate and analyse graphs in $\mathscr{G}_{n,m}$. It is defined as follows. Given positive integers n and m, define $\mathscr{F}(n,m)$ to be the set of functions $f : [2m] \to [n]$, such that each $f \in \mathscr{F}(n,m)$ takes the same probability measure. We can define $\mathscr{C}(n,m)$, the probability space of pseudograhs, as follows. A random pseudograph corresponding to f, called $G(f)$, is defined on the vertex set $[n]$, to have edge set $\{(u,v) \mid u = f(2i-1), v = f(2i) : \quad \text{for all } 1 \le i \le m\}$. Then

$$\mathscr{C}(n,m) = \{G(f), \quad \text{for all } f \in \mathscr{F}(n,m), \text{ where each } f \text{ takes the same probability measure.}\}$$

Theorem 2.6 (Chvátal). *Simple graphs are equiprobable in $\mathscr{C}(n,m)$, and for $G \in \mathscr{C}(n,m)$ and $c = 2m/n$,*

$$\mathbf{P}(G \textit{ is simple}) = \frac{\binom{N}{m} m! 2^m}{n^{2m}} = \exp(-c/2 - c^2/4) + o(1)$$

where $N = \binom{n}{2}$.

Proof:

Consider $\mathscr{F}(n,m)$, the set of functions maps from $[2m]$ to $[n]$. Clearly $|\ \mathscr{F}(n,m)\ | = n^{2m}$. Consider a simple graph G, with n vertices and m edges. Every such graph G, corresponds to $m!2^m$ number of $f \in \mathscr{F}(n,m)$, since there are $m!$ ways to place the order of the m edges, and for each ordering, there are 2^m ways to place the order of the end-vertices of each edge. Finally, there are $\binom{N}{m}$ ways to choose m edges from an n-vertex graph. So $\mathbf{P}(G \text{ is simple}) = \frac{\binom{N}{m} m!2^m}{n^{2m}}$. Let $K = \frac{\binom{N}{m} m!2^m}{n^{2m}}$. Then since $N = n(n-1)/2 = \frac{n^2}{2}(1-\frac{1}{n})$, and $m = cn/2$,

$$\begin{aligned}
K &= \frac{\binom{N}{m} m!2^m}{n^{2m}} = \frac{[N]_m 2^m}{n^{2m}} \\
&= \frac{N(N-1)\dots(N-m+1)2^m}{n^{2m}} \\
&= \frac{n^2(1-1/n)(n^2(1-1/n)-2)(n^2(1-1/n)-4)\dots(n^2(1-1/n)-cn+2)}{n^{2m}} \\
&= \prod_{i=0}^{cn/2-1}\left(1-\frac{1}{n}-\frac{2i}{n^2}\right).
\end{aligned}$$

So

$$\begin{aligned}
\log K &= \sum_{i=0}^{cn/2-1} \log\left(1-\frac{1}{n}-\frac{2i}{n^2}\right) \\
&= \sum_{i=0}^{cn/2-1} -\left(\frac{1}{n}+\frac{2i}{n^2}+O(\frac{1}{n^2})\right) \\
&= -\frac{1}{n}\frac{cn}{2}-\frac{2}{n^2}\frac{cn}{4}\left(\frac{cn}{2}-1\right)+O(\frac{1}{n}) \\
&= -\frac{c}{2}-\frac{c^2}{4}+o(1).
\end{aligned}$$

Thus

$$K = \exp\left(-\frac{c}{2} - \frac{c^2}{4} + o(1)\right)$$
$$= \exp\left(-\frac{c}{2} - \frac{c^2}{4}\right)(1+o(1)).$$

So, $\mathbf{P}(G \text{ is simple}) = K = \exp(-c/2 - c^2/4) + o(1)$. ∎

2.2 Non-uniform models for random regular graphs

The problem of pairing model is that generating graphs by pairing model becomes slow when d is large. However, for large constant d, we can generate graphs by using near-uniform models, which is much faster than the pairing model. In this section, we discuss some of them.

2.2.1 d-process

This process applies when d is a constant, and starts from $G_0 = \overline{K_n}$. Given G_t, choose u.a.r. a pair of non-adjacent vertices u and v, whose degrees are both less than d, and set $G_{t+1} := G_t + \{uv\}$. The process stops when no such vertex pair exists. The following theorem explains why the algorithm of generating d-regular graphs by using the d-process is fast.

Theorem 2.7 (Ruciński and Wormald [9]) *For fixed $d \geq 1$, in a random d-process, the final graph is d-regular a.a.s.*

Since the final graph is d-regular almost surely, we can generate random d-regular graphs by running a random d-process.

The following theorem shows the connectivity property of graphs generated in a random d-process.

Theorem 2.8 (Ruciński and Wormald [9]) *For fixed $d \geq 3$, the final graph is connected a.a.s.*

The analysis of d-process is complicated. The difficulty is discussed in [6]. Ruciński and Wormald gave the following result for short cycle distribution in a 2-process.

Theorem 2.9 (Ruciński and Wormald [15]) *In a 2-process, let $l \geq 3$ be fixed. The number of l-cycles in G_n is asymptotically Poisson. For $l = 3$, the mean converges to*

$$\frac{1}{2}\int_0^\infty \frac{(\log(1+x))^2 dx}{xe^x} \approx 0.188735349357788830.$$

2.2.2 d-star process

This process is defined when d is a constant, and starts from $G_0 = \overline{K_n}$. Given G_t, choose u.a.r. a vertex u with minimum degree in G_t, choose u.a.r. $k = d - d(u)$ vertices, $u_1, u_2, \ldots, u_k$, which are non-adjacent to u, and with degrees strictly less than d. Set $G_{t+1} := G_t + \{uu_1, uu_2, \ldots, uu_k\}$. The process stops when for some u of minimum degree chosen, there are less than $d - d(u)$ other vertices which are non-adjacent to u, and with degree less than d.

We have similar connectivity property of the final graph in a random d-star process as in the random d-process.

Theorem 2.10 (Greenhill, Ruciński and Wormald [10]) *For fixed $d \geq 3$, the final graph is connected a.a.s.*

Fox fixed d which is large enough, we have the higher connectivity property for the final graph.

Theorem 2.11 (Greenhill, Ruciński and Wormald [10]) *For fixed d large enough, the final graph is d-connected a.a.s.*

Robalewska and Wormald showed that the final graph of random d-star process is d-regular asymptotically almost surely. For more details, see [7].

2.2.3 Pegging operation

We define a new algorithm, called the *pegging algorithm*, which simply repeats a random operation we call pegging, to generate random d-regular graphs, where d is a constant. We define the pegging operation on a d-regular graph, where d is even, as follows.

Pegging Operation. Input: (G,d), Output: $\tilde{G}$

1: Let $c := d/2$. choose c non-adjacent edges in $E(G)$ u.a.r., denoted by $E_1 = \{u_1u_2, u_3u_4, \ldots, u_{2c-1}u_{2c}\} \subset E$.

2: $G := (G \backslash E_1) \cup \{u\} \cup E_2$, where u is a new vertex added in G, and $E_2 = \{uu_1, uu_2, uu_3, \ldots uu_{2c-1}, uu_{2c}\}$.

3: $\tilde{G} := G$. End.

See Figure 2.1 as an example of pegging operation with $d = 4$. The pegging algorithm starts from a d-regular graph G_0, for example, K_{d+1}, and repeatedly apply the pegging operations. It is easy to check that the graph resulting from taking the pegging operations defined above is still d-regular. Then G_t is defined to be the graph resulting from t successive pegging operations on G_0, and G_t contains $n_t = n_0 + t$ vertices.

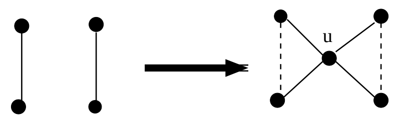

Figure 2.1: *Pegging operation when $d = 4$*

We will present results of our examination of the short cycle distribution of these graphs in Chapter 3.

2.3 Sampling random structures

Markov chains are useful to sample random structures. Usually a Markov chain is well designed such that the states in the chain represents the random structures we want to sample, and the stationary distribution of the chain is the same as the distribution from which we are going to sample the random structures. So we will wait until this chain is close to its stationary distribution and then we can sample the random structures. In this section, we will discuss this method.

2.3.1 Preliminaries for Markov chains

A *Markov chain* is a random process $\{X_t\}_{t\geq 0}$, such that the distribution of X_{t+1} is determined only by X_t, the previous step. Formally,

$$\mathbf{P}(X_{t+1} = x | X_t = x_t, \ldots, X_1 = x_1, X_0 = x_0) = \mathbf{P}(X_{t+1} = x | X_t = x_t).$$

Define the *transition probability matrix* $\mathbf{P}$ as follows. The entry p_{ij} of $\mathbf{P}$ is the probability of going from state i to state j in a single step

$$p_{ij} = \Pr(X_1 = j \mid X_0 = i).$$

Then the n-step transition probabilities are

$$p_{ij}^{(n)} = \mathbf{P}(X_n = j \mid X_0 = i).$$

Thus n-step transition probability matrix is $\mathbf{P}^{(n)} = \mathbf{P}\mathbf{P}\ldots\mathbf{P}$ which is a n-fold matrix product.

The n-step transition satisfies the Chapman-Kolmogorov equation,

$$p_{ij}^{(n)} = \sum_{r\in\Omega} p_{ir}^{(k)} p_{rj}^{(n-k)} \qquad \text{for any } 0 < k < n.$$

A state j is said to be *accessible* from state i if there exits $t \geq 0$, such that

$$\mathbf{P}(X_t = j | X_0 = i) > 0.$$

A *communicating class* C is a set of states such that any two states in C are accessible from each other. A Markov chain is said to be *irreducible* if its state space is a communicating class. This means that it is possible to get to any state from any state in an irreducible Markov chain.

A state i has period k if starting from state i, any return to state i must occur in some multiple of k time steps. Formally, the period of a state is defined as

$$k = \gcd\{n : \mathbf{P}(X_n = i | X_0 = i) > 0\}.$$

If $k = 1$, then the state is said to be *aperiodic*; otherwise, the state is said to be periodic with period k. It can be shown that every state in a communicating class must have the same period.

An irreducible Markov chain is said to be *ergodic* if its states are aperiodic. A Markov chain is ergodic if it is irreducible and aperiodic. The vector π is a stationary distribution if $\sum_{j \in \Omega} \pi_j = 1$ and satisfies $\pi_j = \sum_{i \in \Omega} \pi_i p_{ij}$ for all j.

Theorem 2.12 (The convergence theorem [16]). *If an irreducible Markov chain has a stationary distribution π, then this stationary distribution is unique. Moreover, if this Markov chain is also ergodic, then $\mathbf{P}^t(i, j) \to \pi(j)$ as $t \to \infty$, for all $i, j \in \Omega$.*

Consider a random walk on a finite graph $G = (V, E)$, where the walk starts at vertex v_0. At each step t, assuming the walk is at position v_t, it will go to one of the neighbors of v_t at step $t+1$, with probability $1/d(v_t)$. The sequence $\{v_0, v_1, v_2, \ldots\}$ is a Markov chain with stationary distribution $\pi(j) = d(v_j)/ \mid E \mid$. If G is non-bipartite, then this Markov chain is ergodic, and the distribution of v_t converges to the stationary distribution.

Also consider the pegging operation defined in section 2.2.3. Let's start from a d-regular graph G_0, where d is even, and assume at step t, we have graph G_t, then we get graph G_{t+1} by taking a pegging operation, or a reverse operation of pegging. The sequence $\{G_0, G_1, G_2, \ldots\}$ is a Markov chain. Cooper, Dyer and Greenhill have studies this chain in [2], in which they define this chain to model the SWAN network. They proved that the distribution of G_t, conditional on certain size of G_t, will converges to the uniform distribution.

The graph sequences $\{G_t\}_{t\geq 0}$ generated from d process, d-star process, and pegging algorithms are also Markov chains, but these Markov chains are a bit special, since no state in the chain will ever recur once it has occurred. Thus they have no stationary distributions and do not have a convergence property.

While generating graphs from a Markov chain, people usually wait for enough time till the current distribution is close to the stationary distribution of the Markov chain. This brings us to the problem of mixing time.

2.3.2 Mixing time

A finite state ergodic Markov chain has a unique stationary distribution π, and the distribution of the chain at time t converges to π as t tends to infinity, no matter which initial state it starts. Mixing time refers to the idea of how large t should be to guarantee that the distribution at time t is approximately π. We define the total variation distance to be the distance between the current distribution of the chain, denoted as σ_{X_t}, and the stationary distribution π to be

$$d_{TV}(\sigma_{X_t}, \pi) = \max_{A\subset\Omega}\{\mathbf{P}(X_t \in A) - \pi(A)\}. \tag{2.3.1}$$

then the mixing time of the chain is defined to be the following function τ of some small positive constant ε, where the total variation distance is at most τ.

$$\tau(\varepsilon) = \min_t\{d_{TV}(\sigma_{X_t}, \pi) \leq \varepsilon\}.$$

Tools for proving mixing rate include conductance and multicommodity flow [20], and the method of coupling and path coupling [20, 19]. In broader uses of the Markov chain Monte Carlo method, rigorous justification of simulation results would require a theoretical bound on mixing time.

We now have a more detailed look at the coupling method, since we will use it in later in Chapter 3.

Let M be a Markov chain on a finite state space Ω. A coupling of M is a random process (X_t, Y_t) on Ω^2 such that each of X_t, Y_t is marginally a copy of M, namely

$$\sum_{\sigma_2} \mathbf{P}\Big((X_{t+1}, Y_{t+1}) = (\sigma_1, \sigma_2) \mid (X_t, Y_t) = (\omega_1, \omega_2)\Big) = \mathbf{P}(X_{t+1} = \sigma_1 \mid X_t = \omega_1) \qquad \text{for all } \omega_2 \in \Omega$$

$$\sum_{\sigma_1} \mathbf{P}\Big((X_{t+1}, Y_{t+1}) = (\sigma_1, \sigma_2) \mid (X_t, Y_t) = (\omega_1, \omega_2)\Big) = \mathbf{P}(Y_{t+1} = \sigma_2 \mid Y_t = \omega_2) \qquad \text{for all } \omega_1 \in \Omega$$

Then Doeblin [14] gave the following powerful coupling lemma.

Lemma 2.1 (Coupling Lemma) *Let X_t, Y_t be a coupling for* M *such that Y_0 has the stationary distribution π. Then if X_t has distribution p_t,*

$$d_{TV}(p_t, \pi) \leq \mathbf{P}(X_t \neq Y_t).$$

Proof Let $A_t \subset \Omega$ maximizes in (3.1.1). Then Y_t has stationary distribution as Y_0.

$$\begin{aligned} d_{TV}(p_t, \pi) &= \mathbf{P}(X_t \in A_t) - \mathbf{P}(Y_t \in A_t) \\ &\leq \mathbf{P}(X_t \in A_t) - \mathbf{P}(X_t = Y_t \wedge X_t \in A_t) \\ &= \mathbf{P}(X_t \in A_t) + \mathbf{P}(X_t \neq Y_t \vee X_t \notin A_t) - 1 \\ &\leq \mathbf{P}(X_t \in A_t) + \mathbf{P}(X_t \neq Y_t) + \mathbf{P}(X_t \notin A_t) - 1 \\ &= \mathbf{P}(X_t \neq Y_t). \quad \blacksquare \end{aligned}$$

Chapter 3

Short cycles distribution in random d-regular graphs generated by pegging

3.1 Main Result

In this chapter, we study the random d-regular graphs produced by the pegging algorithm, for even d. To present our methods, we focus on $d = 4$ in particular. The algorithm starts from a 4-regular graph G_0 with n_0 vertices. As the general definition of pegging operation above, at each step, choose randomly two non-adjacent edges. Delete these two edges, and create a new vertex and then join this vertex to the four end vertices of the deleted edges. So G_t contains $n_t = n_0 + t$ vertices and $2n_t$ edges.

Let $Y_{t,k}$ denote the number of k-cycles in G_t. For simplicity, we let Y_t to be $Y_{t,3}$, since we will first study the distribution of triangles in G_t as t goes to infinity. Then we generalize the analysis to the distribution of the number of k-cycles for any fixed k.

Note that initially, the number of triangles might be as big as $2n_0$. However, as we will show later, in such an extreme case, $E(Y_t)$ will decrease quickly in the early stage of the algorithm. Our first lemma will show that the expected number of triangles will be bounded above by some constant C, for large enough C, which is independent of n_0, after sufficiently many steps.

Lemma 3.1 *Given constant C, such that C is sufficiently large, there exists* $\tau_3 = \tau_3(n_0, Y_0, C)$*, such that* $\mathbf{E}(Y_t) \le C$*, for all* $t \ge \tau_3$.

Lemma 3.2 *For any* $t > \tau_3^{4/3}$,

$$\mathbf{E}(Y_t) = 3 + O\left(n_t^{-3/4}\right).$$

Lemma 3.3 *Given sufficiently large constant C, there exists* $\tau_k = \tau_k\left(n_0, Y_{0,k}, C\right)$*, such that* $\mathbf{E}\left(Y_{t,k}\right) < C$*, provided* $t > \tau_k$*, for all* $k \ge 3$*. More precisely, for any* $t \ge \tau_k^{4/3}$,

$$\mathbf{E}\left(Y_{t,k}\right) = \frac{3^k - 9}{2k} + O\left(n_t^{-3/4}\right).$$

We will show that $Y_{t,k}$ has a limiting distribution, as $t \to \infty$, which is Poisson with mean $\frac{3^k-9}{2k}$. Let $\sigma_{t,k}$ be the distribution of $Y_{t,k}$, let π be any distribution. Then the *total variation distance* between $\sigma_{t,k}$ and π is defined as

$$d_{TV}\left(\sigma_{t,k}, \pi\right) = \max_{A_t \subset \mathbf{N}^+ \cup \{0\}} \left\{\sigma_{t,k}(A_t) - \pi(A_t)\right\}. \tag{3.1.1}$$

A similar definition of total variation distance is

$$d_{TV}\left(\sigma_{t,k}, \pi\right) = \frac{1}{2} \sum_{x \in \mathbf{N}^+ \cup \{0\}} \mid \sigma_{t,k}(x) - \pi(x) \mid.$$

The standard definition of the mixing time $\tau(\varepsilon)$ in Markov chain with probability space Ω has been given in Chapter 2. But the random process $\{Y_t\}_{t \ge 0}$ is not a Markov chain, since the distribution of Y_t depends not only Y_{t-1}, but in fact on the underlying graph G_{t-1}. However, we can define *pseudo-mixing time* quite similarly to measure how fast the distribution of $Y_{t,k}$ is getting close to its limiting distribution π_k^*, if π_k^* exists. We define the pseudo-mixing time of the sequence $\{\sigma_{t,k}\}_{t \ge 0}$ to be

$$\tau^*(\varepsilon) = \min\{T \ge 0 : \; d_{TV}\left(\sigma_{t,k}, \pi_k^*\right) \le \varepsilon \quad \text{for all } t \ge T\ \}. \tag{3.1.2}$$

Note that the asterisk on π and τ is just to distinguish them from the usual stationary distribution and mixing time.

We can prove that, as ε goes to 0, the pseudo-mixing time is at most $O(1/\varepsilon)$. More specifically, we get the following theorem. Currently, we have only proved results for triangles, but we conjecture the following two theorems hold for any fixed k as well. The theorems are written in a general form.

Theorem 3.1 *For $k = 3$,*

$$\lim_{t\to\infty} \mathbf{Pr}\left(Y_{t,k} = c\right) = e^{-\mu_k} \frac{\mu_k^c}{c!}.$$

where $\mu_k = \frac{3^k-9}{2k}$.

The pseudo-mixing time satisfies $\tau_k^*(\varepsilon) = O\left(\varepsilon^{-1}\right)$.

This result can be extended to random d-regular graphs generated by pegging operations, where d is any even integer. Let $Y_{t,d}$ be the number of triangles, and $Y_{t,d,k}$ be the number of k-cycles in G_t, where G_t is a random d-regular graph generated by pegging operations.

Theorem 3.2 *Let G_t be a random d-regular graph generated by pegging operations. Given sufficiently large constant C, there exists $\tau_{d,k} = \tau_{d,k}\left(n_0, Y_{0,d,k}, C\right)$, such that $\mathbf{E}\left(Y_{t,d,k}\right) < C$, provided $t > \tau_{d,k}$, for all $k \geq 3$. More precisely, for any $t \geq \tau_k^{4/3}$,*

$$\mathbf{E}\left(Y_{t,d,k}\right) = \frac{(d-1)^k - (d-1)^2}{2k} + O\left(n_t^{-3/4}\right).$$

For $k = 3$, the limiting distribution of $Y_{t,d,k}$ is derived similarly as follows

$$\lim_{t\to\infty} \mathbf{P}\left(Y_{t,d,k} = c\right) = e^{-\mu_{d,k}} \frac{\mu_{d,k}^c}{c!} \qquad \text{where } \mu_{d,k} = \frac{(d-1)^k-(d-1)^2}{2k}.$$

The pseudo-mixing time satisfies $\tau_{d,k}^*(\varepsilon) \leq O\left(\varepsilon^{-1}\right)$.

We can extend coupling lemma to measure the total variation distance between two distributions on two different Markov chains, or even two different random processes.

Let $\{X_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ be two random processes defined on the same sample space Ω_t, and the same σ-field in each step t. Then a coupling of $\{X_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ is a random process (X_t, Y_t) on Ω_t^2 such that

$$\sum_{\sigma_2} \mathbf{P}\Big((X_{t+1}, Y_{t+1}) = (\sigma_1, \sigma_2) \mid (X_t, Y_t) = (\omega_1, \omega_2)\Big) = \mathbf{P}(X_{t+1} = \sigma_1 \mid X_t = \omega_1) \qquad \text{for all } \omega_2 \in \Omega_t$$

$$\sum_{\sigma_1} \mathbf{P}\Big((X_{t+1}, Y_{t+1}) = (\sigma_1, \sigma_2) \mid (X_t, Y_t) = (\omega_1, \omega_2)\Big) = \mathbf{P}(Y_{t+1} = \sigma_2 \mid Y_t = \omega_2) \qquad \text{for all } \omega_1 \in \Omega_t$$

We derive this following extended coupling lemma.

Lemma 3.4 (Extended Coupling Lemma) $\{X_t\}_{t\geq 0}$ *and* $\{Y_t\}_{t\geq 0}$ *are two random processes in the same probability space* Ω_t. *Let* $\sigma_{X,\,t}$ *and* $\sigma_{Y,\,t}$ *denote the distribution of* X_t *and* Y_t *in step* t *respectively, and let* $\{(X_t, Y_t)\}_{t\geq 0}$ *be a coupling. Then*

$$d_{TV}(\sigma_{X,\,t}, \sigma_{Y,\,t}) \leq \mathbf{P}(X_t \neq Y_t).$$

The proof is almost the same as that of the coupling lemma.

3.2 Proofs

Proof of Lemma 3.1

Our analysis is now based on the underlying graph G_t at step $t+1$. We need to choose two non-adjacent edges e_1 and e_2 to do a pegging operation. There is $2n_t$ choices for e_1, and $2n_t - 7$ choices for e_2, which are non-adjacent to e_1. So the ways to choose an ordered pair (e_1, e_2) is $2n_t(2n_t - 7)$. So the total number of ways to do a pegging operation at step $t+1$ is

$$N = \frac{2n_t(2n_t - 7)}{2} = n_t(2n_t - 7). \tag{3.2.1}$$

Consider creating a new triangle. Given an edge e of G_t not in a triangle, a new triangle is created containing e if and only if we peg two edges x and y both adjacent to e, but incident with different end-points of e. Since G_t is 4-regular, the number of ways to choose x and y is precisely 9. So the expected number of new triangles created should be at least $9(2n_t - 3Y_t)/N$ where N is the number of ways to do the pegging operation. And $9 \cdot 2n_t/N$ is obvious an upper bound.

To destroy a triangle, one edge is in the triangle, and there are $2n_t - 7$ choices for another edge to be pegged. So for each triangle in G_t, the probability for it to be destroyed is $3(2n_t - 7)/N$, and thus the expected number of existing triangles being destroyed is $3Y_t(2n_t - 7)/N = 3Y_t/n_t$.

It follows that, the expected value of $Y_{t+1} - Y_t$, given G_t, satisfies

$$\frac{18}{2n_t - 7} - \frac{3Y_t}{n_t}\left(1 + \frac{9}{2n_t - 7}\right) \leq \mathbf{E}(Y_{t+1} - Y_t \mid G_t) \leq \frac{18}{2n_t - 7} - \frac{3Y_t}{n_t}. \tag{3.2.2}$$

Thus

$$\mathbf{E}(Y_{t+1} - Y_t \mid G_t) = \frac{9 - 3Y_t}{n_t} + O\left(\frac{1 + Y_t}{n_t^2}\right). \tag{3.2.3}$$

By taking expectation of both sides of (3.2.2) we get

$$\begin{aligned} \mathbf{E}(Y_{t+1}-Y_t) &\leq \frac{18}{2n_t-7}-\frac{3\mathbf{E}(Y_t)}{n_t} \\ &\leq -\frac{3}{n_t}\mathbf{E}(Y_t)+\frac{9+\alpha}{n_t} \end{aligned}$$

for some $\alpha = O(1/n_0)$, and for all $t \geq 0$.

Iteratively,

$$\begin{aligned} \mathbf{E}(Y_t) &\leq \left(1-\frac{3}{n_{t-1}}\right)\mathbf{E}(Y_{t-1})+\frac{9+\alpha}{n_{t-1}} \\ &\leq \mathbf{E}(Y_0)\prod_{i=0}^{t-1}\left(1-\frac{3}{n_i}\right)+\sum_{i=0}^{t-1}\frac{9+\alpha}{n_i}\prod_{j=i+1}^{t-1}\left(1-\frac{3}{n_j}\right) \\ &\leq Y_0\exp\left(-\sum_{i=0}^{t-1}\frac{3}{n_i}\right)+(9+\alpha)\sum_{i=0}^{t-1}\exp\left(-\sum_{j=i+1}^{t-1}\frac{3}{n_j}\right)\frac{1}{n_i} \\ &\leq Y_0\exp\left(-3\left(\log n_t-\log n_0\right)\right) \\ &\quad +(9+\alpha)\sum_{i=0}^{t-1}\left(\exp\left(\log n_t-\log n_{i+1}\right)\frac{1}{n_i}\right) \\ &= \left(\frac{n_0}{n_t}\right)^3 Y_0+(9+\alpha)\sum_{i=0}^{t-1}\frac{1}{n_i}\frac{n_{i+1}^3}{n_t^3}. \end{aligned}$$

Since $\frac{n_0+i+1}{n_0+i} < \beta = 1+O(1/n_0)$, so $(9+\alpha)\beta < 9+\alpha'$, for some $\alpha' = O(1/n_0)$.

So

$$\begin{aligned} \mathbf{E}(Y_t) &\leq \left(\frac{n_0}{n_t}\right)^3 Y_0+\frac{9+\alpha'}{n_t^3}\sum_{i=0}^{t-1}n_i^2 \\ &\leq \left(\frac{n_0}{n_t}\right)^3 Y_0+\frac{9+\alpha'}{n_t^3}\left(\frac{1}{3}\left(n_t^3-n_0^3\right)\right) \\ &\leq \left(\frac{n_0}{n_t}\right)^3 Y_0+\frac{9+\alpha'}{3}\left(1-\left(\frac{n_0}{n_t}\right)^3\right) \\ &\leq \left(\frac{n_0}{n_t}\right)^3 Y_0+\frac{9+\alpha'}{3}. \end{aligned}$$

The first term tends to 0 as $t \to \infty$.

So, as C sufficiently large, $\mathbf{E}(Y_t)$ will be bounded above by C, for all $t \geq \tau_3$, where τ_3 depends only on n_0, Y_0, and C. ∎

Proof of Lemma 3.2

From (3.2.2),

$$\frac{18}{2n_t - 7} - \frac{3Y_t}{n_t}\left(1 + \frac{9}{2n_t - 7}\right) \leq \mathbf{E}(Y_{t+1} - Y_t \mid Y_t) \leq \frac{18}{2n_t - 7} - \frac{3Y_t}{n_t}.$$

$Y_t \geq 0$ for all $t \geq 0$.

Then for some $\alpha_1 = O(1/n_{t_0})$

$$\mathbf{E}(Y_{t+1} - Y_t \mid Y_t) \leq \frac{9 + \alpha_1}{n_t} - \frac{3Y_t}{n_t} \qquad \text{for all } t > t_0.$$

By taking expectation of both sides,

$$\mathbf{E}(Y_{t+1} - Y_t) \leq \frac{9 + \alpha_1}{n_t} - 3\frac{\mathbf{E}(Y_t)}{n_t}.$$

Iteratively, we obtain

$$\begin{aligned}
\mathbf{E}(Y_t) &\leq \left(1 - \frac{3}{n_{t-1}}\right)\mathbf{E}(Y_{t-1}) + \frac{9 + \alpha_1}{n_{t-1}} \\
&\leq \mathbf{E}(Y_{t_0})\exp\left(-\sum_{i=t_0}^{t-1}\frac{3}{n_i}\right) + (9 + \alpha_1)\sum_{i=t_0}^{t-1}\exp\left(-\sum_{j=i+1}^{t-1}\frac{3}{n_j}\right)\frac{1}{n_i} \\
&\leq \mathbf{E}(Y_{t_0})\left(\frac{n_{t_0}}{n_t}\right)^3 + (9 + \alpha_1)\sum_{i=t_0}^{t-1}(n_{i+1} - n_t)^3 \\
&\leq \mathbf{E}(Y_{t_0})\left(\frac{n_{t_0}}{n_t}\right)^3 + \frac{9 + \alpha_1'}{n_t^3}\sum_{i=t_0}^{t-1} n_i^2 \\
&\leq \mathbf{E}(Y_{t_0})\left(\frac{n_{t_0}}{n_t}\right)^3 + \frac{9 + \alpha_1'}{3}
\end{aligned}$$

for some $\alpha_1' = O(1/n_{t_0})$.

Similarly we can also get the lower bound from (3.2.2), such that

$$\mathbf{E}(Y_t) \geq \mathbf{E}(Y_{t_0})\left(\frac{n_{t_0}}{n_t}\right)^3 + \frac{9 + \beta_1'}{3}$$

for some $\beta_1' = O(1/n_{t_0})$.

So, we obtain

$$\mathbf{E}(Y_t) = 3 + \mathbf{E}(Y_{t_0}) \left(\frac{n_{t_0}}{n_t}\right)^3 + O(1/n_{t_0}).$$

By Lemma 3.1, $\mathbf{E}(Y_t) < C$ for some sufficiently large constant C, where $t \geq \tau_3 = \tau_3(n_0, Y_0, C)$. So for any $t_0 \geq \tau_3$, $\mathbf{E}(Y_{t_0}) < C$.

Thus we get

$$\mathbf{E}(Y_t) = 3 + O(1/n_{t_0}) + O\left(\left(\frac{n_{t_0}}{n_t}\right)^3\right) \qquad \text{for all } t \geq t_0 \geq \tau_3.$$

Choose $n_{t_0} = n_t^{3/4}$, then Lemma 3.2 follows.

Proof of Lemma 3.3

We prove that $\mathbf{E}\left(Y_{t,k}\right) = \frac{3^k-9}{2k} + O_k(1/n_{t_0}) + O_k\left(\left(\frac{n_{t_0}}{n_t}\right)^3\right)$ for all $t \geq t_0 \geq \tau_k$.

It is true for $k = 3$, from Lemma 3.2. Assume it is also true of all integers smaller than k, namely, for any integer $i < k$, there exists constant A_i, and B_i, depending only on i, such that

$$\mathbf{E}(Y_{t,i}) \leq \frac{3^i - 9}{2i} + \frac{A_i}{n_{t_0}} + B_i \left(\frac{n_{t_0}}{n_t}\right)^3.$$

The number of k-paths in G_t starting from a fixed vertex v is at most $4 \cdot 3^{k-1}$, so the number of k-paths in G_t is at most $4 \cdot 3^{k-1} n_t/2$. There are $\sum_{i=1}^k Y_{t,i}$ cycles of size at most k in G_t. So delete an edge in each of those cycles. We need to delete at most $\sum_{i=1}^k Y_{t,i}$ edges, each edges contributes to at most $k3^{k-1}$ $k-cycles$. So G_t contains at least $4 \cdot 3^{k-1} n_t/2 - k3^{k-1} \sum_{i=1}^k Y_{t,i}$ different k-paths. There are two ways to create a k-cycle. The first one is to choose the two end edges of a k-path and do the pegging operation. The probability to do that is at most

$$\frac{2 \cdot 3^{k-1}}{2n_t} \left(1 + O\left(\frac{1}{n_t}\right)\right)$$

and at least

$$\frac{2 \cdot 3^{k-1} n_t - k3^{k-1} \sum_{i=1}^k Y_{t,i}}{2n_t^2} \left(1 + O\left(\frac{1}{n_t}\right)\right).$$

The other way is to do pegging on two non-adjacent edges such that one of them is contained in a $(k-1)$-cycle. The probability to do this is $(k-1)Y_{t,k-1}(2n_t-7)/N = (k-1)Y_{t,k-1}/n_t$. Let $Z_{t,k}$ denote the number of new $k-cycles$ from G_t to G_{t+1} created is

$$\begin{aligned}
\mathbf{E}\left(Z_{t,k} \mid G_t\right) &\geq \frac{2\cdot 3^{k-1}n_t - k3^{k-1}\sum_{i=1}^{k} Y_{t,i}}{2n_t^2}\left(1+O\left(\frac{1}{n_t}\right)\right)+\frac{(k-1)\,Y_{t,k-1}}{n_t}\\
\mathbf{E}\left(Z_{t,k} \mid G_t\right) &\leq \frac{3^{k-1}}{n_t}\left(1+O\left(\frac{1}{n_t}\right)\right)+\frac{(k-1)\,Y_{t,k-1}}{n_t}.
\end{aligned}$$

So,

$$\begin{aligned}
\mathbf{E}\left(Z_{t,k}\right) &\geq \frac{2\cdot 3^{k-1}n_t - k3^{k-1}\left(\sum_{i=1}^{k-1}\mathbf{E}(Y_{t,i})+\mathbf{E}\left(Y_{t,k}\right)\right)}{2n_t^2}\left(1+O\left(\frac{1}{n_t}\right)\right)+\frac{(k-1)\,\mathbf{E}\left(Y_{t,k-1}\right)}{n_t}\\
\mathbf{E}\left(Z_{t,k}\right) &\leq \frac{2\cdot 3^{k-1}}{2n_t}\left(1+O\left(\frac{1}{n_t}\right)\right)+\frac{(k-1)\,\mathbf{E}\left(Y_{t,k-1}\right)}{n_t}.
\end{aligned}$$

By induction,

$$\mathbf{E}\left(Y_{t,k-1}\right)=\frac{3^{k-1}-9}{2\,(k-1)}+\frac{A_{k-1}}{n_{t_0}}+B_{k-1}\left(\frac{n_{t_0}}{n_t}\right)^3. \tag{3.2.4}$$

Similar to the case of triangles, the expected number of $k-cycles$ destroyed is $kY_{t,k}/n_t$.

Then

$$\mathbf{E}\left(Y_{t+1,k}\right)-\mathbf{E}\left(Y_{t,k}\right)=\frac{3^{k-1}}{n_t}\left(1+O\left(\frac{\mathbf{E}(Y_{t,k})}{n_t}\right)\right)-\frac{k\mathbf{E}\left(Y_{t,k}\right)}{n_t}+\frac{(k-1)\,\mathbf{E}\left(Y_{t,k-1}\right)}{n_t}. \tag{3.2.5}$$

By induction, $\mathbf{E}\left(Y_{t,k-1}\right)<C$ provided $t\geq\tau_3\left(n_0,Y_{0,k-1},C\right)$. Similarly as the proof in Lemma 3.1, $\mathbf{E}\left(Y_{t,k}\right)$ will be bounded above by C, after some constant time. So, there exists $\tau_k=\tau_k\left(n_0,Y_{0,k},C\right)$, such that $\mathbf{E}\left(Y_{t,k}\right)<C$ for large enough constant C, provided $t>\tau_k$. Choose $t_0\geq\tau_k$.

By induction, for all $t \geq t_0$, and for some $\alpha_k = O(1/n_{t_0})$, $\alpha'_k = O(1/n_{t_0})$,

$$
\begin{aligned}
\mathbf{E}\left(Y_{t,k}\right) &\leq \mathbf{E}\left(Y_{t_0,k}\right)\prod_{i=t_0}^{t-1}\left(1-\frac{k}{n_i}\right)+\sum_{i=t_0}^{t-1}\left(\frac{(k-1)\,\mathbf{E}\left(Y_{i,k-1}\right)+3^{k-1}+\alpha_k}{n_i}\prod_{j=i+1}^{t-1}\left(1-\frac{k}{n_j}\right)\right) \\
&\leq \mathbf{E}\left(Y_{t_0,k}\right)\left(\frac{n_{t_0}}{n_t}\right)^k+\frac{1}{k}\left(3^{k-1}+\alpha_k\right) \\
&\quad +(k-1)\sum_{i=t_0}^{t-1}\left(\frac{1}{n_i}\left(\frac{n_{i+1}}{n_t}\right)^k\left(\frac{3^{k-1}-9}{2(k-1)}+A_{k-1}\frac{1}{n_{t_0}}+B_{k-1}\left(\frac{n_{t_0}}{n_i}\right)^3\right)\right) \\
&\leq \mathbf{E}\left(Y_{t_0,k}\right)\left(\frac{n_{t_0}}{n_t}\right)^k+\frac{1}{k}\left(3^{k-1}+\alpha_k\right) \\
&\quad +(k-1)\left(1-\alpha'_k\right)\left(\frac{3^{k-1}-9}{2(k-1)}\frac{1}{n_t^k}\sum_{i=t_0}^{t-1}n_i^{k-1}+\frac{1}{n_t^k}\frac{A_{k-1}}{n_{t_0}}\sum_{i=t_0}^{t-1}n_i^{k-1}+\frac{B_{k-1}n_{t_0}}{n_t^k}\sum_{i=t_0}^{t-1}n_i^{k-4}\right) \\
&\leq \mathbf{E}\left(Y_{t_0,k}\right)\left(\frac{n_{t_0}}{n_t}\right)^k+\frac{1}{k}\left(3^{k-1}+\alpha_k\right)+(1-\alpha'_k)\frac{3^{k-1}-9}{2k}\left(1-\left(\frac{n_{t_0}}{n_t}\right)^k\right) \\
&\quad +(1-\alpha'_k)\left(\frac{k-1}{k}A_{k-1}\left(\frac{1}{n_{t_0}}-\frac{n_{t_0}^{k-1}}{n_t^k}\right)\right)+(1-\alpha'_k)\frac{k-1}{k-3}B_{k-1}\left(\frac{n_{t_0}}{n_t^3}-\frac{n_{t_0}^k-2}{n_t^k}\right)
\end{aligned}
$$

Thus there exists constant A_k, and B_k, depending only on k, such that

$$
\begin{aligned}
\mathbf{E}\left(Y_{t,k}\right) &\leq \frac{1}{k}3^{k-1}+\frac{3^{k-1}-9}{2k}+\frac{A_k}{n_{t_0}}+B_k\left(\frac{n_{t_0}}{n_t}\right)^3 \\
&= \frac{3^k-9}{2k}+O_k\left(\frac{1}{n_{t_0}}\right)+O_k\left(\left(\frac{n_{t_0}}{n_t}\right)^3\right).
\end{aligned}
$$

Similarly, we can derive the lower bound as $E(Y_{t,k}) \geq \frac{3^k-9}{2k}+O_k\left(\frac{1}{n_{t_0}}\right)+O_k\left(\left(\frac{n_{t_0}}{n_t}\right)^3\right)$.

Choose $n_{t_0} = n_t^{3/4}$, then Lemma 3.3 follows. ∎

Proof of Theorem 3.1

For simplicity, we first consider the case of $k = 3$.

A pegging operation can create or destroy at most six triangles in one step, since G_t is 4-regular. In some cases, pegging creates more than one new triangle, and will also destroys some existing triangles.

We first show that increasing or decreasing the number of triangles by at least 2 in one step, is of probability $O\left(1/n_t^2\right)$.

Figure 3.1 shows a case that if we do pegging operation on the two dashed edges, then two new triangles will be created and an existing triangle will be destroyed. So the number of triangles increase only by 1.

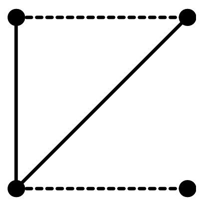

Figure 3.1: *two new triangles created, one existing triangle deleted*

Figure 3.2 shows a case in which three new triangles will be created and two existing triangles will be deleted, if a pegging operation is done on the two dashed edges. So the number of triangles still increases only by 1.

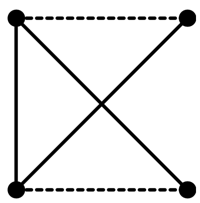

Figure 3.2: *three new triangles created, two existing triangles deleted*

The only way to create two triangles without destroying other triangles is that the four end-vertices joined by the two chosen non-adjancent edges form a 4-cycle as shown in Figure 3.3.

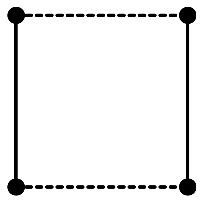

Figure 3.3: *two new triangles created*

By Lemma 3.3, we can choose ε small enough, and let $t_0 = O(1/\varepsilon)$. So, for all $t \geq t_0$, the expected number of k-cycles is bounded above by some large enough constant.

The expected number of ways to choose two non-adjacent edges both contained in a 4-cycle is at most $2\mathbf{E}\left(Y_{t,4}\right)$, so the probability of pegging this way is at most $2\mathbf{E}\left(Y_{t,4}\right) / \left(2n_t^2\right)$, and hence, $O\left(1/n_t^2\right)$.

It is easy to check that the probability of creating i triangles, where $3 \leq i \leq 6$, is even smaller. Since creating more than 2 triangles also requires the occurrence of 4-cycles, whose expected number is bounded by constant. So the probability of increasing Y_t by 2 is at most $O\left(1/n_t^2\right)$. This implies that the probability of increasing Y_t by more than 2 is $O\left(1/n_t^2\right)$.

We now show that the existence of two triangles sharing an common edge as shown in Figure 3.4 is of probability at most $O\left(1/n_t\right)$. For convenience, we call this structure C_3^*. Let $Y_{t,3}^*$ denote the number of C_3^* in G_t. The expected number of C_3^* being destroyed in one step is $5Y_{t,3}^*\left(2n_t - 7\right)/N = 5Y_{t,3}^*/n_t$. The only way to create a C_3^* by pegging is shown in Figure 3.5, where the two dashed edges, both of which are adjacent to one of the triangles in G_t are pegged. So the expected number of C_3^* created in one step is at most $12Y_t/N = \left(6Y_t/n_t^2\right)\left(1 + O\left(1/n_t\right)\right)$.

So

$$\mathbf{E}\left(Y_{t+1,3}^* - Y_{t,3}^* \mid Y_{t,3}^*\right) \leq \frac{6Y_t}{n_t^2}\left(1 + O\left(\frac{1}{n_t}\right)\right) - \frac{5Y_{t,3}^*}{n_t}.$$

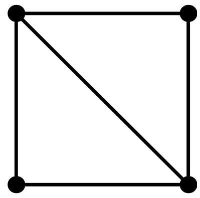

Figure 3.4: C_3^*

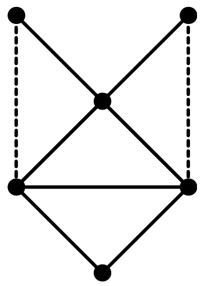

Figure 3.5: *a pegging to create a* C_3^*

By taking expectation of both sides,

$$\mathbf{E}\left(Y^*_{t+1,3}\right) - \mathbf{E}\left(Y^*_{t,3}\right) \leq \frac{6\mathbf{E}\left(Y_t\right)}{n_t^2}\left(1 + O\left(\frac{1}{n_t}\right)\right) - \frac{5\mathbf{E}\left(Y^*_{t,3}\right)}{n_t}.$$

Since $\mathbf{E}(Y_t) = O(1)$, for all $t \geq \tau_3$.

$$\mathbf{E}(Y^*_{t,3}) \leq (1 - \frac{5}{n_{t-1}})\mathbf{E}(Y^*_{t-1,3}) + \frac{C}{n^2_{t-1}} \qquad \text{for some constant } C.$$

So

$$
\begin{aligned}
\mathbf{E}\left(Y_{t,3}^*\right) &\le \left(\frac{n_{t_0}}{n_t}\right)^5 \mathbf{E}\left(Y_{t_0,3}^*\right) + C\sum_{i=t_0}^{t-1} \frac{1}{n_i^2} \prod_{j=i+1}^{t-1} \left(1-\frac{5}{n_j}\right) \\
&\le \left(\frac{n_{t_0}}{n_t}\right)^5 \mathbf{E}\left(Y_{t_0,3}^*\right) + C\sum_{i=t_0}^{t-1} \frac{1}{n_i^2} \left(\frac{n_{i+1}}{n_t}\right)^5 \\
&\le \left(\frac{n_{t_0}}{n_t}\right)^5 \mathbf{E}\left(Y_{t_0,3}^*\right) + \frac{2C}{n_t^5}\sum_{i=t_0}^{t-1} n_i^3 \\
&\le \left(\frac{n_{t_0}}{n_t}\right)^5 \mathbf{E}\left(Y_{t_0,3}^*\right) + \frac{2C}{n_t} \\
&= O\left(\frac{1}{n_t}\right)
\end{aligned}
$$

Then we obtain $\mathbf{E}\left(Y_{t,3}^*\right) = O(1/n_t)$.

This can be easily extended to $\mathbf{E}\left(Y_{t,k}^*\right) = O(1/n_t)$, for all $k \ge 3$, as t sufficiently large, where C_k^* denotes the structure that two k-cycles share one and only one edge, and $Y_{t,k}^*$ denotes the number of C_k^* in G_t.

There are also only two ways to destroy two triangles in one step. Namely, to select an edge for pegging that is contained in two triangles, or choose two non-adjacent edges in two different triangles. In the first case, the expected number of ways to do this is at most $\mathbf{E}\left(Y_{t,3}^*\right)(2n_t - 7) = O(1)$. In the second case, it is at most $\mathbf{E}(Y_t)^2 = O(1)$. So the probability that the number of triangles decrease by 2 is $O\left(1/n_t^2\right)$.

The probability of creating a triangle and destroying another triangle in one step is also small. The only way to achieve that is the case shown in Figure 3.6. Here, first choose the dashed edge that is contained in a triangle, and any grey edge. So one triangle is deleted, and one of the edges that is adjacent to the dashed edge will be contained in a new triangle. For each edge in a triangle, there are at most 14 grey edges to choose. Thus, there are at most $14 \cdot 3\mathbf{E}(Y_t)$ such expected pegging operations. So the probability of this occuring is $\left(14 \cdot 3 \cdot \mathbf{E}(Y_t)/2n_t^2\right)(1 + O(1/n_t))$, and hence $O\left(1/n_t^2\right)$.

Let L_i be the event that the number of triangles decreases by i in the next step, and R_i be the event that the number of triangles increases by i in the next step. Then

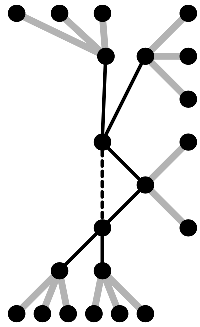

Figure 3.6: *one new triangle created, one existing triangle deleted*

Then we obtain

$$\begin{aligned}\mathbf{P}(Y_{t+1}=j) &= \mathbf{P}(Y_t=j)\left(1-\sum_{i=1}^{6}(\mathbf{P}(L_i \mid Y_t=j)+\mathbf{P}(R_i \mid Y_t=j))\right)\\ &\quad+\sum_{i=1}^{6}\mathbf{P}(Y_t=j-i)(\mathbf{P}(R_i \mid Y_t=j-i))\\ &\quad+\sum_{i=1}^{6}\mathbf{P}(Y_t=j+i)(\mathbf{P}(L_i \mid Y_t=j+i)) \qquad (3.2.6)\end{aligned}$$

From (3.2.2), we know that the expected number of triangles created is $9/n_t+\varepsilon_1(G_t)$, and the expected number of triangles destroyed is $3Y_t/n_t+\varepsilon_2(G_t)$, where $\varepsilon_1(G_t)$ and $\varepsilon_2(G_t)$ are error terms depends on G_t which is of order $O((1+Y_t)/n_t^2)$. As shown in Figure 3.6, the probability to create and destroy triangles in a single step is $O((1+Y_t)/n_t^2)$. So

$$\sum_{i=1}^{6} i\mathbf{P}(L_i \mid Y_t)=\frac{3Y_t}{n_t}+O\left(\frac{1+Y_t}{n_t^2}\right)$$

$$\sum_{i=1}^{6} i\mathbf{P}(R_i \mid Y_t)=\frac{9}{n_t}+O\left(\frac{1+Y_t}{n_t^2}\right).$$

Let $\mathscr{G}$ be any σ-field such that $\mathscr{G}\subseteq\sigma(Y_t)$. It is also obvious that

$$\sum_{i=1}^{6} i\mathbf{P}(L_i \mid Y_t,\mathscr{G})=\frac{3Y_t}{n_t}+O\left(\frac{1+Y_t}{n_t^2}\right)$$

$$\sum_{i=1}^{6} i\mathbf{P}(R_i \mid Y_t, \mathscr{G}) = \frac{9}{n_t} + O\left(\frac{1+Y_t}{n_t^2}\right). \tag{3.2.7}$$

By previous arguments, we also know that

$$\begin{aligned}
\mathbf{P}(L_i) &= \sum_{j\in\mathbf{N}} \mathbf{P}(L_i \mid Y_t = j)\mathbf{P}(Y_t = j) = O\left(\frac{1}{n_t^2}\right) \qquad \text{for all } 2 \le i \le 6 \\
\mathbf{P}(R_i) &= \sum_{j\in\mathbf{N}} \mathbf{P}(R_i \mid Y_t = j)\mathbf{P}(Y_t = j) = O\left(\frac{1}{n_t^2}\right) \qquad \text{for all } 2 \le i \le 6 \\
\mathbf{P}(L_1) &= \sum_{j\in\mathbf{N}} \mathbf{P}(L_1 \mid Y_t = j)\mathbf{P}(Y_t = j) = \frac{9}{n_t} + O\left(\frac{1}{n_t^2}\right) \\
\mathbf{P}(R_1) &= \sum_{j\in\mathbf{N}} \mathbf{P}(R_1 \mid Y_t = j)\mathbf{P}(Y_t = j) = \frac{3\mathbf{E}(Y_t)}{n_t} + O\left(\frac{1}{n_t^2}\right) \qquad (3.2.8)
\end{aligned}$$

Now we define another random process to be a random walk on the nonnegative integers. We define the behavior of the random walk as following:

$$X_{t+1} = \begin{cases} X_t - 1 \text{ with probability } 3X_t/n_t \\ X_t \text{ with probability } 1 - 3X_t/n_t - 9/n_t \\ X_t + 1 \text{ with probability } 9/n_t. \end{cases} \tag{3.2.9}$$

Let $\mathbf{Po}(\mu)$ denote Poisson distribution with mean μ. We show that the Markov chain $\{X_t\}_{t\ge 0}$ has a stationary distribution as $\mathbf{Po}(3)$.

Assume X_t has Poisson distribution with mean 3, then

$$\mathbf{P}(X_t = i) = e^{-3}\frac{3^i}{i!} \qquad \text{for all } i \in \mathbf{N}^+ \cup \{0\}$$

$$\begin{aligned}
\mathbf{P}(X_{t+1} = j) &= \sum_{i\in\mathbf{N}^+\cup\{0\}} \mathbf{P}(X_t = i)\mathbf{P}_{ij} \\
&= e^{-3}\frac{3^{j-1}}{(j-1)!}\frac{9}{n_t} + e^{-3}\frac{3^j}{j!}\left(1 - \frac{9}{n_t} - \frac{3j}{n_t}\right) + e^{-3}\frac{3^{j+1}}{(j+1)!}\frac{3(j+1)}{n_t} \\
&= e^{-3}\frac{3^j}{j!}.
\end{aligned}$$

Thus $\mathbf{Po}(3)$ is invariant, and by definition it is the stationary distribution.

Let X_t has its stationary distribution at step t_0. We apply the coupling method to X_t and Y_t. Now we define another random walk on the nonnegative integers to be a copy of $\{Y_t\}_{t\geq 0}$. We still name it $\{Y_t\}_{t\geq 0}$, which is now a random walk on integers, though it has exactly the same behavior as the original one. Formally, for any integer $1 \leq i \leq 6$,

$$\begin{aligned}\mathbf{P}(Y_{t+1} = Y_t + i) &= \mathbf{P}(R_i \mid Y_t)\\ \mathbf{P}(Y_{t+1} = Y_t - i) &= \mathbf{P}(L_i \mid Y_t).\end{aligned}$$

Now X_t and Y_t are defined on the same probability space, thus we can set up the joint distribution of (X_t, Y_t). Assume we have (X_t, Y_t) at step t, we set the transition probability shown in the following tables. For example, the entry of the intersection of the second row and second column in Table 3.2, and Table 3.3 shows the probability that (X_{t+1}, Y_{t+1}) takes the value of $(X_t - 1, Y_t - i)$, for any $2 \leq i \leq 6$.

Table 1: $Y_t \neq X_t$

	$Y_t - i\ (2 \leq i \leq 6)$	$Y_t - 1$	Y_t	$Y_t + 1$	$Y_t + i\ (2 \leq i \leq 6)$
$X_t - 1$	0	0	$3X_t/n_t$	0	0
X_t	$\mathbf{P}(L_i \mid Y_t)$	$\mathbf{P}(L_1 \mid Y_t)$	p	$\mathbf{P}(R_1 \mid Y_t)$	$\mathbf{P}(R_i \mid Y_t)$
$X_t + 1$	0	0	$9/n_t$	0	0

where $p = 1 - 9/n_t - 3X_t/n_t - \sum_{i=1}^{6}(\mathbf{P}(L_i \mid Y_t) + \mathbf{P}(R_i \mid Y_t))$.

Let Z_t be the number of triangles created in G_t, as we defined in the proof of Lemma 3.3. Let M_t be the number of triangles destroyed in G_t. From 3.2.2, we know

$$\mathbf{E}(Z_t \mid G_t) \leq \frac{9}{n_t} + O\left(\frac{1}{n_t^2}\right), \mathbf{E}(M_t \mid G_t) \leq \frac{3Y_t}{n_t} + O\left(\frac{1}{n_t^2}\right).$$

By taking expectation of both sides, and conditional on the value of Y_t,

$$\mathbf{E}(Z_t \mid Y_t) \leq \frac{9}{n_t} + O\left(\frac{1}{n_t^2}\right), \mathbf{E}(M_t \mid G_t) \leq \frac{3Y_t}{n_t} + O\left(\frac{1}{n_t^2}\right).$$

As shown in Figure 3.6, triangles created and destroyed in a single step is of probability $O(Y_t/n_t^2)$. Thus, $\sum_{i=1}^{6} i\mathbf{P}(L_i \mid Y_t) = \frac{3Y_t}{n_t} + O((1+Y_t)/n_t^2)$, and $\sum_{i=1}^{6} i\mathbf{P}(R_i \mid Y_t) = \frac{9}{n_t} + O((1+Y_t)/n_t^2)$.

So for some $a_t(Y_t) = O((1+Y_t)/n_t^2)$, $b_t(Y_t) = O((1+Y_t)/n_t^2)$, $\mathbf{P}(L_1 \mid Y_t) \le 3Y_t/n_t + b_t(Y_t)$, and $\mathbf{P}(R_1 \mid Y_t) \le 9/n_t + a_t(Y_t)$. Then it is easy to choose $\hat{b}_t(Y_t) > 0$, $\bar{b}_t(Y_t) > 0$, $\hat{a}_t(Y_t) > 0$, $\bar{a}_t(Y_t) > 0$, such that $\hat{b}_t(Y_t) = O((1+Y_t)/n_t^2)$, $\hat{a}_t(Y_t) = O((1+Y_t)/n_t^2)$, $\mathbf{P}(L_1 \mid Y_t) = 3Y_t/n_t + \hat{b}_t - \bar{b}_t(Y_t)$, $\mathbf{P}(R_1 \mid Y_t) = 9/n_t + \hat{a}_t - \bar{a}_t(Y_t)$.

The we define the following joint distribution according to the case $Y_t = X_t$.

Table 2: $Y_t = X_t$

	$Y_t - i\ (2 \le i \le 6)$	$Y_t - 1$	Y_t	$Y_t + 1$	$Y_t + i\ (2 \le i \le 6)$
$X_t - 1$	0	$3Y_t/n_t - \bar{b}_t(Y_t)$	$\bar{b}_t(Y_t)$	0	0
X_t	$\mathbf{P}(L_i \mid Y_t)$	$\hat{b}_t(Y_t)$	p	$\hat{a}_t(Y_t)$	$\mathbf{P}(R_i \mid Y_t)$
$X_t + 1$	0	0	$\bar{a}_t(Y_t)$	$9/n_t - \bar{a}_t(Y_t)$	0

where $p = 1 - 9/n_t - 3Y_t/n_t - \hat{a}_t(Y_t) - \hat{b}_t(Y_t) - \sum_{i=2}^{6} (\mathbf{P}(L_i \mid Y_t) + \mathbf{P}(R_i \mid Y_t))$.

We can check that the marginal satisfies (3.2.9) and (3.2.6).

Let

$$D_t = \mid Y_t - X_t \mid \tag{3.2.10}$$

If $Y_t > X_t$, from Table 1 we get

$$\begin{aligned} \mathbf{E}(D_{t+1} - D_t \mid X_t, Y_t, s.t. Y_t > X_t) &\le \frac{3X_t}{n_t} - \mathbf{P}(L_1 \mid Y_t)) + \sum_{i=2}^{6} i\mathbf{P}(L_i \mid Y_t)) + \sum_{i=1}^{6} i\mathbf{P}(R_i \mid Y_t) - \frac{9}{n_t} \\ &= \frac{3X_t}{n_t} - \sum_{i=1}^{6} i\mathbf{P}(L_i \mid Y_t)) + \sum_{i=1}^{6} i\mathbf{P}(R_i \mid Y_t) - \frac{9}{n_t} + 2\sum_{i=2}^{6} i\mathbf{P}(L_i \mid Y_t)) \end{aligned}$$

Taking expectation of both sides, and applying (3.2.7), by Tower Property, we obtain

$$\begin{aligned}
\mathbf{E}(D_{t+1}-D_t \mid Y_t > X_t) &= \mathbf{E}(\mathbf{E}(D_{t+1}-D_t \mid X_t, Y_t, s.t. Y_t > X_t) \mid Y_t > X_t) \\
&\leq \frac{3\mathbf{E}(X_t \mid Y_t > X_t)}{n_t} - \frac{3\mathbf{E}(Y_t \mid Y_t > X_t)}{n_t} + O\left(\frac{1+\mathbf{E}(Y_t \mid Y_t > X_t)}{n_t^2}\right) \\
&\quad +2\sum_{i=2}^{6} \mathbf{E}(i\mathbf{P}(L_i \mid Y_t > X_t))) \\
&= -\frac{3}{n_t}\mathbf{E}(D_t \mid Y_t > X_t) + O\left(\frac{1+\mathbf{E}(Y_t \mid Y_t > X_t)}{n_t^2}\right) + \sum_{i=2}^{6} 2\mathbf{E}(i\mathbf{P}(L_i \mid Y_t > X_t))).
\end{aligned}$$

If $Y_t < X_t$, from Table 1 we get

$$\begin{aligned}
\mathbf{E}(D_{t+1}-D_t \mid X_t, Y_t, s.t. Y_t < X_t) &\leq -\frac{3X_t}{n_t} + \sum_{i=1}^{6} i\mathbf{P}(L_i \mid Y_t) - \mathbf{P}(R_1 \mid Y_t) + \sum_{i=2}^{6} i\mathbf{P}(R_i \mid Y_t) + \frac{9}{n_t} \\
&= -\frac{3X_t}{n_t} + \sum_{i=1}^{6} i\mathbf{P}(L_i \mid Y_t) - \sum_{i=1}^{6} i\mathbf{P}(R_i \mid Y_t) + \frac{9}{n_t} + 2\sum_{i=2}^{6} i\mathbf{P}(R_i \mid Y_t)
\end{aligned}$$

Taking expectation of both sides, and applying (3.2.7), by Tower Property, we obtain

$$\begin{aligned}
\mathbf{E}(D_{t+1}-D_t \mid Y_t < X_t) &= \mathbf{E}(\mathbf{E}(D_{t+1}-D_t \mid X_t, Y_t, s.t. Y_t < X_t) \mid Y_t < X_t) \\
&\leq -\frac{3\mathbf{E}(X_t \mid Y_t < X_t)}{n_t} + \frac{3\mathbf{E}(Y_t \mid Y_t < X_t)}{n_t} + O\left(\frac{1+\mathbf{E}(Y_t \mid Y_t < X_t)}{n_t^2}\right) \\
&\quad +2\sum_{i=2}^{6} \mathbf{E}(i\mathbf{P}(R_i \mid Y_t < X_t)) \\
&= -\frac{3}{n_t}\mathbf{E}(D_t \mid Y_t < X_t) + O\left(\frac{1+\mathbf{E}(Y_t \mid Y_t > X_t)}{n_t^2}\right) + 2\sum_{i=2}^{6} \mathbf{E}(i\mathbf{P}(R_i \mid Y_t < X_t)).
\end{aligned}$$

If $Y_t = X_t$, then $D_t = 0$, from Table 2 we get

$$\begin{aligned}
&\mathbf{E}(D_{t+1}-D_t \mid X_t, Y_t, s.t. Y_t = X_t) \\
= \;& \hat{b}_t(Y_t) + \overline{b}_t(Y_t) + \hat{a}_t(Y_t) + \overline{a}_t(Y_t) + \sum_{i=2}^{6} (i\mathbf{P}(L_i \mid Y_t) + i\mathbf{P}(R_i \mid Y_t)) \\
= \;& \frac{3Y_t}{n_t} - \mathbf{P}(L_1 \mid Y_t) + 2\hat{b}_t(Y_t) + \frac{9}{n_t} - \mathbf{P}(R_1 \mid Y_t) + 2\hat{a}_t(Y_t) + \sum_{i=2}^{6} (i\mathbf{P}(L_i \mid Y_t) + i\mathbf{P}(R_i \mid Y_t)) \\
= \;& \frac{3Y_t}{n_t} + 2\hat{b}_t(Y_t) + \frac{9}{n_t} + 2\hat{a}_t(Y_t) - \sum_{i=1}^{6} (\mathbf{P}(L_1 \mid Y_t) + \mathbf{P}(R_1 \mid Y_t)) + \sum_{i=2}^{6} (i\mathbf{P}(L_i \mid Y_t) + i\mathbf{P}(R_i \mid Y_t)).
\end{aligned}$$

Taking expectation of both sides, and applying (3.2.7) and (3.2.8), by Tower Property, we get

$$
\begin{aligned}
& \mathbf{E}(D_{t+1} - D_t \mid Y_t = X_t) \\
= \; & \mathbf{E}(\mathbf{E}(D_{t+1} - D_t \mid Y_t, X_t, s.t. Y_t = X_t) \mid Y_t = X_t) \\
= \; & \frac{3\mathbf{E}(Y_t \mid Y_t = X_t)}{n_t} + \frac{9}{n_t} - \sum_{i=1}^{6} (\mathbf{P}(L_1 \mid X_t = Y_t) + \mathbf{P}(R_1 \mid X_t = Y_t)) + O(1/n_t^2) \\
& + \sum_{i=2}^{6} (i\mathbf{P}(L_i \mid Y_t = X_t) + i\mathbf{P}(R_i \mid Y_t = X_t)) \\
= \; & O\left(\frac{1 + \mathbf{E}(Y_t \mid X_t = Y_t)}{n_t^2}\right) + \sum_{i=2}^{6} (i\mathbf{P}(L_i \mid Y_t = X_t) + i\mathbf{P}(R_i \mid Y_t = X_t)).
\end{aligned}
$$

Since for all $2 \leq i \leq 6$,

$$
\begin{aligned}
& \mathbf{E}(\mathbf{P}(L_i \mid Y_t = X_t)) \\
= \; & \frac{\sum_j \mathbf{P}(L_i \mid Y_t = X_t = j)\mathbf{P}(Y_t = X_t = j)}{\sum_j \mathbf{P}(Y_t = X_t = j)} \\
\leq \; & \frac{\left(\sum_j \mathbf{P}(L_i \mid Y_t = X_t = j)\right)\left(\sum_j \mathbf{P}(Y_t = X_t = j)\right)}{\sum_j \mathbf{P}(Y_t = X_t = j)} \\
= \; & \sum_j \mathbf{P}(L_i \mid Y_t = X_t = j) \\
= \; & \sum_j \mathbf{P}(L_i \mid Y_t = j) \\
= \; & O\left(\frac{1}{n_t^2}\right)
\end{aligned}
$$

Similarly, we can show that

$$
\begin{aligned}
& \mathbf{E}(\mathbf{P}(R_i \mid Y_t = X_t)) = O\left(\frac{1}{n_t^2}\right), \mathbf{E}(\mathbf{P}(L_i \mid Y_t < X_t)) = O\left(\frac{1}{n_t^2}\right), \\
& \mathbf{E}(\mathbf{P}(R_i \mid Y_t < X_t)) = O\left(\frac{1}{n_t^2}\right), \mathbf{E}(\mathbf{P}(L_i \mid Y_t > X_t)) = O\left(\frac{1}{n_t^2}\right), \\
& \mathbf{E}(\mathbf{P}(R_i \mid Y_t > X_t)) = O\left(\frac{1}{n_t^2}\right).
\end{aligned}
$$

So

$$
\begin{aligned}
\mathbf{E}(D_{t+1} - D_t \mid Y_t = X_t) &= -\frac{3}{n_t}\mathbf{E}(D_t \mid X_t = Y_t) + O\left(\frac{1 + \mathbf{E}(Y_t \mid X_t = Y_t)}{n_t^2}\right) \\
\mathbf{E}(D_{t+1} - D_t \mid Y_t > X_t) &= -\frac{3}{n_t}\mathbf{E}(D_t \mid Y_t > X_t) + O\left(\frac{1 + \mathbf{E}(Y_t \mid Y_t > X_t)}{n_t^2}\right) \\
\mathbf{E}(D_{t+1} - D_t \mid Y_t < X_t) &= -\frac{3}{n_t}\mathbf{E}(D_t \mid Y_t < X_t) + O\left(\frac{1 + \mathbf{E}(Y_t \mid Y_t > X_t)}{n_t^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}(D_{t+1}-D_t) &= \mathbf{E}(D_{t+1}-D_t \mid X_t<Y_t)\,\mathbf{P}(X_t<Y_t)+\mathbf{E}(D_{t+1}-D_t \mid X_t=Y_t)\,\mathbf{P}(X_t=Y_t) \\
&\quad +\mathbf{E}(D_{t+1}-D_t \mid X_t>Y_t)\,\mathbf{P}(X_t>Y_t) \\
&= -\frac{3}{n_t}\mathbf{E}(D_t \mid X_t<Y_t)\,\mathbf{P}(X_t<Y_t)-\frac{3}{n_t}\mathbf{E}(D_t \mid X_t=Y_t)\,\mathbf{P}(X_t=Y_t) \\
&\quad -\frac{3}{n_t}\mathbf{E}(D_t \mid X_t>Y_t)\,\mathbf{P}(X_t>Y_t)+O\left(\frac{1+\mathbf{E}(Y_t)}{n_t^2}\right) \\
&= -\frac{3}{n_t}\mathbf{E}(D_t)+O\left(\frac{1}{n_t^2}\right).
\end{aligned}
$$

$$
\mathbf{E}(D_{t+1}) = \left(1-\frac{3}{n_t}\right)\mathbf{E}(D_t)+O\left(\frac{1}{n_t^2}\right).
$$

Since $\mathbf{E}(Y_t) < C$ for some constant C, for all $t \geq t_0$

$$
\mathbf{E}(D_{t+1}) \leq \left(1-\frac{3}{n_t}\right)\mathbf{E}(D_t)+\frac{\zeta}{n_t^2}
$$

for some positive constant ζ, where ζ depends only on the value of C.

Iteratively,

$$\begin{aligned}
\mathbf{E}(D_t) &\leq \left(\prod_{i=t_0}^{t-1}\left(1-\frac{3}{n_i}\right)\right)\mathbf{E}(D_{t_0})+\zeta\frac{1}{n_{t-1}^2}+\zeta\left(1-\frac{3}{n_{t-1}}\right)\frac{1}{n_{t-2}^2} \\
&\quad +\zeta\left(1-\frac{3}{n_{t-1}}\right)\left(1-\frac{3}{n_{t-2}}\right)\frac{1}{n_{t-3}^2}+\ldots+\zeta\prod_{i=t_0+1}^{t-1}\left(1-\frac{3}{n_i}\right)\frac{1}{n_{t_0}^2} \\
&\leq \mathbf{E}(D_{t_0})\exp\left(-3\sum_{i=t_0}^{t-1}\frac{1}{n_i}\right)+\zeta\sum_{i=t_0}^{t-1}\frac{1}{n_i^2}\exp\left(-3\sum_{j=i+1}^{t-1}\frac{1}{n_j}\right) \\
&\leq \mathbf{E}(D_{t_0})\exp\left(-3\left(\log n_t-\log n_{t_0}\right)\right)+\zeta\sum_{i=t_0}^{t-1}\frac{1}{n_i^2}\exp\left(-3\left(\log n_t-\log n_{i+1}\right)\right) \\
&\leq \mathbf{E}(D_{t_0})\left(\frac{n_{t_0}}{n_t}\right)^3+\zeta\sum_{i=t_0}^{t-1}\frac{1}{n_i^2}\left(\frac{n_{i+1}}{n_t}\right)^3 \\
&\leq \mathbf{E}(D_{t_0})\left(\frac{n_{t_0}}{n_t}\right)^3+\frac{2\zeta}{(n_t)^3}\sum_{i=t_0}^{t-1}(n_i) \\
&\leq \left(\frac{n_{t_0}}{n_t}\right)^3(C-3)+\frac{2\zeta}{(n_t)^3}(t-t_0)\left(n_0+\frac{t+t_0-1}{2}\right) \\
&\leq \left(\frac{n_{t_0}}{n_t}\right)^3(C-3)+\frac{2\zeta}{n_t}.
\end{aligned}$$

Let

$$\begin{aligned}
&\left(\frac{n_{t_0}}{n_t}\right)^3(C-3)\leq\frac{\varepsilon}{2} \\
&\frac{2\zeta}{n_t}\leq\frac{\varepsilon}{2}.
\end{aligned}$$

By Lemma 3.4, we obtain

$$\begin{aligned}
d_{TV}\left(\sigma_{t,3},\mathbf{Po}(3)\right) &\leq \mathbf{P}(Y_t\neq X_t)\leq\mathbf{E}(D_t) \\
&\leq \left(\frac{n_{t_0}}{n_t}\right)^3(C-3)+\frac{2\zeta}{n_t}.
\end{aligned} \tag{3.2.11}$$

We only need to choose $t_0>\max\{\tau_3,\tau_4\}$, such that the expected number of triangles and 4-cycles are bounded. So, t_0 is a constant. Let

$$d_{TV}\left(\sigma_{t,3},\mathbf{Po}(3)\right)<\varepsilon$$

We obtain

$$\tau^*(\varepsilon) = O(1/\varepsilon).$$

$\blacksquare$

Proof of Theorem 3.2

Let G_t be a random d-regular graph generated by pegging operations, for even d. Then G_t contains $n_t = n_0 + t$ vertices, and $dn_t/2$ edges. Then N, the number of ways to do a pegging operation, is asymptotically $\binom{dn_t/2}{d/2}$. There are two ways to create a k-cycle. One is to choose the two end edges of a k-path, and other $d/2-2$ non-adjacent edges, and do the pegging. The other way is to choose an edge contained in a $(k-1)$-cycle, and other $d/2-1$ non-adjacent edges, and do the pegging.

In the first case, the number of k-paths in G_t is asymptotically $d(d-1)^{k-1}n_t/2$. So the number of ways to do pegging is asymptotically

$$\frac{d(d-1)^{k-1}n_t}{2}\binom{\frac{dn}{2}}{\frac{d}{2}-2} \sim \frac{d(d-1)^{k-1}n_t(dn_t/2)^{d/2-2}}{2(d/2-2)!}.$$

In the second case, the number of ways to do pegging is asymptotically

$$(k-1)Y_{t,k-1,d}\binom{\frac{dn_t}{2}}{\frac{d}{2}-1} \sim \frac{(k-1)Y_{t,k-1,d}(dn_t/2)^{d/2-1}}{(d/2-1)!}.$$

The way to destroy an existing k-cycle is to choose an edge contained in a k-cycle, and another $d/2-1$ non-adjacent edges, and do the pegging. So the number of ways to destroy an existing k-cycle is asymptotically

$$kY_{t,k,d}\binom{\frac{dn_t}{2}}{\frac{d}{2}-1} \sim \frac{kY_{t,k,d}(dn_t/2)^{d/2-1}}{(d/2-1)!}.$$

So, we obtain the expected value of $Y_{t+1,k,d} - Y_{t,k,d}$

$$\begin{aligned}
&\mathbf{E}(Y_{t+1,k,d} - Y_{t,k,d} \mid Y_{t,k,d}) \\
= \quad & \frac{d(d-1)^{k-1}n_t(dn/2)^{d/2-2}}{2(d/2-2)!N} + \frac{(k-1)Y_{t,k-1,d}(dn_t/2)^{d/2-1}}{(d/2-1)!N} - \frac{kY_{t,k,d}(dn_t/2)^{d/2-1}}{(d/2-1)!N}.
\end{aligned}$$

Take the expectation of both sides

$$
\begin{aligned}
& \mathbf{E}(Y_{t+1,k,d} - Y_{t,k,d}) \\
\sim\ & \frac{d(d-1)^{k-1} n_t (dn/2)^{d/2-2}}{2(d/2-2)!N} + \frac{(k-1)\mathbf{E}(Y_{t,k-1,d})(dn_t/2)^{d/2-1}}{(d/2-1)!N} - \frac{k\mathbf{E}(Y_{t,k,d})(dn_t/2)^{d/2-1}}{(d/2-1)!N} \\
=\ & \frac{(dn_t/2)^{d/2-2} dn_t}{(d/2-2)!N}\left(\frac{(d-1)^{k-1}}{2} + \frac{(k-1)\mathbf{E}(Y_{t,k-1,d})}{d-2} - \frac{k\mathbf{E}(Y_{t,k,d})}{d-2}\right) \\
\sim\ & \frac{d-2}{n_t}\left(\frac{(d-1)^{k-1}}{2} + \frac{(k-1)\mathbf{E}(Y_{t,k-1,d})}{d-2} - \frac{k\mathbf{E}(Y_{t,k,d})}{d-2}\right) \\
=\ & \frac{(d-2)(d-1)^{k-1}}{2n_t} + \frac{(k-1)\mathbf{E}(Y_{t,k-1,d})}{n_t} - \frac{k\mathbf{E}(Y_{t,k,d})}{n_t} \\
& \mathbf{E}(Y_{t,2,d}) = 0.
\end{aligned}
$$

Note this recursive function is exactly the same as (3.2.5) but the first term. So reproducing the proof of Lemma 3.2 and 3.3, we obtain

$$
\mathbf{E}\left(Y_{t,k,d}\right) = \frac{(d-1)^k - (d-1)^2}{2k} + O\left(\frac{1}{n_{t_0}}\right) + O\left(\left(\frac{n_{t_0}}{n_t}\right)^3\right)
$$

for sufficiently large t_0 which is depends only on n_0, $Y_{0,k,d}$, and C.

We derive the same mixing rate as stated in Theorem 3.1. The proof and method used is precisely the same. Then Theorem 3.2 follows. ∎

3.3 More discussion about k-cycles

We see in the previous section that in the random process of $\{Y_t\}_{t\geq 0}$, where Y_t is the number of triangles in step t, we coupled $\{Y_t\}_{t\geq 0}$ to $\{X_t\}_{t\geq 0}$, such that the transition with probability $O(1/n_t^2)$ is omitted. Those $O(1/n_t^2)$ error terms accumulates in each step, and will contributes $O(1/n_t)$ to $\mathbf{E}(D_t)$ when t goes to infinity. So $\{Y_t\}_{t\geq 0}$ and $\{X_t\}_{t\geq 0}$ will have the same limiting distribution.

We investigate an attempt to apply the same method to couple two sequences of vectors $\{Y_{t,3}, Y_{t,4}, Y_{t,5}, \cdots, Y_{t,k}\}_{t\geq 0}$ and $\{X_{t,3}, X_{t,4}, X_{t,5}, \cdots, X_{t,k}\}_{t\geq 0}$ for any $k \geq 3$, such that the transitions in the $\{X_{t,3}, X_{t,4}, X_{t,5}, \cdots, X_{t,k}\}_{t\geq 0}$ are obtained from those in the $\{Y_{t,3}, Y_{t,4}, Y_{t,5}, \cdots, Y_{t,k}\}_{t\geq 0}$ by omitting all transitions with probability $O(1/n^2)$ and adjusting the rest to compensate. The two random processes would have the same limiting distribution. We did not carry this out completely. It is too complicated to list all cases when we couple the two vectors, so we are looking for a more general way to describe this argument.

We can define the following random walk

$$(X_{t+1,3}, X_{t+1,4}, \cdots, X_{t+1,k})$$
$$= \begin{cases} (X_{t,3}, X_{t,4}, \cdots, X_{t,i}+1, \cdots, X_{t,k}) \text{ with probability } 3^{i-1}/n_t, \text{ for all } 3 \leq i \leq k. \\ (X_{t,3}, X_{t,4}, \cdots, X_{t,i}-1, X_{t,i+1}+1 \cdots, X_{t,k}) \text{with probability } iX_{t,i}/n_t, \text{ for all } 3 \leq i \leq k-1. \\ (X_{t,3}, X_{t,4}, \cdots, X_{t,k-1}, X_{t,k}-1) \text{ with probability } kX_{t,k}/n_t. \end{cases}$$

Start $\{X_{t,3}, X_{t,4}, X_{t,5}, \cdots, X_{t,k}\}_{t\geq 0}$ with independent Poisson at $t=0$, with means $\mu_3, \mu_4, \cdots, \mu_k$, where $\mu_i = \frac{3^i-9}{2i}$, for all $3 \leq i \leq k$. Now we show by induction that $X_{t,3}, X_{t,4}, X_{t,5}, \cdots, X_{t,k}$ are independent Poisson for all $t \geq 0$. Assuming they are independent Poisson at some time $t \geq 0$, we have

$$\mathbf{P}(X_{t,3} = x_3, X_{t,4} = x_4, \cdots, X_{t,k} = x_k) = \exp\left(-\sum_{i=3}^{k} \mu_i\right) \prod_{i=3}^{k} \frac{\mu_i^{x_i}}{x_i!}.$$

and

$$
\begin{aligned}
& \mathbf{P}(X_{t+1,3}=x_3, X_{t+1,4}=x_4, \cdots, X_{t+1,k}=x_k) \\
= & \left(1-\sum_{i=3}^{k}\frac{3^{i-1}}{n_t}-\sum_{i=3}^{k}\frac{kx_k}{n_t}\right)\mathbf{P}(X_{t,3}=x_3, X_{t,4}=x_4, \cdots, X_{t,k}=x_k) \\
& +\sum_{i=3}^{k}\frac{3^{i-1}}{n_t}\mathbf{P}(X_{t,3}=x_3,, \cdots, X_{t,i}=x_i-1, \cdots, X_{t,k}=x_k) \\
& +\sum_{i=3}^{k-1}\frac{i(x_i+1)}{n_t}\mathbf{P}(X_{t,3}=x_3,, \cdots, X_{t,i}=x_i+1, X_{t,i+1}=x_{i+1}-1, \cdots, X_{t,k}=x_k) \\
& +\frac{k(x_k+1)}{n_t}\mathbf{P}(X_{t,3}=x_3,, \cdots, X_{t,k-1}=x_{k-1}, X_{t,k}=x_k+1) \\
= & \exp\left(-\sum_{i=3}^{k}\mu_i\right) \\
& \prod_{i=3}^{k}\frac{\mu_i^{x_i}}{x_i!}\left(1-\sum_{i=3}^{k}\frac{3^{i-1}}{n_t}-\sum_{i=3}^{k}\frac{kx_k}{n_t}+\sum_{i=3}^{k}\frac{x_i}{\mu_i}\frac{3^{i-1}}{n_t}+\sum_{i=3}^{k-1}\frac{\mu_i}{x_i+1}\frac{x_{i+1}}{\mu_{i+1}}\frac{i(x_i+1)}{n_t}+\frac{\mu_k}{x_k+1}\frac{k(x_k+1)}{n_t}\right) \\
= & \exp\left(-\sum_{i=3}^{k}\mu_i\right) \\
& \prod_{i=3}^{k}\frac{\mu_i^{x_i}}{x_i!}\frac{1}{n_t}\left(n_t-\sum_{i=3}^{k}3^{i-1}-\sum_{i=3}^{k}kx_k+\sum_{i=3}^{k}\frac{2i\cdot 3^{i-1}x_i}{3^i-9}+\sum_{i=3}^{k-1}\frac{(3^i-9)(i+1)x_{i+1}}{3^{i+1}-9}+\frac{3^k-9}{2}\right) \\
= & \exp\left(-\sum_{i=3}^{k}\mu_i\right)\prod_{i=3}^{k}\frac{\mu_i^{x_i}}{x_i!}\frac{1}{n_t}\left(n_t-\frac{3^k-9}{2}-\sum_{i=3}^{k}kx_k+\sum_{i=4}^{k}\frac{2i\cdot 3^{i-1}x_i+(3^{i-1}-9)ix_i}{3^i-9}+3x_3+\frac{3^k-9}{2}\right) \\
= & \exp\left(-\sum_{i=3}^{k}\mu_i\right)\prod_{i=3}^{k}\frac{\mu_i^{x_i}}{x_i!}\frac{1}{n_t}\left(n_t-\sum_{i=3}^{k}kx_k+\sum_{i=4}^{k}\frac{ix_i(2\cdot 3^{i-1}+3^{i-1}-9)}{3^i-9}+3x_3\right) \\
= & \exp\left(-\sum_{i=3}^{k}\mu_i\right)\prod_{i=3}^{k}\frac{\mu_i^{x_i}}{x_i!} \\
= & \mathbf{P}(X_{t,3}=x_3, X_{t,4}=x_4, \cdots, X_{t,k}=x_k).
\end{aligned}
$$

So the limiting distribution of $\{X_{t,3}, X_{t,4}, X_{t,5}, \cdots, X_{t,k}\}_{t\geq 0}$ is independent Poisson with means $\mu_3, \mu_4, \cdots, \mu_k$, where $\mu_i = \frac{3^i-9}{2i}$, for all $3 \leq i \leq k$. By applying the extended coupling Method in Lemma 3.4, we expect to get $\mathbf{E}(D_{t,k}) = O(1/n_t)$, which goes to 0 as t goes to infinity, so $Y_{t,3}, Y_{t,4}, Y_{t,5}, \cdots, Y_{t,k}$ have limiting distribution of independent Poisson random variables.

We will present an example to couple (Y_3, Y_4) and (X_3, X_4). Define $D_{t,4} := (1/3) \mid Y_{t,3} - X_{t,3} \mid$

$+(1/4) \mid Y_{t,4} - X_{t,4} \mid$. The following tables shows an attempt at coupling (Y_3, Y_4) and (X_3, X_4). But the data shown in the table are not exactly what will happen in the coupling procedure because the random walk of (Y_3, Y_4) depends on the graph. So we are "cheating" here. The data shown there is what we would obtain if we calculate the expectation of $D_{t+1,4}$, where the expectation is averaging over all $(Y_{t,3}, Y_{t,4})$, conditional on each case. For a rigorous proof one would have to do some careful accounting as was done for the case of triangles. Also we only give the main terms there. Every term would have some error terms of $O(1/n_t^2)$.

case 1: $Y_{t,3} < X_{t,3}$ and $Y_{t,4} < X_{t,4}$.

	$(Y_{t,3}+1, Y_{t,4})$	$(Y_{t,3}, Y_{t,4})$	$(Y_{t,3}-1, Y_{t,4}+1)$	$(Y_{t,3}, Y_{t,4}+1)$	$(Y_{t,3}, Y_{t,4}-1)$
$(X_{t,3}+1, X_{t,4})$		$9/n_t$			
$(X_{t,3}, X_{t,4})$	$9/n_t$	p	$3Y_{t,3}/n_t$	$27/n_t$	$4Y_{t,4}/n_t$
$(X_{t,3}-1, X_{t,4}+1)$		$3X_{t,3}/n_t$			
$(X_{t,3}, X_{t,4}+1)$		$27/n_t$			
$(X_{t,3}, X_{t,4}-1)$		$4X_{t,4}/n_t$			

case 2: $Y_{t,3} = X_{t,3}$ and $Y_{t,4} < X_{t,4}$.

	$(Y_{t,3}+1, Y_{t,4})$	$(Y_{t,3}, Y_{t,4})$	$(Y_{t,3}-1, Y_{t,4}+1)$	$(Y_{t,3}, Y_{t,4}+1)$	$(Y_{t,3}, Y_{t,4}-1)$
$(X_{t,3}+1, X_{t,4})$	$9/n_t$				
$(X_{t,3}, X_{t,4})$		p		$27/n_t$	$4Y_{t,4}/n_t$
$(X_{t,3}-1, X_{t,4}+1)$			$3Y_{t,3}/n_t$		
$(X_{t,3}, X_{t,4}+1)$		$27/n_t$			
$(X_{t,3}, X_{t,4}-1)$		$4X_{t,4}/n_t$			

case 3: $Y_{t,3} < X_{t,3}$ and $Y_{t,4} = X_{t,4}$.

	$(Y_{t,3}+1, Y_{t,4})$	$(Y_{t,3}, Y_{t,4})$	$(Y_{t,3}-1, Y_{t,4}+1)$	$(Y_{t,3}, Y_{t,4}+1)$	$(Y_{t,3}, Y_{t,4}-1)$
$(X_{t,3}+1, X_{t,4})$	$9/n_t$				
$(X_{t,3}, X_{t,4})$		p			
$(X_{t,3}-1, X_{t,4}+1)$		$3(X_{t,3}-Y_{t,3})/n_t$	$3Y_{t,3}/n_t$		
$(X_{t,3}, X_{t,4}+1)$				$27/n_t$	
$(X_{t,3}, X_{t,4}-1)$					$4Y_{t,4}/n_t$

case 4: $Y_{t,3} = X_{t,3}$ and $Y_{t,4} = X_{t,4}$.

	$(Y_{t,3}+1, Y_{t,4})$	$(Y_{t,3}, Y_{t,4})$	$(Y_{t,3}-1, Y_{t,4}+1)$	$(Y_{t,3}, Y_{t,4}+1)$	$(Y_{t,3}, Y_{t,4}-1)$
$(X_{t,3}+1, X_{t,4})$	$9/n_t$				
$(X_{t,3}, X_{t,4})$		p			
$(X_{t,3}-1, X_{t,4}+1)$			$3Y_{t,3}/n_t$		
$(X_{t,3}, X_{t,4}+1)$				$27/n_t$	
$(X_{t,3}, X_{t,4}-1)$					$4Y_{t,4}/n_t$

We are expecting to get the following inequalities for each cases,

$$\begin{aligned}
&\mathbf{E}(D_{t+1,4} \mid (Y_{t,3}, Y_{t,4}), (X_{t,3}, X_{t,4}), s.t. Y_{t,3} < X_{t,3}, Y_{t,4} < X_{t,4}) \\
&\quad = D_{t,4} - \frac{3(X_{t,3}-Y_{t,3})}{n_t}\left(\frac{1}{3}-\frac{1}{4}\right) - \frac{4(X_{t,4}-Y_{t,4})}{n_t}\frac{1}{4} + O(1/n_t^2) \\
&\quad \leq D_{t,4} - \frac{3D_{t,4}}{4n_t} + O(1/n_t^2)
\end{aligned}$$

$$
\begin{aligned}
&\mathbf{E}(D_{t+1,4} \mid (Y_{t,3},Y_{t,4}),(X_{t,3},X_{t,4}), s.t. Y_{t,3}=X_{t,3}, Y_{t,4}<X_{t,4})\\
&\quad = D_{t,4} - \frac{4(X_{t,4}-Y_{t,4})}{n_t}\frac{1}{4} + O(1/n_t^2)\\
&\quad \le D_{t,4} - \frac{4D_{t,4}}{n_t} + O(1/n_t^2)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{E}(D_{t+1,4} \mid (Y_{t,3},Y_{t,4}),(X_{t,3},X_{t,4}), s.t. Y_{t,3}<X_{t,3}, Y_{t,4}=X_{t,4})\\
&\quad = D_{t,4} - \frac{3(X_{t,3}-Y_{t,3})}{n_t}\left(\frac{1}{3}-\frac{1}{4}\right) + O(1/n_t^2)\\
&\quad \le D_{t,4} - \frac{3D_{t,4}}{4n_t} + O(1/n_t^2)
\end{aligned}
$$

$$
\mathbf{E}(D_{t+1,4} \mid (Y_{t,3},Y_{t,4}),(X_{t,3},X_{t,4}), s.t. Y_{t,3}=X_{t,3}, Y_{t,4}=X_{t,4}) = O(1/n_t^2)
$$

So we would get

$$
\mathbf{E}(D_{t+1,4}) \le \left(1-\frac{3}{4n_t}\right)D_{t,4} + O(1/n_t^2).
$$

Similarly, define $D_{t,k} := \sum_{i=3}^{k}(1/i) \mid Y_{t,i}-X_{t,i} \mid$, we expect to get the following by extended coupling method

$$
\begin{aligned}
&\mathbf{E}(D_{t+1,k} \mid (Y_{t,3},Y_{t,4}),\cdots,(X_{t,k},X_{t,k}), s.t. Y_{t,3}=X_{t,3},\cdots,Y_{t,i-1}=X_{t,i-1}, Y_{t,i}<X_{t,i},\\
&Y_{t,i+1}=X_{t,i+1},\cdots,Y_{t,k}=X_{t,k})\\
\le\ & \left(1-\frac{i}{(i+1)n_t}\right)D_{t,k} + O(1/n_t^2)\\
\le\ & \left(1-\frac{3}{4n_t}\right)D_{t,k} + O(1/n_t^2).
\end{aligned}
$$

$$
\mathbf{E}(D_{t+1,k}) \le \left(1-\frac{3}{4n_t}\right)D_{t,k} + O(1/n_t^2).
$$

So Theorem 3.1, and Theorem 3.2 would follow for any fixed k, and more precisely, the random variables $Y_{t,3}, Y_{t,4}, \ldots, Y_{t,k}$ would be asymptotically independent Poisson. But it becomes complicate to construct the coupling for the two vectors, since there will be lots of cases to discuss. We are looking for a more general way to derive it. Due to time restrictions, we will leave it as future work.

Chapter 4

Conclusion

In this essay, we briefly discussed several commonly used models to generate random regular graphs. We studied the pegging algorithm, the application of which is to model the SWAN network. In Chapter 3, we presented our result of short cycle distribution in the random d-regular graphs generated by the pegging algorithm. We derived the expected number of k-cycles for any fixed k, and we proved that the number of triangles is asymptotically a Poisson random variable. We also presented our conjecture that the set of random variables $Y_{t,k}$, the number of k-cycles, where $k \geq 3$ is in some finite integer set $I \subset \mathbf{N}$, are asymptotically independent Poisson as t goes to infinity. Finally we discussed the difficulties we met when we were searching for a rigorous proof of the conjecture by coupling two sequences of random variables.

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