Interpolating between maximum degree and maximum density in multigraph edge-colorings

by

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Abstract

The Goldberg-Seymour Conjecture (a proof of which has recently been announced by Chen, Jing, and Zang) asserts that the chromatic index of a multigraph is closely determined by either its maximum degree or a certain maximum density parameter. A multigraph's maximum degree and maximum density also play a central role in other edge-coloring problems, such as arboricity and pseudoarboricity. In this essay, we will discuss how techniques similar to those developed to approach the Goldberg-Seymour Conjecture can be used to study these seemingly unrelated edge-coloring problems. In the hopes of "interpolating" among these edge-coloring problems, we also study bounded degree versions of these problems, specifically when the color classes are bounded degree subgraphs, bounded degree forests, or bounded degree pseudoforests. We will prove exact results in some cases, and will consider conjectures similar to the Goldberg-Seymour Conjecture in unknown cases. This will include a conjecture that strengthens the still unsolved Linear Arboricity Conjecture, and we will explain known results. In the cases that we can prove exact results, we will also prove that the list coloring versions of these problems are the same as the ordinary versions. Finally, we will briefly discuss related problems in matroid coloring and star arboricity.

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Chapter 1

Introduction

1.1 Background and motivation

An edge-coloring of a multigraph G is an assignment of colors to the edges of G. Various interesting edge-coloring problems arise when we restrict what the color classes of the edge-coloring could look like. The most well-known type of edge-coloring described this way is a proper edge-coloring, where no two adjacent edges can receive the same color. In this case the color classes are restricted to being matchings, and the classical problem is to determine the minimum number of colors necessary to find a proper edge-coloring of a multigraph G, called the chromatic index of G and denoted by $\chi'(G)$. But one can ask a similar question if each color class is required to be some other kind of subgraph than a matching, such as a forest, a pseudoforest, a subgraph of bounded degree, or a forest of bounded degree. (It is also common to refer to these as decomposition problems, but it will be beneficial for us to think of them as edge-coloring.) For some of these questions, an exact and satisfactory answer is known, while for others getting even close to an exact answer appears quite difficult, if not NP-hard.

One notable pattern with a lot of answers, bounds, and conjectures on these edge-coloring problems is their similar dependence on two key parameters of the multigraph in question: maximum degree and maximum density. (The exact definition of density will vary depending on context, but in general it refers to the idea of having many edges on relatively few vertices; the parameter $\rho(G)$ below is a good example.) Generally, the maximum degree and maximum density of a multigraph form trivial lower bounds for these edge-coloring problems based on the restrictions, such as how many edges of a certain color can meet at a vertex, or how many edges a color classes can have. An interesting phenomenon in many edge-coloring problems is that these trivial lower bounds are often close to being exact, so there is an intimate connection between certain edge-colorings and maximum degree or maximum density. In this essay, we summarize and relate results and conjectures about various edge-coloring problems that depend closely on maximum degree or maximum density. Because of their similar-looking bounds, we also look for a way to interpolate among these problems. List

coloring analogues will also be studied. In the process, we search for underlying themes in the proofs that could aid in a better understanding of edge-coloring in general.

One of the main motivations for this essay is the celebrated Goldberg-Seymour Conjecture [43, 80] on the chromatic index. For background, there are two easy lower bounds one could derive on the chromatic index $\chi'(G)$ of a multigraph G. One is that $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of G. The other is that $\chi'(G) \geq \lceil \rho(G) \rceil$ where $\rho(G)$ denotes the maximum density parameter

$$\rho(G) = \max_{S \subseteq V(G), |S| \geq 2} \frac{e(S)}{\lfloor |S|/2 \rfloor}.$$

When it comes to simple graphs G, a central theorem of Vizing [90] states that the chromatic index $\chi'(G)$ is either $\Delta(G)$ or $\Delta(G)+1$. A lot of work has gone into distinguishing these two chromatic classes, but in general the problem is NP-hard [87]. For simple graphs G, the maximum density $\lceil \rho(G) \rceil$ seems to play no role in determining the chromatic index $\chi'(G)$, but for more general multigraphs G, it appears to be fundamental. The Goldberg-Seymour Conjecture asserts that for any multigraph G,

$$\chi'(G) \leq \max \{\Delta(G)+1, \lceil \rho(G) \rceil\}.$$

Thus, the conjecture states that there is a pretty tight relationship between $\chi'(G)$ and each of $\Delta(G)$ and $\lceil \rho(G) \rceil$ separately, in the sense that $\chi'(G)$ is almost entirely determined by one of them.

There has been a lot of work devoted to trying to prove the Goldberg-Seymour Conjecture, and a proof has recently been announced in a long and technical paper by Chen, Jing, and Zang [23], though it awaits verification. We will not look into this proof. Instead, in Chapter 2 we will explain in detail structural techniques that have been used to get quite close to the conjecture, specifically the use of Tashkinov trees as developed by Tashkinov [88]. (The use of this technique is fundamental in the announced proof [23].) We will start by going over classical results and proofs about edge-coloring multigraphs such as Vizing's Theorem [90] that $\chi'(G) \leq \Delta(G) + \mu(G)$, where $\mu(G)$ denotes the maximum number of parallel edges in G, and Shannon's Theorem [82] that $\chi'(G) \leq 3\Delta(G)/2$. The structural techniques used to prove these results will have a common theme: We will consider a critical multigraph (where deleting any edge decreases its chromatic index) and a proper edge-coloring of all but one edge of the multigraph. There must be a reason we cannot color the uncolored edge with one of the colors, and by studying why, we can build a "critical" subgraph of the multigraph whose properties let us deduce an upper bound on the chromatic index. This is the approach used to construct Vizing multi-fans and Kierstead paths to prove Vizing's Theorem and Shannon's Theorem, respectively. These critical subgraphs will be generalized by Tashkinov trees, which have proven to be much more useful for attacking the Goldberg-Seymour Conjecture. After giving more background on the Goldberg-Seymour Conjecture, we will use Tashkinov trees to prove an approximation of the conjecture due independently to Scheide [77] and to Chen, Yu, and Zang [24]. A maximum Tashkinov tree will induce an

approximate "dense spot" in the multigraph as asserted by the Conjecture. We will end the chapter by summarizing known results, without proofs, on proper list edge-colorings where the goal is to prove the infamous List Coloring Conjecture (see [18]), which states that the list chromatic index and the chromatic index are equal for any multigraph. The List Coloring Conjecture is the inspiration for all of the list edge-coloring results we will prove.

The hope about the discussed well-studied proper edge-coloring techniques is that they could help inform the study of other edge-coloring problems on multigraphs, particularly ones that seem to depend heavily on the maximum degree or maximum density of the multigraph. This will be the case in Chapter 3 where we study edge-colorings such that the color classes are required to be forests rather than matchings. The minimum number of colors needed to edge-color a multigraph into forests is called the arboricity of the multigraph, and a famous theorem of Nash-Williams [70] (independently proven by Tutte [89]) states that it is exactly given by a maximum density parameter which forms a trivial lower bound. We give a proof of Nash-Williams' Theorem by following the same approach as many proper edge-coloring proofs: using a maximal forest edge-coloring of a critical multigraph to construct a "critical" subgraph whose properties give an upper bound for the arboricity. In this case, the upper bound turns out to match the trivial lower bound. In addition, we can extend this proof without much difficulty to the list setting and prove a result of Seymour [79] that list arboricity and arboricity are the same. The proofs in this chapter can be written in the more general context of matroids, but we stick to the multigraph setting for the purposes of drawing connections to the chromatic index.

In Chapter 4, we will discuss the related edge-coloring problem where the color classes of a multigraph are required to be pseudoforests, known as pseudoarboricity. A theorem of Hakimi [51] states that the pseudoarboricity of a multigraph is given by a maximum density parameter similar to that of Nash-Williams' Theorem on arboricity, and we will prove it along similar lines. This time, though, we will use orientations of multigraphs rather than edge-colorings directly. This approach has similarities to proper edge-colorings in other ways, and in addition to proving Hakimi's Theorem, it will let us quite easily prove that pseudoarboricity and list pseudoarboricity are the same.

Having discussed each of the Goldberg-Seymour Conjecture, Nash-Williams' Theorem, and Hakimi's Theorem, we wish to understand the vague similarities apparent among them, both in the statements and in the techniques of their proofs or proof attempts. This motivates us to look for a sort of "interpolation" among the chromatic index, arboricity, and pseudoarboricity. A natural approach that we take in Chapter 5 is bounded degree edge-colorings. We study edge-colorings where the color classes are subgraphs, forests, or pseudoforests that have maximum degree at most some specified integer t. In these cases, the vague interplay between maximum degree and maximum density becomes more transparent. We will prove exact results in some cases, and in other cases we will give a Goldberg-Seymour-type of conjecture on these parameters. First we will prove an exact result on the edge-coloring problem for subgraphs when the maximum degree t is even, which turns out to be just another way of writing Petersen's famous 2-factor theorem (see [67]). Still, the edge-coloring perspective has some advantages. For one, we will prove the natural list coloring conjecture for this edge-

coloring problem. Moreover, we will observe striking similarities between this parameter and pseudoarboricity, particularly in the use of multigraph orientations, and these observations will be generalized when we study bounded degree pseudoarboricity. We will prove an exact formula for bounded degree pseudoarboricity that resembles the Goldberg-Seymour Conjecture, and we will also prove that the natural list coloring conjecture holds in this setting. For the case of subgraphs for odd t and bounded degree arboricity for general t, it becomes more difficult to prove exact results, and in some cases it is NP-hard to do so. We will pay particular attention to the case of bounded degree arboricity when $t = 2$. In this case, called linear arboricity, the color classes are required to be linear forests, and the famous Linear Arboricity Conjecture [3, 4] attempts to give a near-optimal upper bound in terms of the maximum degree of the multigraph. We strengthen this conjecture by conjecturing that a Goldberg-Seymour-type upper bound holds, and in the process we connect it to the ordinary Goldberg-Seymour Conjecture. Finally, we will survey structural results on bounded degree arboricity that have been used to get close to optimal results in some cases.

We end with two topics that provide a different but related direction of study on edge-coloring multigraphs. In Chapter 6, we will give a brief overview of coloring problems on matroids. Everything we proved about arboricity applies more generally to matroids, and it is worth mentioning these matroid coloring results. Then we will discuss a variation of ordinary coloring for which there are many open problems. The problem is known as joint coloring, where given two matroids on the same ground set we want to color the ground set into the fewest possible monochromatic common independent sets. This problem simultaneously generalizes both arboricity and proper edge-colorings on bipartite multigraphs, and getting even good bounds on it has proven difficult. We mention known results and conjectures, including a version of the list coloring conjecture for matroid joint colorings. In Chapter 7, we will study another variant of arboricity on multigraphs known as star arboricity. In this case, the color classes of the edge-coloring are required to be star forests. Star arboricity has natural lower and upper bounds in terms of ordinary arboricity, but in general it quite hard to determine. We will survey known results and prove basic upper bounds. We will also discuss the natural list coloring conjecture for star arboricity, giving some support for it.

We will conclude with a reflection on the various connections observed and conjectured among the studied edge-coloring problems. Along the way we will discuss possible future work for this meta-problem of trying to interpolate between maximum degree and maximum density as bounds for edge-coloring problems.

Although this essay is mainly expository, we give a number of results as well as proofs of known results that we were unable to find in the literature, namely: in Chapter 3, an edge-coloring formulation of one proof of Nash-Williams' Theorem 23 on arboricity and of Lasoń's proof of Seymour's Theorem 27 on list arboricity; in Chapter 4, the proof of Theorem 34 on list pseudoarboricity; in Chapter 5, Section 5.1, Theorem 42 on the list analogue of the degree t chromatic index for even t; in Chapter 5, Section 5.2, Proposition 50 on a multigraph version of an upper bound for degree t arboricity of sparse graphs; and in Chapter 5, Section 5.3, Theorem 53 on a formula for degree t pseudoarboricity and Theorem 54 on its list analogue.

1.2 Terminology and notation

In this essay, a multigraph will be assumed to be nonempty, finite, undirected, and loopless, but it is allowed to have parallel edges. Let $G = (V, E)$ be a multigraph, which has a vertex set $V = V(G)$ and an edge set $E = E(G)$. We write $v(G) = |V(G)|$ and $e(G) = |E(G)|$. If the edge $e \in E(G)$ has the vertex $v \in V(G)$ as an end-vertex, then e is said to be **incident** to v. If two edges are incident to a common vertex, they are said to be **adjacent edges**. Similarly, if two vertices have an edge between them, they are said to be **adjacent vertices**. For two vertices u, v of G, the set $E_G(u, v)$ denotes the set of all edges in $E(G)$ connecting u and v.

The **degree** $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges in $E(G)$ that are incident to v. The **multiplicity** $\mu_G(u, v)$ of two distinct vertices $u, v \in V(G)$ is the number of edges connecting u and v, i.e., $\mu_G(u, v) = |E_G(u, v)|$. The **maximum degree** $\Delta(G)$ and the **maximum multiplicity** $\mu(G)$ of G are defined as

$$\Delta(G) = \max_{v \in V(G)} d_G(v), \qquad \mu(G) = \max_{u,v \in V(G)} \mu_G(u, v).$$

A multigraph G is called a **simple graph** if $\mu(G) = 1$, i.e., if there are no parallel edges. A multigraph is called k**-regular** if $d_G(v) = k$ for all $v \in V(G)$. We have the general identity

$$2e(G) = \sum_{v \in V(G)} d_G(v).$$

The **average degree** of G is $d(G) = \frac{1}{v(G)} \sum_{v \in V(G)} d_G(v) = 2e(G)/v(G)$.

A subgraph H of a multigraph G is a multigraph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph H is a **spanning subgraph** if $V(H) = V(G)$. Given a vertex subset $S \subseteq V(G)$, the subgraph of G **induced by** S is $G[S] = (S, \{e \in E(G) : e \in E_G(u, v) \text{ for some } u, v \in S\})$. We write $e(S) = e(G[S])$ and $G - S = G[V(G) \setminus S]$. In the case $S = \{v\}$, we let $G - v$ be short for $G - \{v\}$. For an edge subset $F \subseteq E(G)$, we write $G - F = (V(G), E(G) \setminus F)$. In the case $F = \{e\}$, we let $G - e$ be short for $G - \{e\}$.

An **orientation** D of a multigraph G is the multigraph G with each edge replaced by an **arc**, which is the edge together with a direction from one end-vertex to the other. Then D is said to be an **oriented** or **directed multigraph**. For a vertex v of D, its **indegree** $d_D^-(v)$ is the number of incident arcs oriented toward v, and its **outdegree** $d_D^+(v)$ is the number of incident arcs oriented away from v. We use the same subgraph notations as in the undirected case to indicate directed subgraphs of D.

For now, we define the **density** of a vertex subset $S \subseteq V(G)$ to be $e(S)/|S|$, and we define the **maximum density** of G to be $\max_{S \subseteq V(G), S \neq \emptyset} e(S)/|S|$. In general, we will use the terms "density" and "maximum density" non-rigorously to describe the idea of containing lots of edges on relatively few vertices. The exact ratio of edges to vertices in the discussed density parameters will vary depending on the context.

A multigraph G is said to be **k-degenerate** if every subgraph of G has a vertex of degree at most k. A useful property of degeneracy is that we can define a **degeneracy ordering** on $V(G)$. Specifically, we order the vertices of G as $v_1, v_2, \ldots, v_n$, where for $i \in \{n, n-1, \ldots, 1\}$ we recursively set v_i to be a vertex of minimum degree in $G - \{v_{i+1}, \ldots, v_n\}$. If G is k-degenerate, then at each step i we remove a vertex of degree at most k, thus removing at most k edges from G. This implies that $e(G) \leq k(v(G) - 1)$. Every subgraph of G is also k-degenerate, so in fact $e(S) \leq k(|S| - 1)$ for all $S \subseteq V(G)$. Thus, degeneracy provides a way of insisting that a multigraph be nowhere dense, i.e., a measure of a multigraph's sparsity.

Finally, an **edge-coloring** ϕ of a multigraph G is a function from $E(G)$ to some other set. An element in the codomain set of ϕ is referred to as a **color**, and if e is an edge of G, then $\phi(e)$ is said to be the color of the edge e with respect to ϕ. A **k-edge-coloring** of G is an edge-coloring ϕ of G using colors from the set $[k] = \{1, \ldots, k\}$. If ϕ be is edge-coloring of a multigraph G and α is a color used by ϕ on some edge of G, we call the set $\phi^{-1}(\alpha) = \{e \in E(G) : \phi(e) = \alpha\}$ a **color class** of ϕ. Since the edges of a color class all have the same color with respect to ϕ, we describe the subgraph they form as **monochromatic**.

Chapter 2

Proper edge-colorings

We follow the textbook of Stiebitz et al. [86] for the terminology and results of this chapter. Let G be a multigraph. An edge-coloring ϕ of G is said to be a **proper edge-coloring** if no two adjacent edges e, e' of G receive the same color with respect to ϕ, i.e., $\phi(e) \neq \phi(e')$. The **chromatic index** $\chi'(G)$ of G is the minimum integer k for which there exists a proper k-edge-coloring of G. This will be the multigraph parameter of focus for this chapter. Note that $\chi'(G) \leq e(G)$ is finite, since giving each edge of G a different color results in a proper edge-coloring. By the definition of a proper edge-coloring, a color class of a proper edge-coloring ϕ is a collection of edges in which no two edges are adjacent. Such a collection of edges is called a **matching**. Thus, a proper edge-coloring of G can be viewed as a partition of $E(G)$ into matchings.

We now prove some preliminary results on the chromatic index $\chi'(G)$. If ϕ is a proper edge-coloring of a multigraph G, the edges incident to a given vertex must all have different colors. Thus, an easy lower bound for the chromatic index $\chi'(G)$ is $\chi'(G) \geq \Delta(G)$. On the other hand, we can find an upper bound on $\chi'(G)$ by using a greedy coloring procedure. We color the edges of G sequentially in arbitrary order using the colors $\{1, \ldots, k\}$, at each step choosing the smallest color to give to the edge e so that the edge-coloring remains proper (as a partial edge-coloring). Each of the two end-vertices of e is incident to at most $\Delta(G) - 1$ edges other than e. Hence, if $k = 2\Delta(G) - 1$, then there will always be a color to give to e so that the edge-coloring remains proper. This proves that

$$\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1.$$

Thus, $\chi'(G)$ is always within a factor of 2 of $\Delta(G)$. The discussion that follows is a history of attempts to determine the chromatic index $\chi'(G)$ as closely as possible, or at least to improve the greedy upper bound.

The results we will prove will often be about multigraphs that are critical for chromatic index. For a multigraph G, an edge $e \in E(G)$ is said to be a **critical edge** if $\chi'(G - e) < \chi'(G)$, that is, deleting e from G decreases the chromatic index of G. Necessarily in this case, $\chi'(G - e) = \chi'(G) - 1$. Thus, if G is a multigraph with a critical edge e, then $G - e$

has a proper $(\chi'(G)-1)$-edge-coloring. A multigraph G is said to be **critical** if each of its edges is critical. Every multigraph G has a critical subgraph with the same chromatic index, which can be found by iteratively deleting non-critical edges from G.

Now, let ϕ be a proper edge-coloring of a multigraph G. As mentioned before, each of the color classes of ϕ forms a matching. Let α and β be two distinct colors used by ϕ. If we look at the spanning subgraph of G with edge set $\phi^{-1}(\alpha) \cup \phi^{-1}(\beta)$, we see that its connected components consist of edges that alternate in the colors α and β as we walk along the component in any direction. In particular, the subgraph has maximum degree 2, and each of its connected components are either paths or even cycles (which may include the 2-cycle consisting of two parallel edges). In other words, even cycles and paths are the only connected multigraphs with chromatic index two. These maximal dichromatic connected components in a proper edge-coloring are often called **Kempe chains**. If a Kempe chain with respect to ϕ is a path, we call it simply an (α,β)**-alternating path** with respect to ϕ, interpreted to be a maximal such path. Notice that if we flip the colors of the edges in a Kempe chain C, so that edges of color α become β and vice versa, then we get another proper edge-coloring ϕ' of G. In this case, we say that ϕ' is obtained from ϕ by **switching on** C. Switching on Kempe chains will be a frequent operation in the edge-coloring arguments that follow. Almost always, the switching will be done on alternating paths.

Finally, we define the notions of present and missing colors. Let ϕ be a proper k-edge-coloring of a multigraph G. As explained before, we must have $k \geq \Delta(G)$. For a vertex $v \in V(G)$, we say that a color $\alpha \in \{1,\ldots,k\}$ is **present** at v if there is some edge incident to v that is colored α with respect to ϕ. Otherwise, we say that the color α is **missing** at v. Because all edges incident to v must get different colors with respect to ϕ, the number of colors present at v is $d_G(v)$, and the number of colors missing at v is $k - d_G(v)$. Observe that if α is missing at v and β is present at v, then there is an (α,β)-alternating path P that starts at v and ends at some other vertex. If we switch on P, then we get a proper edge-coloring ϕ' in which now α is present at v and β is missing at v. Often we will choose k large enough so that some or all vertices of G will be missing at least one color.

2.1 Fundamentals of proper edge-colorings

We can start proving better bounds on the chromatic index $\chi'(G)$. Recall the general lower $\chi'(G) \geq \Delta(G)$ for every multigraph G. A theorem of König [64] states that this lower bound is tight when G is bipartite. We present a proof that illustrates the utility of alternating paths, a prevalent theme in proper edge-coloring proofs, although it is also common to prove König's Theorem using Hall's matching theorem, which is equivalent.

Theorem 1 (König)**.** *For every bipartite multigraph G, we have $\chi'(G) = \Delta(G)$.*

Proof. Assume not. Let G be a bipartite multigraph with $\chi'(G) = k \geq \Delta(G)+1$. By possibly deleting edges, we may assume that G has a critical edge $e_0 \in E_G(x,y)$. Let ϕ be a

proper $(k-1)$-edge-coloring of $G - e_0$. Since the edge e_0 is uncolored and $k - 1 \geq \Delta(G)$, the end-vertices x and y are each missing a color. Say that α is missing at x and that β is missing at y. If $\alpha = \beta$, then we may give e_0 the color α and obtain a proper $(k-1)$-coloring of G, a contradiction. Otherwise, $\alpha \neq \beta$ and β is present at x. Then there is an (α, β)-alternating path P starting at x. Observe that P cannot contain y. If it did, then P starts with β at x and ends with α at y, which implies that it has even length, and so $P \cup \{e_0\}$ is an odd cycle, which contradicts the hypothesis that G is bipartite. Thus, we can switch on P to get a new proper $(k-1)$-edge-coloring ϕ' of $G - e_0$. Now α is missing at both x and y and so we may give e_0 the color α, again a contradiction. □

On the other hand, if G is an odd cycle, then we have that $\chi'(G) = \Delta(G) + 1$. A surprising theorem of Vizing [90] asserts that $\Delta(G)$ and $\Delta(G) + 1$ are the only two possible values for $\chi'(G)$ when G is a simple graph. Thus, the chromatic index of a simple graph is determined almost entirely by its maximum degree. More generally, Vizing's theorem states that $\chi'(G) \leq \Delta(G) + \mu(G)$ for any multigraph G. Let us study its proof.

Let G be a multigraph, let $e_0 \in E_G(x, y_0)$ be an edge, and let ϕ be a proper edge-coloring of $G - e_0$. A **Vizing multi-fan** at x with respect to ϕ is a sequence $F = (x, e_0, y_0, e_1, y_1, \ldots, e_p, y_p)$ with $p \geq 1$ consisting of distinct edges $e_0, e_1, \ldots, e_p \in E(G)$ and not necessarily distinct vertices $y_0, y_1, \ldots, y_p \in V(G)$ that satisfy:

- for each $i \in \{0, 1, \ldots, p\}$, $e_i \in E_G(x, y_i)$;
- for each $i \in \{0, 1, \ldots, p\}$, $\phi(e_i)$ is missing at y_j for some $0 \leq j < i$.

Viewing F as a subgraph of G, we let $V(F) = \{x, y_0, y_1, \ldots, y_p\}$ and $E(F) = \{e_0, e_1, \ldots, e_p\}$ (see Figure 2.1). The following central result will enable us to use Vizing multi-fans to prove Vizing's Theorem.

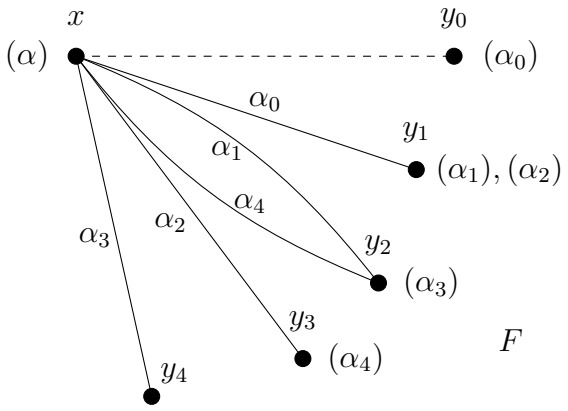

Figure 2.1: A Vizing multi-fan F. The parentheses indicate colors missing at the vertex.

Theorem 2. *Let G be a multigraph with $\chi'(G) = k \geq \Delta(G) + 1$, let e_0 be a critical edge of G, and let ϕ be a proper $(k-1)$-edge-coloring of $G - e_0$. If $F = (x, e_0, y_0, \ldots, e_p, y_p)$ is a Vizing multi-fan at x with respect to ϕ, then*

(a) No vertex y_i is missing a color in common with x.

(b) If α is missing at x and β is missing at y_i for some $0 \leq i \leq p$, then there is an (α, β)-alternating path with end-vertices x and y_i.

(c) No two distinct vertices y_i and y_j are missing a common color.

Proof. For (a), assume not. Choose ϕ and F so that both x and y_i are missing a common color α, with i as small as possible. If $i = 0$, then we can color e_0 with α and get a contradiction. Otherwise, $i \geq 1$ and for the color $\beta = \phi(e_i)$ there is an index $j < i$ such that β is missing at y_j. Recolor e_i with α. This new coloring ϕ' gives a new multi-fan $(x, e_0, y_0, e_1, y_1, \ldots, e_j, y_j)$ at x such that x and y_j are both missing β, which contradicts the minimality of i.

For (b), assume not. Let i be the smallest index for which the statement is false. By (a), α is present at y_j for each $j \in \{0, 1, \ldots, p\}$. By the minimality of i, the (α, β)-alternating path P starting at y_i ends at some vertex $x' \notin \{x, y_0, \ldots, y_i\}$. Since x cannot be in P, none of $e_0, \ldots, e_i$ are in P. Switch on P to get a $(k-1)$-edge-coloring ϕ' of $G - e_0$. Then $F' = (x, e_0, y_0, \ldots, e_i, y_i)$ is a multi-fan at x with respect to ϕ' with α missing at both x and y_i, contradicting (a).

For (c), assume there is a color β missing at two vertices y_i and y_j. Let α be missing at x. By (b), there is an (α, β)-alternating path with end-vertices x and y_i, and an (α, β)-alternating path with end-vertices x and y_j. This is impossible if y_i and y_j are distinct. □

Theorem 3. *Let G be a multigraph with $\chi'(G) = k \geq \Delta(G) + 1$, let $e_0 \in E_G(x, y_0)$ be a critical edge of G, and let ϕ be a proper $(k-1)$-edge-coloring of $G - e_0$. If F is a maximal Vizing multi-fan at x with respect to ϕ, then*

$$\sum_{y \in V(F) \setminus \{x\}} (d_G(y) + \mu_F(x, y)) = 2 + (k-1)(v(F) - 1).$$

Proof. Let F be a maximal Vizing multi-fan at x with respect to ϕ. Let $X = \{\phi(e_1), \ldots, \phi(e_p)\}$ be the set of colors of the edges of F, and let Y be the set of colors missing at one of the vertices $y_0, \ldots, y_p$. By the definition of a multi-fan, we have $X \subseteq Y$. On the other hand, if $\beta \in Y$, then by Theorem 2(a) β is present at x, and by the maximality of F we have $\beta \in X$. Thus, $X = Y$, and in particular $|X| = |Y|$. On one hand, we have

$$|X| = p = \left(\sum_{y \in V(F) \setminus \{x\}} \mu_F(x, y) \right) - 1,$$

where the -1 comes from e_0 being uncolored. On the other hand, since the vertices $y_0, \ldots, y_p$ have different missing colors by Theorem 2(c), we have

$$|Y| = 1 + \sum_{y \in V(F)\setminus\{x\}} (k - 1 - d_G(y)) = 1 + (k-1)(v(F)-1) - \sum_{y \in V(F)\setminus\{x\}} d_G(y).$$

where the extra 1 comes from the fact that the edge e_0 incident to y_0 is uncolored. Equating $|X|$ and $|Y|$, the result follows. □

The equation in Theorem 3 is often referred to as the **fan equation**. From it, we easily derive the classical edge-coloring theorem of Vizing [91] (also discovered by Gupta [49]).

Theorem 4 (Vizing)**.** *For every multigraph G, we have $\chi'(G) \leq \Delta(G) + \mu(G)$. In particular, if G is a simple graph, then $\chi'(G) \leq \Delta(G) + 1$.*

Proof. The multigraph G has a subgraph H with $\chi'(H) = \chi'(G) = k$ and a critical edge $e_0 \in E_H(x, y_0)$. Let ϕ be a proper $(k-1)$-edge-coloring of H, and let F be a maximal Vizing multi-fan at x with respect to ϕ. By the fan equation of Theorem 3,

$$\sum_{y \in V(F)\setminus\{x\}} (d_H(y) + \mu_F(x, y)) = (k-1)(v(F)-1) + 2.$$

The left-hand side is at most $(\Delta(H) + \mu(H))(v(F)-1)$, from which it follows that $k - 1 < \Delta(H) + \mu(H)$. Thus, $\chi'(G) = k \leq \Delta(H) + \mu(H) \leq \Delta(G) + \mu(G)$. □

Vizing's bound is tight for a triangle with $\mu(G)$ parallel edges on each side, as all the edges in this graph must have different colors. For this multigraph, we have $\chi'(G) = 3\mu(G)$ whereas $\Delta(G) = 2\mu(G)$. Thus, if the multiplicity $\mu(G)$ is unconstrained, the gap between $\chi'(G)$ and $\Delta(G)$ can be arbitrarily large, and maximum degree plays a lesser role in determining the chromatic index. At this point, we will see in the next section, maximum density will play a bigger role in determining the chromatic index.

Going back to simple graphs G, Vizing's Theorem 4 states that either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. Moreover, the proof can be turned into a polynomial-time algorithm that finds a proper $(\Delta(G)+1)$-edge-coloring for any simple graph G. If $\chi'(G) = \Delta(G)$, then G is said to be of **class 1**. If $\chi'(G) = \Delta(G) + 1$, then G is said to be of **class 2**. König's Theorem 1 states that every bipartite simple graph is of class 1. Odd cycles are of class 2. However, it is NP-complete to recognize whether an arbitrary simple graph is of class 1 or of class 2, even for cubic (i.e., 3-regular) graphs [56]. The four-color theorem for planar graphs is equivalent to the statement that all bridgeless planar cubic graphs are of class 1 [87]. This provides a good indication of the general difficulty of determining whether $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G)+1$ for simple graphs G. For multigraphs G with $\chi'(G) \geq \Delta(G)+2$, however, the story appears to be different, as we will observe in the next section. For the remainder of this section, we review further background on multigraph edge-coloring that is essential for what is forthcoming.

Let G be a multigraph, let $e_0 \in E_G(y_0, y_1)$ be an edge, and let ϕ be a proper edge-coloring of $G - e_0$. A **Kierstead path** with respect to ϕ is a sequence $P = (y_0, e_0, y_1, e_1, \ldots, e_{p-1}, y_p)$ with $p \geq 1$ consisting of distinct vertices $y_0, y_1, \ldots, y_p \in V(G)$ and edges $e_0, e_1, \ldots, e_{p-1} \in E(G)$ that satisfy:

- for each $i \in \{0, 1, \ldots, p-1\}$, $e_i \in E_G(y_i, y_{i+1})$;
- for each $i \in \{1, 2, \ldots, p-1\}$, $\phi(e_i)$ is missing at y_j for some $0 \leq j < i$.

Viewed as a subgraph, a Kierstead path is indeed a path (see Figure 2.2). The following result was discovered by Kierstead [60].

Theorem 5 (Kierstead)**.** *Let G be a multigraph with $\chi'(G) = k \geq \Delta(G) + 1$, let $e_0 \in E_G(y_0, y_1)$ be a critical edge of G, and let ϕ be a proper $(k-1)$-edge-coloring of $G - e_0$. If $P = (y_0, e_0, y_1, e_1, \ldots, e_{p-1}, y_p)$ is a Kierstead path with respect to ϕ and $d_G(y_j) \leq k - 2$ for each $j \in \{2, \ldots, p\}$, then the vertices of P are all missing different colors.*

Proof. Suppose not. Let $P = (y_0, e_0, y_1, e_1, \ldots, e_{p-1}, y_p)$ be a counterexample of minimum length among all $(k-1)$-edge-colorings ϕ of $G - e_0$. Clearly $p \geq 2$, since if $p = 1$ then y_0 and y_1 are missing a common color α and we can give e_0 the color α, thus giving a contradiction. Because e_0 is uncolored, each of y_0 and y_1 has at least one missing color. Moreover, each of $y_2, \ldots, y_p$ also has at least $k - 1 - d_G(y_j) \geq 1$ missing colors by the degree condition.

Observe that $P' = (y_0, e_0, y_1, e_1, \ldots, e_{p-2}, y_{p-1})$ is a Kierstead path with respect to ϕ that satisfies the degree condition $d_G(y_j) \leq k - 2$, so by the minimality of P, the vertices $y_0, y_1, \ldots, y_{p-1}$ have distinct missing colors. Thus, there is a maximal index $i = \mathrm{ind}(\phi)$ such that y_i and y_p have a common missing color. Choose a minimal counterexample (P, ϕ) such that $i = \mathrm{ind}(\phi)$ is maximum. The claim is that $i = p - 1$.

Assume that $i < p - 1$. Let α be a color missing at both y_i and y_p. Since every vertex in P is missing at least one color, let β be a color missing at y_{i+1}. Since $y_{i+1} \in V(P')$, we have $\alpha \neq \beta$, α is present at y_{i+1}, β is present at y_i, and both α and β are present at y_j for $j \in \{0, 1, \ldots, i-1\}$. This implies that $\phi(e_j) \notin \{\alpha, \beta\}$ for $j \in \{1, \ldots, i\}$, as otherwise by the definition of a Kierstead path one of α or β would be missing at y_j for some $j \in \{0, 1, \ldots, i-1\}$. Further, there is an (α, β)-alternating path Q starting at y_{i+1} and ending at some other vertex v. Let ϕ' be the edge-coloring obtained by switching on Q. If $v = y_i$, then P is a Kierstead path with respect to ϕ' with α missing at both y_{i+1} and y_p. Hence (P, ϕ') is a minimal counterexample with $\mathrm{ind}(\phi') > \mathrm{ind}(\phi)$, contradicting the choice

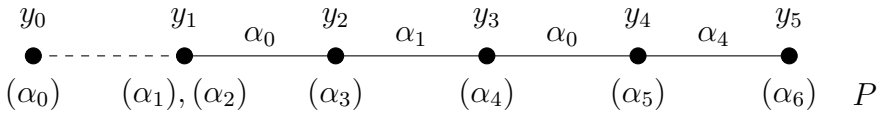

Figure 2.2: A Kierstead path P.

of (P, ϕ). If $v \neq y_i$, then $P'' = (y_0, e_0, y_1, e_1, \ldots, e_i, y_{i+1})$ is a Kierstead path with respect to ϕ', and α is missing at both y_i and y_{i+1}. Since $v(P'') < v(P)$, this makes (P'', ϕ') a smaller counterexample than (P, ϕ), again a contradiction.

By the claim, there is a color α missing at both y_{p-1} and y_p. For the color $\beta = \phi(e_{p-1})$, there exists an index $j < p-1$ such that β is missing at y_j. Recolor e_{p-1} with α. This results in an edge-coloring ϕ' of $G - e_0$ such that $P' = (y_0, e_0, y_1, e_1, \ldots, e_{p-2}, y_{p-1})$ is a Kierstead path with respect to ϕ', and β is missing at both y_j and y_{p-1}. Since $v(P') < v(P)$, this contradicts the minimality of P, and the proof is complete. □

Kierstead's Theorem 5 can be used to reprove Vizing's Theorem 4 that $\chi'(G) \leq \Delta(G) + \mu(G)$. This is done by deriving an equation analogous to the fan equation of Theorem 3.

Theorem 6. *Let G be a multigraph with $\chi'(G) = k \geq \Delta(G) + 2$, let $e_0 \in E_G(y_0, y_1)$ be a critical edge of G, and let ϕ be a proper $(k-1)$-edge-coloring of $G - e_0$. If $P = (y_0, e_0, y_1, e_1, \ldots, e_{p-1}, y_p)$ is a maximal Kierstead path with respect to ϕ, then*

$$\sum_{i=0}^{p-1} (d_G(y_i) + \mu_P(y_i, y_p)) = 2 + (k-1)(v(P) - 1),$$

where $\mu_P(y_i, y_p)$ denotes the number of edges between y_i and y_p whose color is missing at some vertex of P.

Proof. Let $P = (y_0, e_0, y_1, e_1, \ldots, e_{p-1}, y_p)$ be a maximal Kierstead path with respect to ϕ. Note that $d_G(y_j) \leq \Delta(G) \leq k - 2$ for each $j \in \{2, \ldots, p\}$. By Kierstead's Theorem 5, all of the vertices $y_0, y_1, \ldots, y_p$ are missing different colors. In particular, for each color missing at one of $y_0, y_1, \ldots, y_{p-1}$, there is an edge of that color incident to y_p. Since P is maximal, each of these edges must be between y_p and one of $y_0, y_1, \ldots, y_{p-1}$. The number of these edges is $\sum_{i=0}^{p-1} \mu_P(y_i, y_p)$, while the number of colors missing at one of $y_0, y_1, \ldots, y_{p-1}$ is $2 + \sum_{i=0}^{p-1}(k - 1 - d_G(y_i))$, where the extra 2 comes from e_0 being uncolored. Equating these two values, the result follows. □

Proving Vizing's Theorem 4 that $\chi'(G) \leq \Delta(G) + \mu(G)$ from Theorem 6 is basically the same argument as done with the fan equation of Theorem 3; in short, if $\chi'(G) > \Delta(G) + \mu(G)$, then the equation in Theorem 6 will be violated. In another direction, Kierstead's Theorem 5 can be used to prove the following theorem of Goldberg [42]. The **odd girth** $g_o(G)$ of a multigraph G is the length of a shortest odd cycle in G, taken to be ∞ if G is bipartite.

Theorem 7 (Goldberg)**.** *For every multigraph G, we have*

$$\chi'(G) \leq \Delta(G) + 1 + \frac{\Delta(G) - 2}{g_o(G) - 1}$$

Proof. If $\chi'(G) \leq \Delta(G)+1$, we are done. Thus, assume that $\chi'(G) = k \geq \Delta(G)+2$. Let H be a subgraph of G with $\chi'(H) = \chi'(G)$ and a critical edge $e_0 \in E_H(y_0, y_1)$. Note that $g_o(H) \geq g_o(G)$. Let ϕ be a proper $(k-1)$-edge-coloring of H. Then every vertex of H is missing at least one color with respect to ϕ. Let α be missing at y_0 and β be missing at y_1. We have $\alpha \neq \beta$, since otherwise we could give e_0 the color α and get a contradiction. Let Q be the (α,β)-alternating path starting at y_1. If Q does not end at y_0, switch on Q to get α missing at y_1, and color e_0 with α to get a contradiction. Thus, Q ends at y_0. This implies that $Q \cup \{e_0\}$ is an odd cycle, so $v(Q) \geq g_o(H) \geq g_o(G)$. Observe that $P = (y_0, e_0, Q - \{y_0\})$ is a Kierstead path, since each colored edge of P is colored either α or β, and these two colors are missing at either y_0 or y_1. By Kierstead's theorem, the vertices of P are all missing different colors. Thus, the number of colors missing at one of the vertices of P is $2 + \sum_{y \in V(Q)}(k-1-d_H(y))$, and this is at most $k-1$. Hence,

$$k-1 \geq 2 + \sum_{y \in V(Q)} (k-1-d_H(y)) \geq 2 + v(Q)(k-1-\Delta(H)) \geq 2 + g_o(G)(k-1-\Delta(G))$$

Solving this inequality for $k = \chi'(G)$ gives the required inequality. □

Note that we could have written the above proof's original inequality as

$$\sum_{y \in V(Q)} d_G(y) \geq 2 + (k-1)(v(Q)-1)$$

to make it look similar to Theorem 3 and Theorem 6. Now, because the odd girth g_o of any multigraph is at least 3, Goldberg's Theorem 7 implies the following result of Shannon [82].

Theorem 8 (Shannon)**.** *For every multigraph G, we have $\chi'(G) \leq 3\Delta(G)/2$.*

Historically, Shannon proved his result first, then Goldberg proved his, and finally Kierstead proved his. Like Vizing's bound, Shannon's upper bound is tight for some multigraphs. For $\Delta(G)$ even, take G to be the triangle with $\Delta(G)/2$ parallel edges on each side. This multigraph has $3\Delta(G)/2$ edges, and all the edges must get different colors. For $\Delta(G)$ odd, take G to be the triangle with $(\Delta(G)-1)/2$ parallel edges on two sides, and $(\Delta(G)+1)/2$ parallel edges on the other side. In this case we have $(3\Delta(G)-1)/2 = \lfloor 3\Delta(G)/2 \rfloor$ edges that must get different colors. We call these dense multigraphs **Shannon triangles**.

We can summarize the classical bounds on the chromatic index as

$$\chi'(G) \leq \min\left\{\Delta(G) + \mu(G),\ \Delta(G) + 1 + \frac{\Delta(G)-2}{g_o(G)-1}\right\}.$$

There are many other similar bounds that involve other parameters of the multigraph G.

We give some final remarks on the common theme with these edge-coloring proofs. We are studying maximal colorings of critical multigraphs. By taking sufficiently many colors, we may assume that every vertex is missing at least one color. With there being one uncolored

edge, there are restrictions on how the colored edges can be distributed around that uncolored edge. From these restrictions, we can find some kind of subgraph like a Vizing multi-fan or a Kierstead path that points to why the uncolored edge cannot be colored. With some work that may involve certain switches on alternating paths, we can prove that no two vertices in our subgraph can have a common missing color. Taking our subgraph to be maximal, we find that it induces many distinct edges or colors based on the number of colors missing at the vertices in the subgraph. Using estimates on these number of edges, we find an upper bound on the number of colors required for the subgraph, and thus on the number of colors required to color the entire multigraph. This sort of theme will continue when we discuss Tashkinov trees, as well as when we turn to variations of proper edge-colorings such as arboricity.

2.2 The Goldberg-Seymour Conjecture

Let G be a multigraph. Consider a proper k-edge-coloring of G. For any vertex subset $S \subseteq V(G)$, $|S| \geq 2$, each color class in $G[S]$ has size at most $\lfloor |S|/2 \rfloor$ since it is a matching. Thus, at least $\left\lceil \frac{e(S)}{\lfloor |S|/2 \rfloor} \right\rceil$ colors are needed to properly color $G[S]$, and by extension G. Defining

$$\rho(S) = \frac{e(S)}{\lfloor |S|/2 \rfloor}, \quad \rho(G) = \max_{S \subseteq V(G), |S| \geq 2} \rho(S),$$

we see that every multigraph G satisfies

$$\chi'(G) \geq \lceil \rho(G) \rceil$$

This is a maximum density lower bound for $\chi'(G)$, in contrast to the maximum degree lower bound $\chi'(G) \geq \Delta(G)$.

We note that when $v(G) \geq 3$, the maximum in $\rho(G)$ is achievable by a set S of odd cardinality. Specifically, suppose that the maximum is achieved by a set $S \subseteq V(G)$ with even cardinality. If $|S| = 2$, then $\rho(G) = \rho(S) = e(S)$, and letting $v \in V(G) \setminus S$ we have that the set $S' = S \cup \{v\}$ of odd cardinality satisfies $\rho(S') = e(S')/\lfloor |S'|/2 \rfloor \geq e(S) = \rho(G)$. If $|S| \geq 4$, then let v be a vertex of minimum degree in $G[S]$ and let $S' = S \setminus \{v\}$. Then $d_{G[S]}(v) \leq 2e(S)/|S|$, and thus

$$\rho(S') = \frac{2e(S')}{|S'| - 1} = \frac{2e(S) - 2d_{G[S]}(v)}{|S| - 2} \geq \frac{2e(S)}{|S|} = \rho(S) = \rho(G).$$

This demonstrates that

$$\rho(G) = \max_{\substack{S \subseteq V(G), \\ |S| \geq 3 \text{ odd}}} \frac{2e(S)}{|S| - 1}.$$

Note that $\chi'(K_{1,n}) = n$ whereas $\rho(K_{1,n}) = 2$ whenever $n \geq 2$. Thus, the gap between $\chi'(G)$ and $\lceil \rho(G) \rceil$ can be arbitrarily large, just as it was with $\Delta(G)$. This is particularly the

case of simple graphs like $K_{1,n}$ with high maximum degree and low density. However, the situation seems to be different when we look at multigraphs G with $\chi'(G) \geq \Delta(G) + 2$. For such multigraphs, it appears that maximum density $\lceil \rho(G) \rceil$ plays a bigger role in determining the chromatic index number $\chi'(G)$. For example, if G is the triangle with $\mu(G)$ edges on each side, where we saw that $\Delta(G) = 2\mu(G)$ and $\chi'(G) = 3\mu(G)$ are quite far apart, then we have that $\chi'(G) = \lceil \rho(G) \rceil$. These kinds of observations led Goldberg [43] and Seymour [80] independently to formulate a famous conjecture.

Conjecture 9 (Goldberg-Seymour)**.** *For every multigraph G, we have*

$$\chi'(G) \leq \max\{\Delta(G) + 1, \lceil \rho(G) \rceil\}.$$

In other words, the Goldberg-Seymour Conjecture states that if $\chi'(G) \geq \Delta(G) + 1$, then $\chi'(G) = \lceil \rho(G) \rceil$. Thus, if a multigraph G requires $\chi'(G) > \Delta(G) + 1$ colors in a proper edge-coloring, then it must have a vertex subset S whose induced subgraph $G[S]$ is so dense that it trivially requires $\chi'(G)$ colors in a proper edge-coloring. This set S induces a "dense spot". Hence, if the conjecture is true, then $\chi'(G)$ is either closely determined by maximum degree $\Delta(G)$, or completely determined by maximum density $\lceil \rho(G) \rceil$.

The Goldberg-Seymour Conjecture 9 may have recently been proven in a long technical paper by Chen, Jing, and Zang [23], though it currently awaits verification. We will not describe this proof. Instead, in the next section we will prove a weakening of the conjecture using a tool that is central to this long paper as well as other attempted proofs in that direction, namely Tashkinov trees. For multigraphs G with $\rho(G) > \Delta(G)$, Kahn [59] proved the asymptotic bound $\chi'(G) \leq (1 + o(1))\lceil \rho(G) \rceil$. This was improved by Plantholt [75] to

$$\chi'(G) \leq \left(1 + \frac{\log_{3/2}\lceil \rho(G) \rceil}{\lceil \rho(G) \rceil}\right) \lceil \rho(G) \rceil.$$

In addition, Goldberg in the same paper of his conjecture [43] formulated a sharp version of the Goldberg-Seymour Conjecture for multigraphs G with $\rho(G) \leq \Delta(G) - 1$.

Conjecture 10. *For every multigraph G, if $\rho(G) \leq \Delta(G) - 1$ then $\chi'(G) = \Delta(G)$.*

The Goldberg-Seymour Conjecture 9 has a close connection to fractional edge-colorings. A **fractional edge-coloring** w of a multigraph G is an assignment of a nonnegative weight w_M to each matching M of G, in such a way that for every edge $e \in E(G)$ we have $\sum_{M\,:\,e \in M} w_M = 1$, where the sum is over matchings M. The **fractional chromatic index** $\chi'^*(G)$ of G is the minimum of $\sum_M w_M$ taken over all fractional edge-colorings w of G. If ϕ is a proper edge-coloring of G using $\chi'(G)$ colors, then we get a fractional edge-coloring of G by setting $w_M = 1$ if M is a color class of ϕ, and $w_M = 0$ otherwise. In this case, $\sum_M w_M = \chi'(G)$. On the other hand, at every vertex v of G any two edges incident to v cannot lie in one matching M, and so in a fractional edge-coloring w of G we have $\sum_M w_M \geq \sum_{e \sim v} \sum_{M\,:\,e \in M} w_M = \sum_{e \sim v} 1 = d_G(v)$, where $e \sim v$ means that e is incident

to v. Thus, we have demonstrated the following bounds on the fractional chromatic index: $\Delta(G) \leq \chi'^*(G) \leq \chi'(G)$.

Using Edmonds' matching polytope theorem [25], Seymour [81] and Stahl [84] proved the following formula for the fractional chromatic index:

$$\chi'^*(G) = \max\{\Delta(G), \rho(G)\}.$$

This implies that the Goldberg-Seymour Conjecture can be rewritten as

$$\chi'(G) \leq \max\{\Delta(G) + 1, \lceil \chi'^*(G) \rceil\},$$

and this would imply that $\lceil \chi'^*(G) \rceil \leq \chi'(G) \leq \lceil \chi'^*(G) \rceil + 1$. Because of the connection to Edmonds' matching polytope theorem, the fractional chromatic index $\chi'^*(G)$ can be computed in polynomial time. Thus, the truth of the Goldberg-Seymour Conjecture 9 would imply that if $\chi'(G) > \Delta(G) + 1$, then $\chi'(G)$ can be computed in polynomial time. This means that the only difficulty in determining $\chi'(G)$ is to distinguish between the two cases $\chi'(G) = \Delta(G)$ and $\chi'(G) = \Delta(G) + 1$, which we mentioned is an NP-complete problem.

Finally, let us prove a proposition that is relevant for trying to prove the Goldberg-Seymour Conjecture 9, a result that we will also use to prove a weakening of the conjecture in the next section. Recall that an important idea which made Vizing multi-fans and Kierstead paths useful for proving upper bounds on the chromatic index is that all the vertices in these subgraphs are missing different colors (see Theorem 2 and Theorem 5). For a multigraph G and a partial proper edge-coloring ϕ of G, we call a vertex subset $S \subseteq V(G)$ **elementary** with respect to ϕ if no two vertices in S have a common missing color. Let $\partial_G(S)$ denote the set of edges with exactly one end-vertex in S. We say that S is **closed** with respect to ϕ if for every colored edge $f \in \partial_G(S)$ the color $\phi(f)$ is present at every vertex in S. Finally, we say that S is **strongly closed** with respect to ϕ if S is closed with respect to ϕ and $\phi(f) \neq \phi(f')$ for any two distinct colored edges $f, f' \in \partial_G(S)$. Vertex sets that are both elementary and strongly closed are closely connected to the Goldberg-Seymour Conjecture 9, as the following proposition indicates.

Proposition 11. *Let G be a critical multigraph with $\chi'(G) = k \geq \Delta(G) + 1$. Then the following are equivalent.*

(a) $\chi'(G) = \lceil \rho(G) \rceil$.

(b) For every edge $e \in E(G)$ and proper $(k-1)$-edge-coloring ϕ of $G - e$, the set $V(G)$ is elementary with respect to ϕ.

(c) There is an edge $e \in E(G)$ and a proper $(k-1)$-edge-coloring ϕ of $G - e$ such that $V(G)$ is elementary with respect to ϕ.

(d) There is an edge $e \in E(G)$, a proper $(k-1)$-edge-coloring ϕ of $G - e$, and a set $S \subseteq V(G)$ containing both end-vertices of e that is both elementary and strongly closed with respect to ϕ.

Proof. To show that (a) implies (b), assume that $\chi'(G) = k = \lceil \rho(G) \rceil$. Since G is critical, every subgraph H of G satisfies $\lceil \rho(H) \rceil \leq \chi'(H) < \chi'(G) = \lceil \rho(G) \rceil$. Thus, the maximum in $\rho(G)$ is uniquely achieved by the set $V(G)$. By a previous remark, $v(G)$ must be odd. And by the formula for $\rho(G)$, we have that $2e(G) > (k-1)(v(G)-1)$. Now let $e \in E(G)$ be any edge, and let ϕ be a proper $(k-1)$-edge-coloring of $G - e$. The claim is that no two vertices in $V(G)$ have a common missing color with respect to ϕ. By the bound on $2e(G)$ above, the size of the multiset of colors missing at the vertices in $V(G)$ is

$$\begin{aligned} 2 + \sum_{v \in V(G)} (k - 1 - d_G(v)) &= 2 + (k-1)v(G) - 2e(G) \\ &= 1 + k + (k-1)(v(G)-1) - 2e(G) \\ &< 1 + k. \end{aligned}$$

Because $v(G)$ is odd and the vertices at which a given color is present come in pairs, every color is missing at an odd number of vertices of G. In particular, the size of the multiset above is at least k. If one of the colors was missing at more than one vertex, then this would be at least $k+1$, which contradicts the upper bound above. Thus, $V(G)$ is elementary with respect to ϕ.

Clearly (b) implies (c). By taking $S = V(G)$, we see that (c) implies (d). Finally, to show that (d) implies (a), let e, ϕ, and S be as given. We show that $G[S]$ is a desired dense spot in G that shows that $\chi'(G) = \lceil \rho(G) \rceil$. Clearly $|S| \geq 2$. Because e is uncolored, the set of colors missing at one of the vertices in S is nonempty. Consider any such missing color α. Since S is elementary with respect to ϕ, the color α is missing at exactly one vertex. This implies that $|S|$ is odd, and that $\lfloor |S|/2 \rfloor$ edges of $G[S]$ have color α. Next, consider any color β present at every vertex in S (if one exists). Since S is strongly closed, at most one edge in $\partial_G(S)$ has color β. And because $|S|$ is odd, exactly one edge in $\partial_G(S)$ has color β. Thus again $\lfloor |S|/2 \rfloor$ edges of $G[S]$ have color β. This proves that $e(S) = 1 + (k-1)\lfloor |S|/2 \rfloor$, where the extra 1 comes from the uncolored edge e. Since $|S| \geq 2$, it follows that

$$\lceil \rho(G) \rceil \leq \chi'(G) = k \leq \frac{e(S)}{\lfloor |S|/2 \rfloor} + \left(1 - \frac{1}{\lfloor |S|/2 \rfloor}\right) < \rho(S) + 1 \leq \rho(G) + 1.$$

Therefore, $\chi'(G) = \lceil \rho(G) \rceil$ and the maximum in $\rho(G)$ is achieved by the set S. □

As was the case with previous edge-coloring proofs, to prove the Goldberg-Seymour Conjecture 9 it suffices to prove it for all critical multigraphs G. From this fact and the proof of Proposition 11, we can see that the following conjecture is equivalent to the Goldberg-Seymour Conjecture.

Conjecture 12 (Critical Multigraph Conjecture)**.** *Every critical multigraph G with $\chi'(G) = k \geq \Delta(G) + 2$ has an odd number of vertices and satisfies*

$$2e(G) = 2 + (k-1)(v(G)-1).$$

For trying to prove the Goldberg-Seymour Conjecture 9 for critical multigraphs G, Proposition 11 tells us that if we have an edge $e \in E(G)$ and a maximal proper edge-coloring ϕ of $G - e$, then it suffices for us to find a vertex subset $S \subseteq V(G)$ of G that is both elementary and strongly closed with respect to ϕ. Unfortunately, finding such a set S in general appears quite difficult. Even though the properties of S of being both elementary and strongly closed do impose great restrictions on the distribution of colors in $G[S]$, they do not by themselves make it easy to find such a set S in a general multigraph G. Both maximal Vizing multi-fans and maximal Kierstead paths are elementary, but they may not even be closed. In the next section, we will define the notion of a Tashkinov tree, which when maximal will turn out to be both elementary and closed, though not necessarily strongly closed. Still, it will be enough to prove a weakening of the Goldberg-Seymour Conjecture 9.

2.3 Tashkinov trees

Here, we define the notion of a Tashkinov tree and use it to prove a weakening of the Goldberg-Seymour Conjecture 9. This very important notion was introduced by Tashkinov [88] to give an alternative proof to a result of Nishizek and Kashiwagi [72] that supports the Goldberg-Seymour Conjecture. Since then, the methods of Tashkinov have been refined and used by various authors to prove other partial results on the Conjecture. The weakening we will prove is due independently to Scheide [77] and to Chen, Yu, and Zang [24]. As usual, we follow the textbook [86] for terminology and results. The terms "elementary", "closed", and "strongly closed" from the previous section will often be employed here.

Let G be a multigraph, let $e_0 \in E_G(y_0, y_1)$ be an edge, and let ϕ be a proper edge-coloring of $G - e_0$. A **Tashkinov tree** with respect to ϕ is a sequence $T = (y_0, e_0, y_1, e_1, \ldots, e_{p-1}, y_p)$ consisting of distinct vertices $y_0, \ldots, y_p \in V(G)$ and distinct edges $e_0, \ldots, e_{p-1} \in E(G)$ that satisfy:

- for each $i \in \{0, 1, \ldots, p-1\}$, $e_i \in E(y_j, y_{i+1})$ for some $j \leq i$;
- for each $i \in \{1, \ldots, p-1\}$, $\phi(e_i)$ is missing at y_j for some $0 \leq j < i$.

As a subgraph of G, a Tashkinov tree has the structure of a tree as the name implies (see Figure 2.3). Every Vizing multi-fan with respect to ϕ is a Tashkinov tree if all of its vertices

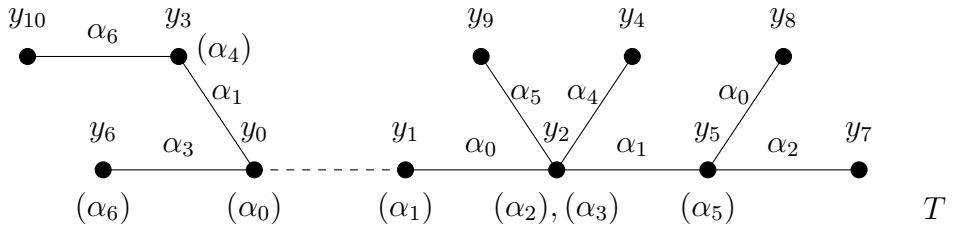

Figure 2.3: A Tashkinov tree T.

are distinct. In addition, every Kierstead path is a Tashkinov tree. Thus, Tashkinov trees are simultaneous generalizations of Vizing multi-fans and Kierstead paths. The following theorem of Tashkinov [88] generalizes Theorem 2 and Theorem 5. Note that we assume $\chi'(G) \geq \Delta(G)+2$ so that every vertex in a proper edge-coloring is missing at least one color.

Theorem 13 (Tashkinov)**.** *Let G be a multigraph with $\chi'(G) = k \geq \Delta(G)+2$, let $e_0 \in E(G)$ be a critical edge, and let ϕ be a proper $(k-1)$-edge-coloring of ϕ. If T is a Tashkinov tree with respect to ϕ, then $V(T)$ is elementary with respect to ϕ (i.e., all the vertices of T are missing different colors).*

The proof of Tashkinov's Theorem is somewhat long and technical, but requires no new ideas (see [86]). Like the proof of Kierstead's Theorem 5, it is a proof by minimal counterexample. This time, we choose a counterexample that, among all choices of e_0, ϕ, T, minimizes the index j such that the sequence $(y_j, e_j, \ldots, e_{p-1}, y_p)$ forms a path. Following this, we choose a counterexample that minimizes the number of vertices of T. The case $j = 0$ is Kierstead's Theorem 5, and the proofs for other cases of j are complicated case-by-case arguments involving many different switches on alternating paths, toward the goal of contradicting the minimality. The complexity of the proof portrays the potential difficulty of simply proving that some vertex set is just elementary, let alone both elementary and strongly closed.

Taking for granted Tashkinov's Theorem 13, we can prove that the vertex set $V(T)$ of a maximal Tashkinov tree T (with respect to a maximal proper coloring of a critical multigraph) has many of the properties we desire from a set S that maximizes $\rho(G)$ in the Goldberg-Seymour Conjecture 9 (as suggested by Proposition 11). For convenience, if T is a Tashkinov tree and y is a vertex of T, let Ty denote the Tashkinov tree T up to the vertex y in its sequence. Also, $\partial_G(T)$ will be short for $\partial_G(V(T))$, which again is the set of all edges of G with exactly one end-vertex in $V(T)$.

Proposition 14. *Let G be a multigraph with $\chi'(G) = k \geq \Delta(G) + 2$, let $e_0 \in E(G)$ be a critical edge, and let ϕ be a proper $(k-1)$-edge-coloring of $G - e_0$. Further, let T be a maximal Tashkinov tree with respect to ϕ, and let $T' = (y_0, e_0, y_1, e_1, \ldots, e_{p-1}, y_p)$ be an arbitrary Tashkinov tree with respect to ϕ. Then*

(a) $V(T)$ is elementary and closed with respect to ϕ;

(b) $v(T) \geq 3$ is odd;

(c) $V(T') \subseteq V(T)$;

(d) There is a Tashkinov tree $\widetilde{T}$ with respect to ϕ such that $V(\widetilde{T}) = V(T)$ and $\widetilde{T}y_p = T'$.

Proof. Write $T = (x_0, f_0, x_1, f_1, \ldots, f_{q-1}, x_q)$, where $f_0 = e_0$.

For (a), we have that $V(T)$ is elementary with respect to ϕ by Tashkinov's Theorem 13. To see that $V(T)$ is closed, let $f \in \partial_G(T)$ and let x be the end-vertex of f not in $V(T)$.

If $\phi(f)$ is missing at some vertex in $V(T)$, then $T'' = (x_0, f_0, x_1, f_1, \ldots, f_{q-1}, x_q, f, x)$ is a larger Tashkinov tree that contains T, which contradicts the maximality of T. Hence, $V(T)$ is closed with respect to ϕ.

For (b), let α be a color missing at x_0. Because $V(T)$ is elementary, α is present at every other vertex in $V(T)$. Further, because $V(T)$ is closed and α is present at every vertex in $V(T) \setminus \{x_0\}$, every edge with color α that is incident to a vertex in $V(T)$ must be between two vertices in $V(T)$. Hence, there are $(v(T)-1)/2$ edges with color α in $G[V(T)]$, and $v(T)$ is odd. Since necessarily $v(T) \geq 2$, we also get that $v(T) \geq 3$.

For (c), suppose that $V(T') \not\subseteq V(T)$. Since $y_0, y_1 \in V(T)$, there exists an $i \geq 1$ such that $y_0, \ldots, y_i \in V(T)$ and $y_{i+1} \notin V(T)$. Since $e_i \in E_G(y_j, y_{i+1})$ for some $j \leq i$ and $\phi(e_i)$ is missing at y_ℓ for some $\ell \leq i$, we have that $T'' = (x_0, f_0, x_1, f_1, \ldots, f_{q-1}, x_q, e_i, y_{i+1})$ is a larger Tashkinov tree containing T, contradicting the maximality of T.

For (d), since T' is a Tashkinov tree, there is a maximal Tashkinov tree $\widetilde{T}$ with respect to ϕ for which $\widetilde{T}y_p = T'$. Then (c) implies that $V(\widetilde{T}) \subseteq V(T)$. On the other hand, taking $\widetilde{T}$ to be the maximal Tashkinov tree and T to be the arbitrary Tashkinov tree with respect to ϕ, (c) also implies that $V(T) \subseteq V(\widetilde{T})$. Hence, $V(\widetilde{T}) = V(T)$. □

From the proof of Proposition 14(a), we see that a Tashkinov tree with respect to ϕ is maximal if and only if its vertex set is closed with respect to ϕ. Part (c) implies that the vertex set S of a maximal Tashkinov tree with respect to ϕ is uniquely determined, not dependent on the particular structure of T. In other words, all maximal Tashkinov trees with respect to ϕ have the same vertex set. What makes Tashkinov trees flexible to work with here is that we can so easily append an extra edge to a Tashkinov tree so long as its color is missing at some previous vertex and one of its end-vertices is not in the tree. This sort of ease can be contrasted with Vizing multi-fans and Kierstead paths, where we are required to maintain the structure of a multi-fan or path.

From the proof of Proposition 14(b), we see that for every color α missing at some vertex of a maximal Tashkinov tree T, there are $(v(T)-1)/2$ edges of color α in $G[V(T)]$. This indicates that $G[V(T)]$ is quite dense, as we desire. Similar to our calculations with Vizing multi-fans and Kierstead paths, we can derive the following inequality on the number of edges induced by $S = V(T)$:

$$e(S) \geq 1 + \left(2 + \sum_{y \in S}(k - 1 - d_G(y))\right) \cdot \frac{|S| - 1}{2}$$

From this we can derive the upper bound

$$\chi'(G) = k < \Delta(G) + 1 + \frac{\rho(S) - 2}{|S|}.$$

The problem is that we do not yet know what is $|S| = v(T)$ or how large it can be, only that it is odd and at least 3 by Proposition 14(b). Thus, we now turn our attention to bounding

the number of vertices of maximal Tashkinov trees, or rather of *maximum* Tashkinov trees, where we take into account all possible choices of edges e_0 and proper $(\chi'(G)-1)$-edge-colorings ϕ of $G-e_0$. We will need two lemmas. For terminology, we say that a color α is **used** on a Tashkinov tree T if it is the color of some edge of T, and otherwise we say that α is **unused** on T.

Lemma 15. *Let G be a multigraph with $\chi'(G) = k \geq \Delta(G)+2$, let $e_0 \in E(G)$ be a critical edge, and let ϕ be a proper $(k-1)$-edge-coloring of $G-e_0$. Further, let T be a maximal Tashkinov tree with respect to ϕ, and let $T' = (y_0, e_0, y_1, e_1, \ldots, e_{p-1}, y_p)$ be an arbitrary Tashkinov tree with respect to ϕ. Suppose that the color α is missing at y_i and the color β is missing at y_j, where $1 \leq i < j \leq p$. Then $\alpha \neq \beta$, there is an (α,β)-alternating path P with end-vertices y_i and y_j, and $V(P) \subseteq V(T)$. Moreover, if α is unused on $T'y_j$ and ϕ' is the coloring obtained from ϕ by switching on P, then T' is a Tashkinov tree with respect to ϕ'.*

Proof. By Proposition 14, $V(T)$ is elementary and closed with respect to ϕ, and $V(T') \subseteq V(T)$. Thus, y_i is the only vertex of T' missing α, and y_j is the only vertex of T' missing β. In particular, $\alpha \neq \beta$. Let P be the (α,β)-alternating path starting at y_i. It ends at some other vertex v. Since $V(T)$ is closed, there is no edge in $\partial_G(T)$ colored α or β. Thus, every edge of P lies in $G[V(T)]$, and so $V(P) \subseteq V(T)$. Since either α or β must be missing at the end-vertex v of P, we deduce that $v = y_j$. Now, let ϕ' be the coloring obtained from ϕ by switching on P. Under ϕ', y_i and y_j exchange the missing colors α and β, and all other missing colors in $V(T)$ stay the same. Since β is missing at y_j with respect to ϕ and $V(T')$ is elementary with respect to ϕ, we have that β is unused on $T'y_j$ with respect to ϕ. Since by hypothesis α is also unused on $T'y_j$ with respect to ϕ, we get that both α and β are unused on $T'y_j$ with respect to ϕ'. We deduce that T' is a Tashkinov tree with respect to ϕ'. □

Lemma 16. *Let G be a multigraph with $\chi'(G) = k \geq \Delta(G)+2$, let $e_0 \in E(G)$ be a critical edge, and let ϕ be a proper $(k-1)$-edge-coloring of $G-e_0$. Then there is a maximal Tashkinov tree T with respect to ϕ such that at most $(v(T)-1)/2$ colors are used on T.*

Proof. Let T' be any maximal Tashkinov tree with respect to ϕ. By Proposition 14(b), $v(T') = 2\ell+1$ for some integer $\ell \geq 1$. To obtain T, we inductively construct, for $i = 1, \ldots, \ell$, a Tashkinov tree $T_{2i} = (y_0, e_0, y_1, e_1, \ldots, e_{2i-1}, y_{2i})$ with respect to ϕ, such that at most i colors are used on T. First, since $v(T') \geq 3$, we can let $T_2 = T'_2$, noting that exactly one color is used on T'_2. Assume we have constructed $T_{2i} = (y_0, e_0, y_1, e_1, \ldots, e_{2i-1}, y_{2i})$ for some $1 \leq i \leq \ell-1$. Since $v(T_{2i}) < v(T')$, by Proposition 14(d) there is a maximal Tashkinov tree $\widetilde{T}$ with respect to ϕ such that $V(\widetilde{T}) = V(T')$ and $\widetilde{T}y_{2i} = T_{2i}$. Hence, there is an edge $f \in \partial_G(T_{2i})$ such that $\phi(f)$ is missing at some vertex in $V(T_{2i})$. Let x be the end-vertex of f not in T_{2i}. Then $T'' = (T_{2i}, f, x)$ is a Tashkinov tree with $v(T'') = 2i+2$. Since $i \geq 1$, there is a color α used on T_{2i}. Because $V(T'')$ is elementary with respect to ϕ by Theorem 13, there is a unique vertex $u \in V(T'')$ at which α is missing. Since there are $v(T'')-2 = 2i$ colored edges in T'', which is even, there is an edge $f' \in \partial_G(T'')$ such that $\phi(f') = \alpha$. Hence, letting x' be the end-vertex of f' not in $V(T'')$, we see that $T_{2i+2} = (T_{2i}, f, x, f', x')$ is a Tashkinov

tree with respect to ϕ such that at most $i+1$ colors are used on T_{2i+2}. This completes the inductive construction. □

Now we can prove bounds on the number of vertices of a maximum Tashkinov tree. For a critical multigraph G, define its **Tashkinov order** $t(G)$ to be the maximum size of a Tashkinov tree in G among all possible choices of critical edges $e_0 \in E(G)$ and proper $(\chi'(G)-1)$-edge-colorings ϕ of $G-e_0$. We refer to all Tashkinov trees in G with size $t(G)$ as maximum Tashkinov trees. Also, for a color α and arbitrary Tashkinov tree T with respect to a maximal proper edge-coloring ϕ, let $\partial_{G,\alpha}(T)$ denote the set of edges in $\partial_G(T)$ that are colored α with respect to ϕ.

Theorem 17. *Let G be a critical multigraph with $\chi'(G) = \Delta(G) + k$ for some $k \geq 2$, and with $\chi'(G) > \lceil \rho(G) \rceil$. Then*

$$2k + 1 \leq t(G) \leq \frac{\Delta(G) - 3}{k - 1} + 1.$$

Proof. Let $e_0 \in E(G)$ be any edge, and let $T = (y_0, e_0, y_1, e_1, \ldots, e_{p-1}, y_p)$ be a maximum Tashkinov tree in G, with respect to some $(\chi'(G)-1)$-edge-coloring ϕ of $G - e_0$. By Proposition 14(b), $v(T) \geq 3$ is odd. Assume that $\chi'(G) > \lceil \rho(G) \rceil$.

By Tashkinov's Theorem 13, $V(T)$ is elementary with respect to ϕ. Since $\chi'(G) > \lceil \rho(G) \rceil$, by Proposition 11, $V(T)$ is not strongly closed with respect to ϕ. In other words, there is a color α found on at least two edges in $\partial_G(T)$, a so-called *defective color*. We can write this as $|\partial_{G,\alpha}(T)| \geq 2$. Note that α is present at every vertex in $V(T)$, since otherwise it is missing at some vertex in $V(T)$ and $\partial_{G,\alpha}(T) \neq \emptyset$, which implies that T is not closed and thus not maximal with respect to ϕ, a contradiction. This further implies that α is unused on T. Next, we observe that there are at least $v(T) + 2$ colors missing at some vertex in $V(T)$ since every vertex is missing at least one color and $V(T)$ is elementary. Also observe that there are exactly $v(T) - 2$ colored edges of T. Hence, there are at least four colors that are both missing at some vertex in $V(T)$ and unused on T, so-called *free colors*. Let β be such a color. Since $V(T)$ is elementary, there is exactly one vertex u at which β is missing. Note that $|\partial_{G,\alpha}(T)| \geq 3$ is odd because $v(T)$ is odd and the edges colored α form a matching, and that $\partial_{G,\beta}(T) = \emptyset$ because β is missing at $u \in V(T)$ and T is maximal.

Since β is missing at $u \in V(T)$ and α is present at every vertex in $V(T)$, there is an (α, β)-alternating path P starting at u. We show that $\partial_{G,\alpha}(T) = E(P) \cap \partial_G(T)$. Because $\partial_{G,\beta}(T) = \emptyset$ and the edges of P are colored only α or β, it suffices to show that $\partial_{G,\alpha}(T) \subseteq E(P)$. Suppose this is false. Let $e' \in \partial_{G,\alpha}(T) \setminus E(P)$, and let ϕ' be the coloring obtained from ϕ by switching on P. Since α and β are unused on T with respect to ϕ, we have that T is a Tashkinov tree with respect to ϕ'. But since $e' \notin E(P)$, we have that $e' \in \partial_G(T)$ is still colored α, i.e., lies in $\partial_{G,\alpha}(T)$, and now α is missing at $u \in V(T)$. This implies that T is not closed and thus not maximal with respect to ϕ', contradicting T being a maximum Tashkinov tree in G.

Now, the path P ends at some vertex v in $V(G) \setminus V(T)$ because $|\partial_{G,\alpha}(T)| \geq 3$ is odd and $\partial_{G,\beta}(T) = \emptyset$. Walking along P from u to v, there is a last vertex v_0 of P that lies in $V(T)$. The claim is that all the colors missing at v_0 are used on T. Suppose not. Let γ be missing at v_0 and unused on T. Because $\partial_{G,\alpha}(T) = E(P) \cap \partial_G(T)$ and $|\partial_{G,\alpha}(T)| \geq 3$, we see that P exits $V(T)$ at least twice. This implies that $v_0 \neq u$. Since β is unused on T but is present at v_0, we have $\beta \neq \gamma$. By Lemma 15, there is a (β, γ)-alternating path P_1 whose end-vertices are u and v_0, and we have $V(P_1) \subseteq V(T)$. Let ϕ_1 be the coloring obtained from ϕ be switching on P_1. By Lemma 15, T is a Tashkinov tree with respect to ϕ_1. Note that each of α, β, γ is still unused on T with respect to ϕ_1. Now β is missing at v_0, and so the (α, β)-alternating path P_2 with respect to ϕ_1 that ends at v only starts at v_0. In particular, all of the vertices of P_2 after v_0 lie outside of T. Let ϕ_2 be the coloring obtained from ϕ_1 by switching on P_2. Then T is also a Tashkinov tree with respect to ϕ_2. But now α is missing at v_0 and still $\partial_{G,\alpha}(T) \neq \emptyset$. Thus, T is not maximal with respect to ϕ_2, and this contradicts T being a maximum Tashkinov tree in G.

Using the proven claim, let us show that at least k colors are used on T with respect to ϕ. If we have $v_0 \in \{y_0, y_1\}$, then at least $\chi'(G) - d_G(v_0) \geq \Delta(G) + k - d_G(v_0) \geq k$ colors are missing at v_0, so by the previous claim at least k colors are used on T. Otherwise, if $v_0 = y_j$ for some $j \in \{2, \ldots, p\}$, then at least $\chi'(G) - 1 - d_G(v_0) \geq k - 1$ colors are missing at v_0, so at least $k - 1$ colors are used on T. And since $\phi(e_{j-1})$ is present at v_0 and used on T, again we find there to be at least k colors are used on T.

Now we can prove the lower bound on $t(G)$. By Lemma 16, there exists a maximum Tashkinov tree T' with respect to ϕ on which at most $(v(T') - 1)/2$ colors are used. Since by the third claim at least k colors are used on T', we get that $(v(T') - 1)/2 \geq k$. Thus,

$$t(G) = v(T') \geq 2k + 1.$$

The proof of the upper bound on $t(G)$ is a standard missing colors argument. Since $V(T)$ is elementary with respect to ϕ by Tashkinov's Theorem 13, the number of colors missing at one of the vertices in $V(T)$ is

$$\begin{aligned} 2 + \sum_{y \in V(T)} (\chi'(G) - 1 - d_G(y)) &= 2 + \sum_{y \in V(T)} (\Delta(G) + k - 1 - d_G(y)) \\ &\geq 2 + (k-1)v(T). \end{aligned}$$

Recall that the defective color α is present at every vertex in $V(T)$. Thus, at most $\chi'(G) - 2$ colors are missing at one of the vertices in $V(T)$, and this implies that $2 + (k-1)v(T) \leq \chi'(G) - 2 = \Delta(G) + k - 2$. It follows that

$$t(G) = v(T) \leq \frac{\Delta(G) + k - 4}{k - 1} = \frac{\Delta(G) - 3}{k - 1} + 1,$$

as required. □

Finally we can prove the promised weakening of the Goldberg-Seymour Conjecture 9 due to Scheide [77] and to Chen, Yu, and Zang [24].

Theorem 18. *For every multigraph G with $\Delta(G) \geq 3$, we have*

$$\chi'(G) \leq \max\left\{\Delta(G) + 1 + \sqrt{\frac{\Delta(G)-3}{2}}, \lceil \rho(G) \rceil\right\}.$$

Proof. As usual, G has a critical subgraph H with $\chi'(H) = \chi'(G)$. If either $\chi'(H) \leq \Delta(H)+1$ or $\chi'(H) = \lceil \rho(H) \rceil$, then the above bounds follow. Assume then that $\chi'(H) = \Delta(H) + k$ for some $k \geq 2$, and that $\chi'(H) > \lceil \rho(H) \rceil$. Then necessarily $\Delta(H) \geq 3$, and Theorem 17 applies to give us that

$$2k + 1 \leq t(H) \leq \frac{\Delta(H) - 3}{k-1} + 1.$$

We obtain the quadratic inequality $k^2 - k - (\Delta(H) - 3)/2 \leq 0$ in k, which we can solve as

$$k \leq \frac{1}{2}\left(1 + \sqrt{2\Delta(H) - 5}\right) \leq 1 + \sqrt{\frac{\Delta(H)-3}{2}} \leq 1 + \sqrt{\frac{\Delta(G)-3}{2}}.$$

Therefore, $\chi'(G) = \Delta(G) + k \leq \Delta(G) + 1 + \sqrt{\frac{\Delta(G)-3}{2}}$. □

The bound proven by Scheide and by Chen, Yu, and Zang is actually slightly better than the one just presented, namely

$$\chi'(G) \leq \max\left\{\Delta(G) + \sqrt{\frac{\Delta(G)-1}{2}}, \lceil \rho(G) \rceil\right\}.$$

This can be obtained by reducing the upper bound on $t(G)$ in Theorem 17 by two. Our proof of Theorem 17 simply used the fact that $V(T)$ is elementary by Tashkinov's Theorem 13, together with the helpful observation that the defective color α is present at every vertex in $V(T)$. But in fact we could have shown that a set slightly larger than $V(T)$ is elementary, namely $V(T) \cup \{v_1, v_2\}$, where v_1 and v_2 are the first two vertices of the (α, β)-alternating path P starting from u that lie outside of $V(T)$. Proving that this larger set is elementary takes a couple more path-switching arguments, but the ideas are the same (see [86]).

2.4 Proper list edge-colorings

In this section, we give a brief overview of list edge-colorings of multigraphs. List colorings will be a frequent variation of the ordinary coloring problems that we will discuss later. Let G be a multigraph and let $\mathcal{C}$ be a universe, or **palette**, of colors (often taken to be $\{1, \ldots, n\}$

for some $n \geq 1$). A **list assignment** for $E(G)$ is a function $L : E(G) \to \mathcal{P}(\mathcal{C})$, where $\mathcal{P}(\mathcal{C})$ is the power set of $\mathcal{C}$. That is, L assigns a list of distinct colors from $\mathcal{C}$ to every edge of G. For an integer $k \geq 1$, a **k-list assignment** for $E(G)$ is a list assignment L for $E(G)$ such that $|L(e)| = k$ for each edge $e \in E(G)$, i.e., every list has size k. For a list assignment L for $E(G)$, an **L-coloring** of G is an edge-coloring ϕ of G such that $\phi(e) \in L(e)$ for each $e \in E(G)$. An L-coloring of G is said to be a **proper L-coloring** if it is proper as an ordinary edge-coloring of G, that is, if each of its color classes forms a matching.

The **list chromatic index** $\chi'_\ell(G)$ of G is the minimum integer k such that for every k-list assignment L for $E(G)$, there is a proper L-coloring ϕ of G. Observe that if $L(e) = \{1, \ldots, k\}$ for each edge $e \in E(G)$, then a proper L-coloring corresponds to a proper k-edge-coloring of G. In particular, if $k = \chi'(G)$ then G will have a proper L-coloring, whereas if $k \leq \chi'(G) - 1$ then it will not. This implies that $\chi'_\ell(G) \geq \chi'(G) \geq \Delta(G)$. On the other hand, it is easy to see that the greedy upper bound on $\chi'(G)$ also works for $\chi'_\ell(G)$, which gives us that $\chi'_\ell(G) \leq 2\Delta(G) - 1$.

The notion of a list coloring of a graph was introduced independently by Vizing [90] and by Erdős, Rubin, and Taylor [31], although in the vertex-coloring setting. In the context of vertex-coloring, the vertex-chromatic number and its list coloring analogue can be arbitrarily far apart independently of one another, particularly if the maximum degree is allowed to grow without bound. This is true even in bipartite graphs. In contrast, as seen above, the list chromatic index is always within a factor of two of the chromatic index. But in general, it is thought that one can do much better than the greedy upper bound. A famous conjecture states that the chromatic index and list chromatic index are always equal.

Conjecture 19 (List Coloring Conjecture)**.** *For every multigraph G, we have $\chi'_\ell(G) = \chi'(G)$.*

The List Coloring Conjecture was suggested independently by various authors including Vizing, Albertson, Collins, Tucker, and Gupta, and it first appeared in print in a paper of Bollobás and Harris [18]. In a surprisingly simple proof somewhat later on, Galvin [38] proved the conjecture for all bipartite multigraphs. This also confirmed the special case that $\chi'_\ell(K_{n,n}) = n$, which was conjectured by Dinitz before the notion of a list coloring was defined (see [31]).

Theorem 20 (Galvin)**.** *For every bipartite multigraph G, we have $\chi'_\ell(G) = \chi'(G) = \Delta(G)$.*

Galvin's proof of Theorem 20 is a clever combination of two seemingly unrelated theorems that are independently easy to prove, one about list coloring vertices in directed graphs with kernels (see [13]) and the other being the stable matching theorem for bipartite graphs (see [37]). Unfortunately, Galvin's proof is quite particular to bipartite multigraphs, and it is difficult to extend to more general classes of multigraphs. In addition to bipartite multigraphs, the List Coloring Conjecture 19 has been proven for complete graphs on an odd number of vertices [50], for cubic bridgeless planar simple graphs [13], and for regular class-1 planar multigraphs [29]. Additionally, using probabilistic methods, Kahn [58] proved that the conjecture holds asymptotically, that is, $\chi'_\ell(G) = (1 + o(1))\chi'(G)$ as $\Delta(G) \to \infty$.

The conjecture is still very much open, even for complete graphs on an even number of vertices.

Let us comment that the mentioned proofs of partial results on the List Coloring Conjecture 19 look quite different from proofs about ordinary edge-colorings discussed in previous sections. The list coloring proofs often involve studying various polynomials corresponding to certain edge-colorings or orientations, and then relying on a combinatorial nullstellensatz result of Alon and Tarsi [13]. Otherwise, the proofs involve being lucky that the structure of the multigraph enables a subtle but straightforward list coloring procedure, as is the case with Galvin's Theorem 20. Or the colorings are probabilistic, as is the case with Kahn's theorem. The problem with ordinary edge-coloring proofs from previous sections is that they all, at some step, involve switching on an alternating path. Because the colors in the lists can vary across different edges, analogues of alternating paths (or merely subgraphs that are easy to recolor) in the list coloring setting are difficult to construct. On the other hand, this problem will not arise when we study arboricity in the list coloring setting.

Chapter 3

Arboricity and list arboricity

3.1 Arboricity

Perhaps the most famous edge-coloring parameter determined by maximum density is arboricity. For a multigraph G, its **arboricity** $a(G)$ is the minimum number of colors needed to partition the edges of G into monochromatic forests. Since every edge is itself a forest, we have that $a(G) \leq e(G)$ and so $a(G)$ is finite. Arboricity was introduced by Nash-Williams [70], who proved that it is completely determined by a maximum density parameter, specifically,

$$a(G) = \max_{S \subseteq V(G), |S| \geq 2} \left\lceil \frac{e(S)}{|S| - 1} \right\rceil.$$

This result was also independently discovered by Tutte [89]. It is easy to see that $a(G) \geq \lceil e(S)/(|S| - 1) \rceil$ for all $S \subseteq V(G)$: in an edge-coloring of G into $a(G)$ forests, each color class induces a forest on vertex set S, and each of them has at most $|S| - 1$ edges, so $e(S) \leq a(G)(|S| - 1)$. Noting that $a(G)$ is an integer, we get the stated lower bound. Nash-Williams states that this trivial lower bound is in fact attained for some $S \subseteq V(G)$. That is, the reason that a multigraph G has arboricity $a(G)$ is because of the existence of a "dense spot" that trivially requires $a(G)$ colors to be partitioned into monochromatic forests.

Nash-Williams' theorem is a special case of powerful theorems in the more general context of matroids (see [73]). However, we seek an intuitive proof of the theorem specifically for multigraphs, one that also provides structural information about the asserted dense spot. We give a proof similar to one by Chen, Matsumoto, Wang, Z. Zhang, and J. Zhang [22], and the recoloring procedure we will use was also described by Gabow and Westermann [36] in the more general matroid sum setting. Our proof can in fact be phrased entirely in terms of matroids, but we stick to the multigraph setting (i.e., the special case of graphic matroids). For the purpose of drawing connections, we will write the proof in the same style as many proofs about the chromatic index: we start with a critical multigraph as well as an edge-coloring of it minus one edge, and we find a desired dense spot by studying and

exploiting the distribution of colors around the uncolored edge. Unlike the chromatic index, in this arboricity setting we get an exact result.

In this section, we say that a multigraph G is **critical** if deleting any edge of G decreases its arboricity $a(G)$. We will refer to an edge-coloring into monochromatic forests as a **forest edge-coloring** of G. Now we can define a critical structure that will form the desired dense spot in G. Let G be a multigraph, let e_0 be an edge of G, and let ϕ be an edge-coloring of $G - e_0$ into monochromatic forests. An **arboretum** A with respect to ϕ is a subgraph of G defined recursively as follows:

- The subgraph consisting only of the uncolored edge e_0 is an arboretum.
- If A is an arboretum, e is an edge of A, and P is a monochromatic path with respect to ϕ between the two end-vertices of e, then $A \cup P$ is an arboretum.

The motivation for this definition is that at each step we add a monochromatic path that prevents us from coloring the uncolored edge. We now prove structural results about arboreta in critical multigraphs, which will lead to a proof of Nash-Williams' Theorem.

Proposition 21. *Let G be a critical multigraph with arboricity $a(G) = k$, let e_0 be an edge of G, and let ϕ be a $(k-1)$-coloring of the edges of $G - e_0$ into monochromatic forests. If A is an arboretum with respect to ϕ, then for every edge e of A and color α there is an α-monochromatic path with respect to ϕ in G between the two end-vertices of e.*

Proof. We use induction on the number of edges of A. If $e(A) = 1$, then e_0 is the only edge in A. For any color α there must be a monochromatic path of color α connecting the end-vertices of e_0, for otherwise we can give e_0 the color α and this would contradict the arboricity of G. Now suppose $e(A) > 1$, and assume the result holds for all arboreta with fewer edges than A. Let $e \in E(A)$ be an edge, and let α be any color. If $e = e_0$ then the same argument in the base case shows that there is an α-monochromatic path connecting the end-vertices of e. Assume e has some color β. If $\alpha = \beta$, then trivially there is a monochromatic path of color α connecting the end-vertices of e. Assume $\alpha \neq \beta$. Suppose for contradiction that there is no α-monochromatic path connecting the end-vertices of e. By the recursive construction of A, there is a sub-arboretum A' in A such that $e \notin E(A')$ but also an edge $e' \in E(A')$ and a β-monochromatic path P in G connecting the two end-vertices of e', such that $e \in E(P)$ and $A' \cup P \subseteq A$. Since there is no α-monochromatic path connecting the two end-vertices of e, we may change the color of e to α to obtain a new good edge-coloring ϕ' of $G - e_0$. Then A' is still an arboretum with respect to ϕ' because $\phi' = \phi$ when restricted to A'. Moreover, A' has fewer edges than A, so by the induction hypothesis, e' has a β-monochromatic path P' with respect to ϕ' that connects its two end-vertices. Note that $P' \neq P$ because P' is β-monochromatic with respect to ϕ' while $e \in E(P)$ is colored α with respect to ϕ'. Also note that P' is β-monochromatic with respect to ϕ. But then $P \cup P'$ contains a β-monochromatic cycle with respect to ϕ, contradicting the assumption that ϕ is a coloring into monochromatic forests. This finishes the induction. □

Proposition 22. *Let G be a critical multigraph with arboricity $a(G) = k$, let e_0 be an edge of G, and let ϕ be a $(k-1)$-coloring of the edges of $G - e_0$ into monochromatic forests. Then a maximal arboretum A with respect to ϕ is the union of $k-1$ monochromatic spanning trees on $V(A)$ in each color, plus the uncolored edge e_0. In particular,*

$$e(A) = 1 + (k-1)(v(A) - 1).$$

Proof. By definition every arboretum is connected and all of its color classes induce forests. Let A be a maximal arboretum with respect to ϕ. We show that each color class α restricted to A induces a connected subgraph, so that it is a spanning tree. Let T be a spanning tree of A. By Proposition 21, for any edge $e \in E(T)$ there is a monochromatic path $P(e)$ in G of color α connecting the two end-vertices of e. Since A is maximal, $P(e) \subseteq A$. Thus, the α-monochromatic graph $\cup_{e \in E(T)} P(e)$ is a spanning subgraph of A, and it is easy to see that it is connected. Therefore, the color class α restricted to A is connected.

The formula for the number of edges $e(A)$ comes from noting that each of the $k-1$ monochromatic spanning trees of A has $v(A) - 1$ edges. □

Theorem 23 (Nash-Williams)**.** *For every multigraph G, we have*

$$a(G) = \max_{S \subseteq V(G), |S| \geq 2} \left\lceil \frac{e(S)}{|S| - 1} \right\rceil.$$

Proof. The fact that $a(G) \geq \lceil e(S)/(|S| - 1) \rceil$ for all $S \subseteq V(G)$ was proven at the beginning of this section. We prove that equality is achieved for some subset $S \subseteq V(G)$. Let H be a critical subgraph of G with $a(H) = a(G) = k$. Let e_0 be any edge of H, let ϕ be a $(k-1)$-edge-coloring of $H - e_0$ into monochromatic forests, and let A be a maximal arboretum with respect to ϕ. We show that we can take $S = V(A)$. Clearly $|S| \geq 2$. By Proposition 22, we have $e(A) = 1 + (k-1)(v(A) - 1)$. Note that A is an induced subgraph of H because it is a union of edge-disjoint spanning trees in each color (plus the uncolored edge). Thus,

$$a(G) = k = \frac{e(A)}{v(A) - 1} + \left(1 - \frac{1}{v(A) - 1}\right) = \frac{e(S)}{|S| - 1} + \left(1 - \frac{1}{|S| - 1}\right).$$

Hence,

$$\frac{e(S)}{|S| - 1} \leq a(G) < \frac{e(S)}{|S| - 1} + 1.$$

It follows that $a(G) = \lceil e(S)/(|S| - 1) \rceil$ as required. □

Thus we have shown that a maximal arboretum A with respect to a maximal partial coloring of critical subgraph of G is a desired dense spot that determines the arboricity $a(G)$, in accordance with Nash-Williams' theorem. It is fortunate that a maximal arboretum determines the exact value of the arboricity the way it does. When it comes to the chromatic index, maximal Vizing multi-fans, Kierstead paths, and Tashkinov trees are all similar critical

structures, but the upper bounds on the chromatic index that they provide are generally not tight. The matroid aspects implicit in the arboricity results above makes arboricity quite special. Notice that arboricity places no restrictions on what the color classes should look like locally at a vertex, only on how big the color classes can be, so it makes sense that maximum degree does not play a role in determining arboricity. Soon we will study restricted versions of arboricity whose parameters will have greater dependency on maximum degree.

3.2 List arboricity

Analogous to list edge-colorings into matchings, we may define a list edge-coloring version of arboricity. In this context, if G is a multigraph and L is a list assignment for $E(G)$, then a **forest L-coloring** of G is an edge-coloring ϕ of G such that $\phi(e) \in L(e)$ for all $e \in E(G)$, and such that each color class with respect to ϕ forms a forest. From this, we define the **list arboricity** $a_\ell(G)$ of G to be the minimum integer k such that G has a forest L-coloring for any k-list assignment L for $E(G)$ (where all the lists have size k). We easily see that $a_\ell(G) \leq e(G)$, so that $a_\ell(G)$ is finite. Similar to the list chromatic index, for any multigraph G we have $a_\ell(G) \geq a(G)$. In a short matroid-style proof, Seymour [79] showed that in fact $a_\ell(G) = a(G)$. Thus, the natural arboricity analogue of the List Coloring Conjecture 19 holds. The nice thing is that we can prove this result using the exact same approach employed in the ordinary arboricity case, with only a little more work. Again, everything in this section can be phrased entirely in terms of matroids, but the focus will be on multigraphs.

As before, we say that a multigraph G is **critical** if deleting any edge of G decreases its list arboricity $a_\ell(G)$. By definition, for every critical multigraph G with list arboricity $a_\ell(G) = k$, there is a $(k-1)$-list assignment L such that G has no forest L-coloring but $G-e$ does have a forest L-coloring, for any $e \in E(G)$. We call such an assignment L a **critical list assignment** for $E(G)$. Thus, for any critical list assignment L for G and edge $e \in E(G)$, there is always a forest L-coloring of $G - e$. Note that we may take any list assignment L to have color palette $\{1, \ldots, n\}$ for some $n \geq 1$, without loss of generality. We now define the analogue of an arboretum in the list coloring setting.

Let G be a multigraph, let L be a k-list assignment for $E(G)$ for some $k \geq 1$, let e_0 be an edge of G, and let ϕ be a forest L-coloring of $G - e_0$ if one exists. A **list arboretum** A with respect to ϕ is a subgraph of G defined recursively as follows:

- The subgraph consisting only of the uncolored edge e_0 is a list arboretum.
- If A is a list arboretum, e is an edge of A, α is a color in $L(e)$, and P is an α-monochromatic path with respect to ϕ between the two end-vertices of e, then $A \cup P$ is a list arboretum.

The following structural result on list arboreta is analogous to Proposition 21, with a nearly identical proof.

Proposition 24. *Let G be a critical multigraph with list arboricity $a_\ell(G) = k$, let L be a critical list assignment for $E(G)$, let e_0 be an edge of G, and let ϕ be a forest L-coloring of $G - e_0$. If A is a list arboretum with respect to ϕ, then for every edge $e \in E(A)$ and color $\alpha \in L(e)$ there is an α-monochromatic path with respect to ϕ in G between the two end-vertices of e.*

An analogue of Proposition 22 for maximal list arboreta is the following. It can be viewed as a special case of the matroid union theorem (see [28, 73]).

Proposition 25. *Let G be a critical multigraph with list arboricity $a_\ell(G) = k$, let L be a critical list assignment for $E(G)$ from the color palette $\{1, \ldots, n\}$, let e_0 be an edge of G, and let ϕ be a forest L-coloring of $G - e_0$. Further, let A be a maximal list arboretum with respect to ϕ. For $i \in \{1, \ldots, n\}$, define the subgraphs $C_i = (V(A), \{e \in E(A) : \phi(e) = i\})$, $Q_i = (V(A), \{e \in E(A) : i \in L(e)\})$. Then for each $i \in \{1, \ldots, n\}$, C_i is a maximal spanning forest of Q_i. In particular,*

$$e(A) = 1 + \sum_{i=1}^{n} (v(A) - c_i(A)),$$

where $c_i(A)$ is the number of connected components in Q_i.

Proof. Fix an $i \in \{1, \ldots, n\}$. Note that C_i is a spanning forest of Q_i because ϕ is a forest L-coloring of A. To show that C_i is a maximal forest in Q_i, let F be any maximal spanning forest in Q_i. Then for each edge $e \in E(F)$ we have $i \in L(e)$. By Proposition 24, there is an i-monochromatic path $P(e)$ with respect to ϕ in G (so that $P(e) \subseteq C_i$) that connects the two end-vertices of e. Since A is maximal, $P(e) \subseteq Q_i$. Thus, the i-monochromatic graph $\cup_{e \in E(F)} P(e)$ is a maximal forest in Q_i, and therefore C_i is a maximal spanning forest of Q_i.

The formula for the number of edges $e(A)$ comes from noting that $\{e_0\}, E(C_1), \ldots, E(C_n)$ is a partition of $E(A)$, and that each C_i has $v(A) - c_i(A)$ edges because it is a maximal spanning forest of Q_i. □

We prove the result of Seymour [79] that $a_\ell(G) = a(G)$ using Proposition 25 and an idea of Lasoń [66]. This proof requires the additional fact that the rank function of a graphic matroid is submodular (see [73]), which we state in more graph theory terms as follows. For a multigraph G, we define its rank function r to be a function on its spanning subgraphs H given by $r(H) = v(G) - c(H)$, where $c(H)$ is the number of connected components in H. Equivalently, $r(H)$ is the number of edges in a maximal spanning forest of H.

Lemma 26. *Let r be the rank function of a multigraph G. Then r is a submodular function; that is, for all spanning subgraphs $X, Y \subseteq G$ we have*

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y).$$

Proof. We first establish the following (equivalent) fact: $(*)$ for all spanning subgraphs $X' \subseteq Y' \subseteq G$ and edges $e \in E(G) \setminus E(Y')$ we have $r(X' \cup \{e\}) - r(X') \geq r(Y' \cup \{e\}) - r(Y')$. For our multigraph rank function, this inequality can be rewritten as $c(X') - c(X' \cup \{e\}) \geq c(Y') - c(Y' \cup \{e\})$. Both the left-hand and right-hand side of this rewritten inequality is either 0 or 1. If the right-hand side is 0, the inequality trivially holds. If the right-hand side is 1, then e must connect two separate components in Y'. Since $X' \subseteq Y'$, e must also connect two separate components in X', so the left-hand side is 1. Thus, $(*)$ always holds.

If $X \subseteq Y$, then the asserted submodularity inequality trivially holds as an equality. Suppose that $E(X - Y) = \{e_1, \ldots, e_m\} \neq \emptyset$. Because $X \cap Y \subseteq Y$, we may use $(*)$ to derive the following series of inequalities:

$$\begin{aligned}
r((X \cap Y) \cup \{e_1\}) - r(X \cap Y) &\geq r(Y \cup \{e_1\}) - r(Y), \\
r((X \cap Y) \cup \{e_1, e_2\}) - r((X \cap Y) \cup \{e_1\}) &\geq r(Y \cup \{e_1, e_2\}) - r(Y \cup \{e_1\}), \\
&\vdots \\
r((X \cap Y) \cup \{e_1, \ldots, e_m\}) - r((X \cap Y) \cup \{e_1, \ldots, e_{m-1}\}) & \\
&\geq r(Y \cup \{e_1, \ldots e_m\}) - r(Y \cup \{e_1, \ldots, e_{m-1}\}).
\end{aligned}$$

Adding the inequalities together, we derive that

$$\begin{aligned}
r((X \cap Y) \cup \{e_1, \ldots, e_m\}) - r(X \cap Y) &\geq r(Y \cup \{e_1, \ldots e_m\}) - r(Y), \\
r((X \cap Y) \cup (X \setminus Y)) - r(X \cap Y) &\geq r(Y \cup (X \setminus Y)) - r(Y), \\
r(X) - r(X \cap Y) &\geq r(X \cup Y) - r(Y),
\end{aligned}$$

and therefore $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$. □

Theorem 27 (Seymour)**.** *For every multigraph G, we have*

$$a_\ell(G) = a(G) = \max_{S \subseteq V(G), |S| \geq 2} \left\lceil \frac{e(S)}{|S| - 1} \right\rceil.$$

Proof (Lasoń). Note that the multigraph G has a subgraph H that is critical for list arboricity with $a_\ell(H) = a_\ell(G) = k$. Let L be a critical list assignment for $E(H)$ from the color palette $\{1, \ldots, n\}$ (without loss of generality), let e_0 be an edge of H, and let ϕ be a forest L-coloring of $H - e_0$. By Proposition 25, there is a maximal list arboretum A in H with

$$e(A) = 1 + \sum_{i=1}^{n} (v(A) - c_i(A)),$$

edges, where $c_i(A)$ is the number of components in the graph $Q_i = \{e \in E(A) : i \in L(e)\}$. Note that $v(A) - c_i(A) = r(Q_i)$ is the rank of the spanning subgraph Q_i with respect to the rank function r of the multigraph A. Using the fact that r is submodular by Lemma 26, we prove that among all list assignments L for $E(H)$ satisfying $L(e) \subseteq \{1, \ldots, n\}$ and

$|L(e)| = k-1$ for all $e \in E(H)$ (not just critical list assignments), the right-hand side of the above formula for $e(A)$ is minimized by the uniform list assignment $L_0(e) = \{1, \ldots, k-1\}$ for all $e \in E(G)$. That is, we show that for any list assignment L,

$$1 + (k-1)(v(A)-1) \leq 1 + \sum_{i=1}^{n} r(Q_i),$$

so that in particular this holds for the critical list assignment L we started with.

The proof is by induction on $e(Q_k)+\ldots+e(Q_n)$. If $e(Q_k)+\ldots+e(Q_n) = 0$, then $L = L_0$ and the result clearly holds. Suppose that $e(Q_k)+\ldots+e(Q_n) \geq 1$. Then there exists an edge $e_1 \in E(A)$ and colors $i \in \{1, \ldots, k-1\}$, $j \in \{k, \ldots, n\}$ such that $i \notin L(e_1)$ and $j \in L(e_1)$. Define a new list assignment L' for $E(G)$ by setting the list color classes to be $Q'_i = Q_i \cup Q_j$, $Q'_j = Q_i \cap Q_j$, and $Q'_m = Q_m$ for all $m \in \{1, \ldots, n\} \setminus \{i, j\}$. In other words, to get L' from L, we replace the color j by the color i in every list that contains j but not i. By the submodularity of the rank function r, we have $r(Q'_i) + r(Q'_j) \leq r(Q_i) + r(Q_j)$. Moreover, $r(Q'_m) = r(Q_m)$ for all $m \in \{1, \ldots, n\} \setminus \{i, j\}$. Since $e(Q'_k)+\ldots+e(Q'_n) < e(Q_k)+\ldots+e(Q_n)$, by the induction hypothesis and submodularity we have

$$1 + (k-1)(v(A)-1) \leq 1 + \sum_{i=1}^{n} r(Q'_i) \leq 1 + \sum_{i=1}^{n} r(Q_i).$$

This finishes the induction.

Returning to the critical list assignment L, we have demonstrated that

$$e(A) = 1 + \sum_{i=1}^{n} (v(A) - c_i(A)) \geq 1 + (k-1)(v(A)-1).$$

Similar to the calculations in the proof of Theorem 23, we may rearrange to find that

$$a_\ell(G) = k \leq \left\lceil \frac{e(A)}{v(A)-1} \right\rceil \leq a(G).$$

Since $a_\ell(G) \geq a(G)$, it follows that $a_\ell(G) = a(G) = \lceil e(A)/(v(A)-1) \rceil$, and that moreover the maximum of $\lceil e(S)/(|S|-1) \rceil$ is attained at $S = V(A)$. □

One reason that the proof of Proposition 21 transfers so easily to the list setting as Proposition 24 is that the proof involves recoloring only a single edge. Compare this with all the proofs presented regarding the chromatic index, which often involve switching along an alternating path in addition to recoloring an edge. This is despite the fact that the equations in Theorem 3, Theorem 6, Proposition 22, and Proposition 29 look so similar. Changing the color of one edge works nicely in the list setting, but switching on alternating paths does not. It would be interesting to explore what makes certain "exchange properties", such as switching on alternating paths, translate to the list color setting better or worse than other exchange properties.

Now, Lasoń in [66] actually proved a slightly stronger result than Theorem 27 that $a_\ell(G) = a(G)$. Instead of every list having the same size k, he allows the list size to vary across the edges. For a multigraph G, let $\ell : E(G) \to \mathbb{N}$ be a **lists size function**, a function that determines the size of a list at a particular edge of G. We call a list assignment L for $E(G)$ an **ℓ-list assignment** if $|L(e)| = \ell(e)$ for all $e \in E(G)$. We say that G is **critical** for lists size function ℓ if there is some ℓ-list assignment L for which G has no forest L-coloring, but that for every such assignment L and edge $e \in E(G)$, $G-e$ does have a forest L-coloring. In this case, we call L a **critical ℓ-list assignment**.

Let G be a multigraph, let ℓ be a lists size function for $E(G)$, let L be an ℓ-list assignment for $E(G)$, let e_0 be an edge of G, and let ϕ be a forest L-coloring of $G - e_0$ if one exists. We define a **list arboretum** A with respect to ϕ in exactly the same way we did when ℓ was a constant function.

The following propositions hold for the same reasons that Proposition 24 and Proposition 25 hold.

Proposition 28. *Let G be a multigraph that is critical for the lists size function ℓ, let L be a critical ℓ-list assignment for $E(G)$, let e_0 be an edge of G, and let ϕ be a proper L-coloring of $G - e_0$. If A is a list arboretum with respect to ϕ, then for every edge $e \in E(A)$ and color $\alpha \in L(e)$ there is an α-monochromatic path with respect to ϕ in G between the two end-vertices of e.*

Proposition 29. *Let G be a multigraph that is critical for the lists size function ℓ, let L be a critical ℓ-list assignment for $E(G)$ from the color palette $\{1, \ldots, n\}$, let e_0 be an edge of G, and let ϕ be a forest L-coloring of $G - e_0$. Further, let A be a maximal list arboretum with respect to ϕ. For $i \in \{1, \ldots, n\}$, define the subgraphs $C_i = (V(A), \{e \in E(A) : \phi(e) = i\})$, $Q_i = (V(A), \{e \in E(A) : i \in L(e)\})$. Then for each $i \in \{1, \ldots, n\}$, C_i is a maximal spanning forest of Q_i. In particular,*

$$e(A) = 1 + \sum_{i=1}^{n} (v(A) - c_i(A)),$$

where $c_i(A)$ is the number of connected components in Q_i.

With minor modifications, Lasoń's proof of Theorem 27 can be adapted to prove the following theorem [66].

Theorem 30 (Lasoń)**.** *For every multigraph G and lists size function $\ell : E(G) \to \mathbb{N}$, the following statements are equivalent:*

(1) G has a forest L-coloring for the ℓ-list assignment $L(e) = \{1, \ldots, \ell(e)\}$, $e \in E(G)$;

(2) G has a forest L-coloring for any ℓ-list assignment L.

Proof. Clearly (2) implies (1). To show that (1) implies (2), suppose that there is an ℓ-list assignment L such that G does not have a forest L-coloring. Then G has a subgraph H that is critical for the lists size function ℓ. Let L be a critical ℓ-list assignment for $E(H)$ from the color palette $\{1, \ldots, n\}$ (without loss of generality), let e_0 be an edge of G', and let ϕ be a forest L-coloring of $H - e_0$. By Proposition 29, H has a maximal list arboretum A with respect to ϕ with

$$e(A) = 1 + \sum_{i=1}^{n} (v(A) - c_i(A)),$$

edges, where $c_i(A)$ is the number of components in $Q_i = (V(A), \{e \in E(A) : i \in L(e)\})$.

The claim again is that among all list assignments L for $E(H)$ that satisfy $L(e) \subseteq \{1, \ldots, n\}$ and $|L(e)| = \ell(e)$ for all $e \in E(H)$ (not just critical list assignments), the right-hand side of the above equality is minimized by the assignment $L_0(e) = \{1, \ldots, \ell(e)\}$ for all $e \in E(H)$. The proof of this claim is exactly the same color-switching/submodularity argument in the proof of Theorem 27, which still works because after each step of the induction, the size of the list at an edge e is still $\ell(e)$. This shows that

$$e(A) \geq 1 + \sum_{i=1}^{n} (v(A) - c'_i(A)),$$

where $c'_i(A)$ is the number of components in $Q'_i = (V(A), \{e \in E(A) : i \in L_0(e)\})$. Based on this inequality, we see that A cannot have a forest L_0-coloring: If ϕ' were an L_0-coloring of A, then each color class i with respect to ϕ' must be a forest with at most $v(A) - c'_i(A)$ edges, and adding the edges together we find that $e(A) \leq \sum_{i=1}^{n} (v(A) - c'_i(A))$, which is a contradiction. Therefore, since A has no forest L_0-coloring, the same is true of G. This completes the proof. □

Chapter 4

Pseudoarboricity and list pseudoarboricity

4.1 Pseudoarboricity

A **pseudoforest** is a multigraph in which every component has at most one cycle (possibly a 2-cycle). It is simply a forest together with at most one additional edge in each component. Analogous to arboricity, the **pseudoarboricity** $pa(G)$ of a multigraph G is the minimum number of colors needed to edge-color G into monochromatic pseudoforests. Similar to how every forest F has at most $v(F) - 1$ edges, every pseudoforest F has at most $v(F)$ edges. Conversely, if a connected multigraph F has at most $v(F)$ edges, then F is a pseudoforest. Because a pseudoforest F can have at most $v(F)$ edges, an easy lower bound for the pseudoarboricity of a multigraph G is

$$pa(G) \geq \max_{S \subseteq V(G), |S| \geq 2} \left\lceil \frac{e(S)}{|S|} \right\rceil.$$

Analogous to Nash-Williams' Theorem 23, a theorem of Hakimi [51] states that this maximum density lower bound is exact:

$$pa(G) = \max_{S \subseteq V(G), |S| \geq 2} \left\lceil \frac{e(S)}{|S|} \right\rceil.$$

In other words, the trivial lower bound for $pa(G)$ is attained at some subset $S \subseteq V(G)$, a "dense spot". This theorem is sometimes formulated in terms of maximum average degree. For a multigraph G, its **maximum average degree** $mad(G)$ is

$$mad(G) = \max_{S \subseteq V(G), |S| \geq 2} d(G[S]) = \max_{S \subseteq V(G), |S| \geq 2} \frac{2e(S)}{|S|}$$

Thus, Hakimi's Theorem states that $pa(G) = \lceil mad(G)/2 \rceil$. Let us prove this theorem.

Similar to forests, the pseudoforests of a multigraph form a matroid, known as the bicircular matroid, and an arboretum-style proof similar to the one we used to prove Nash-Williams' Theorem 23 also works for Hakimi's Theorem. However, we give a slightly different proof of Hakimi's Theorem whose perspective will be useful in later edge-coloring problems. Instead of directly using pseudoforest edge-colorings, we will use the tool of multigraph orientations. The relationship between pseudoarboricity and orientations is encapsulated in the following proposition (see [35]).

Proposition 31. *A multigraph G has pseudoarboricity at most k if and only if G has an orientation such that every vertex has indegree at most k.*

Proof. First we prove the case $k = 1$. Suppose that G has pseudoarboricity one, that is, G is itself a pseudoforest. It is easy to find an orientation of G such that every vertex has indegree at most one (see Figure 4.1). First we orient the cycles of G to form directed cycles. The remaining undirected edges of G form a forest F. We root each component of F either at a vertex of a cycle of G, which is unique if it exists, or at an arbitrary vertex if the latter does not apply. Then we orient each edge of a component of F away from the chosen root vertex of that component. Adding back the directed cycles, we get an orientation of G such that every vertex has indegree at most one. Conversely, suppose that G has an orientation D such that every vertex has indegree at most one. Let X be a connected component of D. Adding up the indegrees of all vertices of X, we find that $e(X) \leq v(X)$, implying that X is a pseudoforest. Therefore, G is a pseudoforest.

Now we prove the result for all $k \geq 1$. Suppose that G has pseudoarboricity at most k. Let $F_1, \ldots, F_k$ be a decomposition of G into k spanning pseudoforests. By the case $k = 1$, each F_i has an orientation D_i such that every vertex of D_i has indegree at most one. By combining the orientations $D_1, \ldots, D_k$, we obtain an orientation D of G such that every vertex has indegree at most k. Conversely, suppose that G has an orientation D such that every vertex has indegree at most k. For each vertex of D, color all of its incoming arcs a different color from the palette $\{1, \ldots, k\}$ to get an edge-coloring of D. Then each color class forms a directed graph such every vertex has indegree at most one. By the case $k = 1$, each color class is a pseudoforest as an undirected graph. Hence, we found an edge-coloring of G into k monochromatic pseudoforests, and therefore G has pseudoarboricity at most k. □

By Proposition 31, we can reduce the problem of pseudoarboricity to the problem of orientations with restricted indegree. From this we can construct a critical structure and use it to prove Hakimi's Theorem similar to how we proved Nash-Williams' Theorem. As usual, a multigraph G is **critical** if deleting any edge from G decreases its pseudoarboricity $pa(G)$. Let G be a multigraph, let e_0 be an edge of G, and let D be an orientation of $G - e_0$. A **pseudoarboretum** A with respect to D is a subgraph of G defined recursively as follows:

- The subgraph consisting only of the undirected edge e_0 is a pseudoarboretum.
- If A is a pseudoarboretum, v is a vertex of A, and e is an incoming arc to v in D, then $A \cup \{e'\}$ is a pseudoarboretum, where e' is the undirected version of e.

The motivation for this definition is similar to that of the arboretum; at each step we simply add all incoming edges that prevent us from orienting the unoriented edge. In a maximal pseudoarboretum A then, every incoming arc to a vertex of A will form an edge in A. Thus, we see that A is the subgraph of G induced by all the vertices v for which there is a directed path in D from v to one of the two end-vertices of e_0. This observation is similar to how a maximal arboretum is a union of edge-disjoint spanning trees in each color (Proposition 22) and thus also an induced subgraph of the multigraph, except it is simpler in this setting.

The following proposition on maximal pseudoarboreta is analogous to Proposition 22 on maximal arboreta. The proof can be viewed simply as an oriented version of the proof of König's Theorem 1.

Proposition 32. *Let G be a critical multigraph with pseudoarboricity $pa(G) = k$, let e_0 be an edge of G, and let D be an orientation of $G - e_0$ such that every vertex has indegree at most $k - 1$. If A is a maximal pseudoarboretum with respect to D, then every vertex of A has indegree exactly $k - 1$ in $D[V(A)]$. In particular,*

$$e(A) = 1 + (k - 1)v(A).$$

Proof. Let A be a maximal pseudoarboretum with respect to D. Suppose for contradiction that a vertex $u \in V(A)$ has indegree less than $k - 1$ in $D[V(A)]$. Since A is maximal, u has indegree less than $k - 1$ in D. By the recursive construction of A, there is a directed path P from u to an end-vertex v of e_0. Flip the direction of all arcs in P to get a new orientation D' of $G - e_0$. Note that every vertex still has indegree at most $k - 1$. But now the end-vertex v of e_0 has indegree less than $k - 1$. Thus, we can orient e_0 toward v and get an orientation of G such that every vertex has indegree at most $k - 1$. This contradicts G having pseudoarboricity k.

The formula for $e(A)$ comes from adding up the indegrees of the vertices of A, and also including the undirected edge e_0. □

Theorem 33 (Hakimi)**.** *For every multigraph G,*

$$pa(G) = \max_{S \subseteq V(G), |S| \geq 2} \left\lceil \frac{e(S)}{|S|} \right\rceil.$$

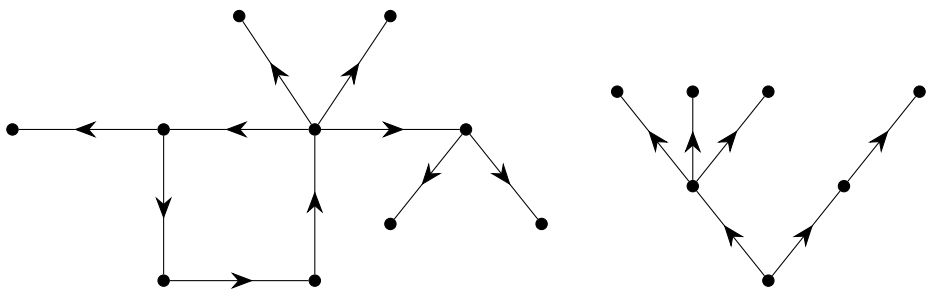

Figure 4.1: An orientation of a pseudoforest where every vertex has indegree at most one.

Proof. The fact that $pa(G) \geq \lceil e(S)/|S| \rceil$ for all $S \subseteq V(G)$ was proven at the beginning of the section. We prove that equality is achieved for some subset $S \subseteq V(G)$. Let H be a critical subgraph of G with the same pseudoarboricity k. Let e_0 be any edge of H, let D be an orientation of $H - e_0$ such that every vertex has indegree at most $k-1$, and let A be a maximal pseudoarboretum with respect to D. We show that we can take $S = V(A)$. By Proposition 32, we have $e(A) = 1 + (k-1)v(A)$. Since A is an induced subgraph of G, we derive that

$$pa(G) = k = \frac{e(A)}{v(A)} + \left(1 - \frac{1}{v(A)}\right) = \frac{e(S)}{|S|} + \left(1 - \frac{1}{|S|}\right).$$

Hence,

$$\frac{e(S)}{|S|} \leq pa(G) < \frac{e(S)}{|S|} + 1.$$

It follows that $pa(G) = \lceil e(S)/|S| \rceil$, as required. □

The density formulas for arboricity and pseudoarboricity given by Nash-Williams' Theorem 23 and Hakimi's Theorem 33 are quite similar:

$$a(G) = \max_{S \subseteq V(G), |S| \geq 2} \left\lceil \frac{e(S)}{|S| - 1} \right\rceil, \quad pa(G) = \max_{S \subseteq V(G), |S| \geq 2} \left\lceil \frac{e(S)}{|S|} \right\rceil.$$

To see how close they are, let $S \subseteq V(G)$ be a maximizer for $a(G)$. We have

$$\frac{e(S)}{|S| - 1} = \frac{e(S)}{|S|} + \frac{e(S)}{|S|(|S| - 1)} \leq \frac{e(S)}{|S|} + \frac{\mu(G)}{|S|(|S| - 1)} \binom{|S|}{2} = \frac{e(S)}{|S|} + \frac{\mu(G)}{2}.$$

Hence, $a(G) \leq pa(G) + \lceil \mu(G)/2 \rceil$. In other words, if we have an edge-coloring of G into pseudoforests, then we only need at most $\lceil \mu(G)/2 \rceil$ extra colors in order for G to have an edge-coloring into forests.

4.2 List pseudoarboricity

As we did for the chromatic index and arboricity, we may define a list coloring version of pseudoarboricity. If G is a multigraph and L is a list assignment for $E(G)$, then a **pseudoforest L-coloring** of G is an edge-coloring ϕ of G such that $\phi(e) \in L(e)$ for all $e \in E(G)$, and such that each color class with respect to ϕ forms a pseudoforest. We define the **list pseudoarboricity** $pa_\ell(G)$ to be the minimum integer k such that G has a pseudoforest L-coloring for any k-list assignment L for $E(G)$ (where all the lists have size k). As usual, it is immediate that $pa_\ell(G) \geq pa(G)$. Similar to Seymour's Theorem 27 that $a_\ell(G) = a(G)$, we can prove that $pa_\ell(G) = pa(G)$. We can give a similar list arboretum-style proof of this result because of the matroid structure of pseudoforests, but actually we can prove this result much more easily simply using Proposition 31.

Theorem 34. *For every multigraph G, we have*

$$pa_\ell(G) = pa(G) = \max_{S \subseteq V(G), |S| \geq 2} \left\lceil \frac{e(S)}{|S|} \right\rceil .$$

Proof. The second equality is Hakimi's Theorem 33. For the first equality, let G be a multigraph with pseudoarboricity k, and let L be any k-list assignment for $E(G)$. By Proposition 31, G has an orientation D such that every vertex has indegree at most k. Give every arc of D the list of the corresponding edge of G. Because each list has size k, for every vertex in D we can give all of its incoming arcs a different color from the respective lists. Doing this at all vertices, we obtain an L-coloring of G such that for every color class, every vertex has indegree at most one with respect to D. By Proposition 31, each color class is a pseudoforest. Therefore, G has a pseudoforest L-coloring. □

We can see from the proof that we only needed the list at an arc to be large enough to guarantee that the arc can get a different color from the other arcs with the same head. Thus, the worst possible scenario is that the lists at all of these arcs are as overlapping as possible. This intuition can be made rigorous to prove a local lists size version of Theorem 34 completely analogous to Lasoń's Theorem 30.

Theorem 35. *For every multigraph G and lists size function $\ell : E(G) \to \mathbb{N}$, the following statements are equivalent:*

(1) G has a pseudoforest L-coloring for the ℓ-list assignment $L(e) = \{1, \ldots, \ell(e)\}$, $e \in E(G)$;

(2) G has a pseudoforest L-coloring for any ℓ-list assignment L.

Recall that we used the submodularity of the rank function of a multigraph to prove list arboricity results such as Seymour's Theorem 27. Although we did not use submodularity in the proof of Theorem 34 on list pseudoarboricity, the indegree function $d^-(S)$ of a directed multigraph D is a submodular function. (For a vertex subset $S \subseteq V(D)$, the value $d^-(S)$ is the number of arcs of D pointing from a vertex in $V(D) \setminus S$ to a vertex in S.)

Chapter 5

Bounded degree edge-colorings

We have explained three classical edge-coloring problems: proper edge-colorings, forest edge-colorings, and pseudoforest edge-colorings. One curious aspect of the Goldberg-Seymour Conjecture 9 on the chromatic index,

$$\chi'(G) \leq \max\left\{\Delta(G)+1, \max_{S\subseteq V(G),|S|\geq 2}\left\lceil \frac{e(S)}{\lfloor |S|/2\rfloor}\right\rceil\right\},$$

is that the density formulas for arboricity and pseudoarboricity given by Nash-Williams' Theorem 23 and Hakimi's Theorem 33,

$$a(G) = \max_{S\subseteq V(G),|S|\geq 2}\left\lceil \frac{e(S)}{|S|-1}\right\rceil, \qquad pa(G) = \max_{S\subseteq V(G),|S|\geq 2}\left\lceil \frac{e(S)}{|S|}\right\rceil,$$

are almost present within its statement. In fact, the only difference appears to be a factor of 2 buried within floors and ceilings, as well as a maximum degree consideration. But it is not obvious how these edge-coloring problems translate to one another, from matchings to forests or pseudoforests and vice versa. The only obvious relations are that $pa(G) \leq a(G) \leq \chi'(G)$ and $pa(G) \leq \lceil \chi'(G)/2\rceil$, where the latter holds because we can pair up color classes in a proper edge-coloring to obtain a pseudoforest edge-coloring. But these do not explain why these edge-coloring parameters seem so similar, both in the formulas and proof techniques. This motivates us to look for a sort of "interpolation" between the chromatic index and each of arboricity and pseudoarboricity, in the process trying to understand why maximum degree $\Delta(G)$ suddenly shows up in the chromatic index.

The approach we take is to study these edge-coloring problems with the added restriction that the maximum degree of the vertices in each of color classes be at most some specified integer t. This is quite natural because a matching can be viewed as a subgraph with maximum degree $t = 1$, as a forest with maximum degree 1, or as a pseudoforest with maximum degree 1. Taking $t \to \infty$ then lets us recover ordinary arboricity and pseudoarboricity. Such an approach has been described by other authors, but in this writing we will go into greater detail on the various structural results we could obtain, as well as the fascinating connections

to be found when comparing the results side-by-side. The discussed interpolated parameters turn out to be quite interesting in their own right.

Let G be a multigraph and let $t \geq 1$ be a fixed integer. A **degree t coloring** of G is an edge-coloring of G such that every vertex in a given monochromatic subgraph has degree at most t. If every vertex has degree exactly t in each color, then each of the color classes is a spanning t-regular subgraph, which is better known as a **t-factor**, and the coloring is a so-called **t-factorization**. We define the **degree t chromatic index** $\chi'_t(G)$ of G to be the minimum number of colors needed in a degree t coloring of G. Next, we define the **degree t arboricity** $a_t(G)$ to be the minimum number of colors needed in a degree t coloring of G such that every monochromatic subgraph is a forest. We define the **degree t pseudoarboricity** $pa_t(G)$ in the analogous way for pseudoforests. Ordinary arboricity $a(G)$ and pseudoarboricity $pa(G)$ can be thought of as $a_t(G)$ and $pa_t(G)$ with $t = \infty$. Finally, let $\chi'_{t,\ell}(G)$, $a_{t,\ell}(G)$, and $pa_{t,\ell}(G)$ be the list coloring analogues of the defined parameters. The following facts are immediate from the definitions:

- $\chi'_1(G) = a_1(G) = pa_1(G) = \chi'(G)$,
- $\chi'_t(G) \leq pa_t(G) \leq a_t(G)$ for all $t \geq 1$,
- $\chi'_{t+1}(G) \leq \chi'_t(G)$, $a_{t+1}(G) \leq a_t(G)$, $pa_{t+1}(G) \leq pa_t(G)$ for all $t \geq 1$,
- $\chi'_t(G) = 1$, $a_t(G) = a(G)$, $pa_t(G) = pa(G)$ for all $t \geq \Delta(G)$.

The above facts similarly hold for the list analogues. Also, notice that $\chi'_2(G) = pa_2(G)$ because a subgraph of maximum degree at most 2 is necessarily a pseudoforest of maximum degree at most 2.

5.1 Bounded degree subgraphs and list analogues

Bounded degree edge-colorings were first studied explicitly in general by Hakimi and Kariv [52], who proved analogues of Vizing's Theorem 4 and Shannon's Theorem 8 in this setting. Actually, they worked on the more general edge-coloring problem where the number of edges of the same color incident to a given vertex v is at most some specified function $f(v)$ which could vary across the vertices (we could call this a degree f coloring), but here we will assume f is a constant function. Our proofs will be shorter based on this assumption. We start our study of bounded degree edge-colorings by proving the following straightforward bounds on the degree t chromatic index $\chi'_t(G)$ and its list analogue $\chi'_{t,\ell}(G)$ (see [61]).

Proposition 36. *For every multigraph G and integer $t \geq 1$,*

$$\left\lceil \frac{\Delta(G)}{t} \right\rceil \leq \chi'_t(G) \leq \left\lceil \frac{\chi'(G)}{t} \right\rceil \quad \text{and} \quad \left\lceil \frac{\Delta(G)}{t} \right\rceil \leq \chi'_{t,\ell}(G) \leq \left\lceil \frac{\chi'_\ell(G)}{t} \right\rceil.$$

Proof. For the lower bounds, first observe that in an edge-coloring of G into monochromatic subgraphs of maximum degree t, every vertex has at most t incident edges of a given color, which implies that $\Delta(G) \leq \chi'_t(G) \cdot t$ and thus $\chi'_t(G) \geq \Delta(G)/t$. Since $\chi'_{t,\ell}(G) \geq \chi'_t(G)$, we also get that $\chi'_{t,\ell}(G) \geq \Delta(G)/t$. These imply the stated lower bounds because $\chi'_t(G)$ and $\chi'_{t,\ell}(G)$ are integers.

For the upper bound on $\chi'_t(G)$, take an edge-coloring of G into $k = \chi'(G)$ matchings $M_1, \ldots, M_k$, partition the set $\{M_1, \ldots, M_k\}$ into $\lceil k/t \rceil$ sets of size at most t, and make all edges within a partition class the same color. The resulting edge-coloring of G with $\lceil k/t \rceil$ colors will have each color class with maximum degree t.

Finally, for the upper bound on $\chi'_{t,\ell}(G)$, let L be any list assignment for $E(G)$ with $\lceil \chi'_\ell(G)/t \rceil$ colors in each list. Define the list assignment L' for $E(G)$ by $L'(e) = L(e) \times [t]$ for all $e \in E(G)$. Since $\lceil \chi'_\ell(G)/t \rceil \cdot t \geq \chi'_\ell(G)$, there exists an L'-coloring ϕ' of G into matchings. Define the map ϕ of $E(G)$ by setting $\phi(e) = c$ if $\phi'(e) = (c, i)$. It is evidently an L-coloring, and every color class will have maximum degree t. Hence, G has a degree t L-coloring. □

By applying Vizing's Theorem 4, we obtain the following analogue of Vizing's Theorem for degree t colorings that was noted by Hakimi and Kariv [52].

Corollary 37. *For every multigraph G and integer $t \geq 1$, we have*

$$\left\lceil \frac{\Delta(G)}{t} \right\rceil \leq \chi'_t(G) \leq \left\lceil \frac{\Delta(G) + \mu(G)}{t} \right\rceil.$$

This corollary implies that for every simple graph G and integer $t \geq 1$, we have

$$\left\lceil \frac{\Delta(G)}{t} \right\rceil \leq \chi'_t(G) \leq \left\lceil \frac{\Delta(G) + 1}{t} \right\rceil.$$

In particular, if G is simple and $\Delta(G)$ is not an integer multiple of t, then $\chi'_t(G) = \lceil \Delta(G)/t \rceil$. Also, by Shannon's Theorem 8 we get that $\chi'_t(G) \leq \lceil \frac{3}{2}\Delta(G)/t \rceil$.

The above bounds on $\chi'_t(G)$ are inexact because Vizing's Theorem 4 cannot exactly determine $\chi'(G) = \chi'_1(G)$, that is, the case $t = 1$. However, the story is different with the case $t = 2$, as Petersen's famous 2-factor theorem suggests (see [67]).

Theorem 38 (Petersen)**.** *Every $2k$-regular multigraph can be decomposed into k 2-factors.*

Proof. It suffices to prove this result for connected multigraphs. Let G be a connected $2k$-regular multigraph. Because every degree of G is even and G is connected, G has an Euler tour. This tour gives an orientation D of G in which every vertex of D has both indegree and outdegree k. Construct the bipartite graph H whose vertex set consists of two copies v_1, v_2 of every vertex v of D, and for every arc from a vertex u to a vertex v of D we put an edge between u_1 and v_2 in H. Since every vertex of D has both indegree and outdegree k, H is a k-regular bipartite multigraph. By König's Theorem 1, H can be edge-colored into k

perfect matchings (matchings that touch every vertex of H). Merging the two copies v_1, v_2 of the vertex v back to a single vertex for every $v \in V(G)$, we obtain an edge-coloring of G into k 2-factors. $\square$

Using Petersen's theorem, we may perform a similar argument to Proposition 36 and determine the exact value of $\chi'_t(G)$ when t is even, a result that was also noted by Hakimi and Kariv [52].

Theorem 39. *For every multigraph G and even integer $t \geq 2$, we have $\chi'_t(G) = \lceil \Delta(G)/t \rceil$.*

Proof. We have shown that $\chi'_t(G) \geq \lceil \Delta(G)/t \rceil$ in Proposition 36. To prove the upper bound, note that G is a subgraph of some $\Delta(G)$-regular multigraph H. Suppose first that $\Delta(G) = 2k$ is even. Then by Petersen's Theorem 38, H can be decomposed into k 2-factors $H_1, \ldots, H_k$. We may partition the set $\{H_1, \ldots, H_k\}$ into $\lceil 2k/t \rceil$ sets of size at most $t/2$, and make all edges within a partition class the same color. The resulting edge-coloring of H with $\lceil 2k/t \rceil = \lceil \Delta(G)/t \rceil$ colors will have each color class with maximum degree t. Thus, $\chi'_t(G) \leq \chi'_t(H) \leq \lceil \Delta(G)/t \rceil$.

Now suppose that $\Delta(G) = 2k + 1$ is odd. Then H has an even number of vertices, and so we may add a perfect matching to H to form a $(2k + 2)$-regular multigraph H'. Following the same argument in the previous paragraph, we obtain that $\chi'_t(G) \leq \chi'_t(H') \leq \lceil (\Delta(G) + 1)/t \rceil = \lceil \Delta(G)/t \rceil$. The last equality holds because $\Delta(G)$ is odd and t is even. $\square$

On the other hand, when t is odd it is conjectured that a bound similar to the Goldberg-Seymour Conjecture 9 holds. Such a conjecture was originally stated by Nakano, Nishizeki, and Saito [69] in the more general degree f coloring setting mentioned above, but in our constant function setting we believe a slightly stronger conjecture holds.

Conjecture 40. *For every multigraph G and odd integer $t \geq 1$, we have*

$$\chi'_t(G) \leq \max\left\{ \left\lceil \frac{\Delta(G) + 1}{t} \right\rceil, \max_{S \subseteq V(G), |S| \geq 2} \left\lceil \frac{e(S)}{\lfloor t|S|/2 \rfloor} \right\rceil \right\},$$

The only difference with the conjecture of Nakano, Nishizeki, and Saito is that the "+1" in our maximum degree upper bound lies outside the fraction in their conjecture. The stated maximum density parameter comes from noting that every color class in a degree t coloring has at most $\lfloor t|S|/2 \rfloor$ edges. Using the same argument shown in the ordinary Goldberg-Seymour Conjecture 9 (the case $t = 1$), one can show that this maximum density parameter can always be attained by a vertex subset $S \subseteq V(G)$ with $|S|$ odd.

As with the Goldberg-Seymour Conjecture 9, our Conjecture 40 can be linked to fractional edge-colorings. Define the maximum density parameter

$$\rho_t(G) = \max_{\substack{S \subseteq V(G), F \subseteq \partial_G(S) \\ t|S| + |F| \geq 2}} \frac{e(S) + |F|}{\lfloor (t|S| + |F|)/2 \rfloor},$$

where $\partial_G(S)$ is the set of edges of G with exactly one end-vertex in S. We can generalize fractional edge-colorings to fractional degree t edge-colorings as follows. A *fractional degree t coloring* w of G is an assignment of a nonnegative weight w_F to the edge set F of each subgraph of G of maximum degree at most t, such that $\sum_{F:e\in F} w_F = 1$ for each edge $e \in E(G)$. The *fractional degree t chromatic index* $\chi_t'^*(G)$ of G is then the minimum of $\sum_F w_F$ over all fractional degree t colorings w of G. The following formula for the fractional degree t chromatic index $\chi_t'^*(G)$ was proven in Stiebitz et al. [86]:

$$\chi_t'^*(G) = \max\left\{\Delta(G)/t, \rho_t(G)\right\}.$$

Zhang, Yu, and Liu [97] claimed to have proven the more natural fractional relaxation of Conjecture 40, where we do not consider the set $F \subseteq \partial_G(S)$ in the maximum of $\rho_t(G)$, but Glock [41] showed that their formula and proof are incorrect. Still, Glock also showed that this mistake makes no difference for Conjecture 40: using either density parameter leads to the same upper bound in the conjecture. (This was shown with the "+1" outside the maximum degree fraction, but the same argument also works with the "+1" in the fraction.)

Note that Conjecture 40 would imply that we could extend Shannon's Theorem 8 on the chromatic index $\chi'(G)$ as

$$\chi_t'(G) \le \left\lceil \frac{3}{3t-1}\Delta(G) \right\rceil$$

for all odd integers $t \ge 1$, since this upper bound is the largest possible value of the stated maximum density parameter, attained by taking $G[S]$ to be $\Delta(G)$-regular and $|S| = 3$. This upper bound would be better than the Shannon-type upper bound proven by Hakimi and Kariv [52], which is more similar to the Shannon-type upper bound we derived from Proposition 36.

We can check that this upper bound is the correct value of the degree t chromatic index for the Shannon triangle G (the triangle with $\Delta(G)/2$ parallel edges on each side): Write $\frac{\Delta(G)}{2} = q \cdot \frac{3t-1}{2} + r$ for some integers $q, r \ge 0$ with $0 \le r < \frac{3t-1}{2}$. For the Shannon triangle with $\lceil \frac{t}{2} \rceil + \lfloor \frac{t}{2} \rfloor + \lfloor \frac{t}{2} \rfloor = \frac{3t-1}{2}$ parallel edges on each side, we can find a degree t coloring using 3 colors by making $\lceil \frac{t}{2} \rceil$ edges on one side of the triangle have color 1 and $\lfloor \frac{t}{2} \rfloor$ of them each have color 2 or 3, and then permuting this color arrangement on the other two sides of the triangle. For a Shannon triangle with $r < \frac{3t-1}{2}$ parallel edges on each side, we can similarly find a degree t coloring using $\left\lceil \frac{3r}{(3t-1)/2} \right\rceil$ colors. This implies that we can find a degree t coloring of G using

$$\chi_t'(G) \le 3q + \left\lceil \frac{3r}{(3t-1)/2} \right\rceil = 3 \cdot \frac{\Delta(G) - 2r}{3t-1} + \left\lceil \frac{6r}{3t-1} \right\rceil < \frac{3}{3t-1}\Delta(G) + 1$$

colors. Thus the bound holds for the Shannon triangle.

Finally, we note something interesting about our stated Conjecture 40: if we assume the ordinary Goldberg-Seymour Conjecture 9 (the case $t = 1$), then Proposition 36 gets us

pretty close to Conjecture 40 for all odd integers $t \geq 1$: it would imply that

$$\chi'_t(G) \leq \max\left\{\left\lceil\frac{\Delta(G)+1}{t}\right\rceil, \left\lceil\frac{1}{t}\left\lceil\max_{S\subseteq V(G),|S|\geq 2}\frac{e(S)}{\lfloor |S|/2\rfloor}\right\rceil\right\rceil\right\}.$$

In particular, we can derive an approximate version of Conjecture 40 using a known approximate version of the ordinary Goldberg-Seymour Conjecture 9 such as Theorem 18, although the approximation gets worse as t gets larger. There is more work to be done here.

Now, recall that the List Coloring Conjecture 19 asserts that $\chi'_\ell(G) = \chi'(G)$ for every multigraph G. It is natural to generalize the conjecture to degree t colorings (see [61]).

Conjecture 41. *For every multigraph G and integer $t \geq 1$, we have $\chi'_{t,\ell}(G) = \chi_{t,\ell}(G)$.*

Let us prove this conjecture when $t \geq 2$ is even. We follow the proof of Petersen's Theorem 38, but apply Galvin's Theorem 20 in place of König's Theorem 1.

Theorem 42. *For every multigraph G and even integer $t \geq 2$, we have $\chi'_{t,\ell}(G) = \chi'_t(G) = \lceil\Delta(G)/t\rceil$.*

Proof. First we prove the theorem for the case $t = 2$. Let G be a multigraph. As usual, G is a subgraph of some $\Delta(G)$-regular multigraph G'. If $\Delta(G)$ is odd, then G' has an even number of vertices, and so we can add to it a perfect matching and obtain a $(\Delta(G)+1)$-regular multigraph H. If $\Delta(G)$ is even, then let $H = G'$. In either case, H is a $2k$-regular multigraph, where $k = \lceil\Delta(G)/2\rceil$ is an integer. Because G is a subgraph of H, we have $\chi'_{2,\ell}(G) \leq \chi'_{2,\ell}(H)$. Thus, it suffices to show that $\chi'_{2,\ell}(H) \leq \lceil\Delta(G)/2\rceil$.

Let L be any list assignment for $E(H)$ with $\lceil\Delta(G)/2\rceil$ colors in each list. Because H is $2k$-regular, each of its components has an Euler tour, and thus H has an orientation D such that every vertex has both indegree and outdegree k. Construct the bipartite graph H' whose vertex set consists of two copies v_1, v_2 of every vertex v of H, and for every arc from a vertex u to a vertex v of D we put an edge between u_1 and v_2 in H'. We also give that edge the same list as its corresponding edge in H. Observe that H' is a k-regular bipartite graph where $k = \lceil\Delta(G)/2\rceil$ is the size of each list. By Galvin's Theorem 20, H' has an L-coloring into matchings. Merging the two copies v_1, v_2 of the vertex v back to a single vertex for every vertex $v \in V(H)$, we obtain a degree 2 L-coloring of H. Therefore, we have $\chi'_{2,\ell}(H) \leq \lceil\Delta(G)/2\rceil$, as required.

Now, to prove the theorem for all even $t \geq 2$, we prove that $\chi'_{t,\ell}(G) \leq \lceil 2\chi'_{2,\ell}(G)/t\rceil$, following the same approach as Proposition 36. Let G be a multigraph, and let L be a list assignment for $E(G)$ with $\lceil 2\chi'_{2,\ell}(G)/t\rceil$ colors in each list. Define the list assignment L' for $E(G)$ by $L'(e) \times [t/2]$ for all $e \in E(G)$. Since $\lceil 2\chi'_{2,\ell}(G)/t\rceil \cdot t/2 \geq \chi'_{2,\ell}(G)$, there exists a degree 2 L'-coloring of ϕ' of G. Define the L-coloring ϕ of G by setting $\phi(e) = c$ if $\phi'(e) = (c, i)$. It is easy to see that ϕ is a degree t L-coloring of G, thus proving that $\chi'_{t,\ell}(G) \leq \lceil 2\chi'_{2,\ell}(G)/t\rceil$.

From this result we immediately get the upper bound $\chi'_{t,\ell}(G) \leq \lceil\Delta(G)/t\rceil$: If $\Delta(G)$ is even, then $\chi'_{t,\ell}(G) \leq \lceil 2\chi'_{2,\ell}(G)/t\rceil = \lceil 2(\Delta(G)/2)/t\rceil = \lceil\Delta(G)/t\rceil$. And if $\Delta(G)$ is odd,

then $\chi'_{t,\ell}(G) \leq \lceil 2\chi'_{2,\ell}(G)/t \rceil = \lceil 2((\Delta(G)+1)/2)/t \rceil = \lceil (\Delta(G)+1)/t \rceil = \lceil \Delta(G)/t \rceil$. The last equality holds because $\Delta(G)$ is odd and t is even. It follows that $\chi'_{t,\ell}(G) = \chi'_t(G)$ by Theorem 39. $\square$

The proofs of Theorem 39 and 42 are remarkably similar to the proofs of Theorem 33 and Theorem 34 on pseudoarboricity and list pseudoarboricity. Of course, as observed before, the degree 2 chromatic index $\chi'_2(G) = pa_2(G)$ is simply a restricted version of pseudoarboricity $pa(G)$, but it is quite a special one. Recall Proposition 31 that a multigraph has pseudoarboricity at most k if and only if it has an orientation such that every vertex has indegree at most k. Likewise, by Euler's theorem on Euler tours, a multigraph has maximum degree at most $2k$ if and only it has an orientation such that every vertex has both indegree and outdgree at most k. The proof of Proposition 31 was easy in that we can find a pseudoforest edge-coloring from an orientation by a greedy coloring procedure. The proof of Petersen's Theorem 38 required us to use König's Theorem 1 on an associated bipartite multigraph. The difference is that pseudoforests only require that the color class vertex indegrees be at most one, whereas in degree 2 colorings we also require that the color class vertex outdegrees to be at most one, which a greedy procedure does not guarantee. Nevertheless, the edge-coloring given by König's Theorem 1 leads directly to Petersen's Theorem 38, whereas deriving Hakimi's Theorem 33 for pseudoarboricity required the extra step of studying appropriate orientations of critical multigraphs as in Theorem 32, and the proof of this theorem is quite similar to that of König; in each case we "switch" on a path leading to the uncolored edge to derive a contradiction. Finally, the proof of Theorem 34 on list pseudoarboricity again only required a greedy list coloring procedure, whereas Theorem 42 on the degree t list chromatic index required Galvin's Theorem 20. These connections will be expanded upon soon when we discuss bounded degree pseudoarboricity. Orientations play a prominent role not only in the presented proofs, but also in the proof of Galvin's Theorem 20 and in the polynomial method proofs of Alon and Tarsi. There are several further connections to be explored in this topic.

It appears that the truth of Conjecture 41 is unknown for odd t. Again, the truth of the original List Coloring Conjecture 19 (the case $t = 1$) would imply that this conjecture nearly holds for every $t \geq 1$ when the multiplicity $\mu(G)$ is bounded. Specifically, if the List Coloring Conjecture holds, then by Proposition 36 and Vizing's Theorem 4, we have

$$\left\lceil \frac{\Delta(G)}{t} \right\rceil \leq \chi'_t(G) \leq \chi'_{t,\ell}(G) \leq \left\lceil \frac{\Delta(G) + \mu(G)}{t} \right\rceil,$$

which implies that $\chi'_{t,\ell}(G) \leq \chi'_t(G) + \lceil \mu(G)/t \rceil$. In particular, if G is a simple graph, then $\chi'_{t,\ell}(G) \leq \chi'_t(G) + 1$ for all integers $t \geq 1$. On the other hand, Proposition 36 and Kahn's theorem on the asymptotic List Coloring Conjecture 19 do imply the asymptotic version of Conjecture 41. That is, for fixed t and $\mu(G)$ we have

$$\chi'_{t,\ell}(G) = (1 + o(1))\frac{\Delta(G)}{t}$$

as $\Delta(G) \to \infty$.

5.2 Bounded degree arboricity and list analogues

Now we study the degree t arboricity parameters $a_t(G)$ and $a_{t,\ell}(G)$. Unlike the case for ordinary arboricity, there is no straightforward matroid structure on the independence system of forests of maximum degree t. Also, the parameters $a_t(G)$ and $a_{t,\ell}(G)$ appear more difficult to determine than $\chi'_t(G)$ and $\chi'_{t,\ell}(G)$ do. We have that

$$a_t(G) \geq \chi_t(G) \geq \lceil \Delta(G)/t \rceil$$

for all $t \geq 1$. However, at least for $t \geq 3$, $a_t(G)$ no longer has a general upper bound close to $\Delta(G)/t$. In fact, for all $t \geq 1$ (assuming $e(G) \geq 1$),

$$a_t(G) \geq a(G) \geq \frac{e(G)}{v(G)-1} > \frac{e(G)}{v(G)} = \frac{d(G)}{2},$$

where $d(G)$ is the average degree of G. In particular, if G is $\Delta(G)$-regular, then $a_t(G) \geq \lceil (\Delta(G)+1)/2 \rceil$. Due to this lower bound, the case $t = 2$ is of particular interest. A **linear forest** is a forest in which all components are paths. The **linear arboricity** $la(G)$ of G is the minimum number of colors needed in an edge-coloring of G into monochromatic linear forests. Thus, $la(G) = a_2(G)$. Linear arboricity was introduced by Harary in [54]. Our observations demonstrate that $la(G) \geq \lceil \Delta(G)/2 \rceil$ in general, and that $la(G) \geq \lceil (\Delta(G)+1)/2 \rceil$ if G is $\Delta(G)$-regular. A famous conjecture of Akiyama, Exoo, and Harary [4] states that for simple graphs, the latter lower bound is achievable.

Conjecture 43 (Linear Arboricity Conjecture)**.** *For every simple graph G, we have that $la(G) \leq \lceil (\Delta(G)+1)/2 \rceil$.*

Note that every simple graph G is a subgraph of a $\Delta(G)$-regular simple graph, possibly on more vertices. Thus, the Linear Arboricity Conjecture is equivalent to the statement that $la(G) = \lceil (\Delta(G)+1)/2 \rceil$ for every $\Delta(G)$-regular simple graph G. The conjecture was later extended to multigraphs by Aït-djafer [3].

Conjecture 44 (Linear Arboricity Conjecture for Multigraphs)**.** *For every multigraph G, we have that $la(G) \leq \lceil (\Delta(G)+\mu(G))/2 \rceil$.*

Recall that the degree 2 chromatic index of a multigraph G is given by $\chi'_2(G) = \lceil \Delta(G)/2 \rceil$ by Theorem 39. Hence, the Linear Arboricity Conjecture for Multigraphs implies that $la(G) \leq \chi'_2(G) + \lceil \mu(G)/2 \rceil$. That is, if we can color a multigraph into monochromatic subgraphs consisting only of cycles or paths, then we need at most $\lceil \mu(G)/2 \rceil$ more colors to make the monochromatic subgraphs consist only of paths. This is similar to the relationship between arboricity and pseudoarboricity discussed before, where we calculated that $a(G) \leq pa(G) + \lceil \mu(G)/2 \rceil$ by combining Nash-Williams' Theorem 23 and Hakimi's Theorem 33. Arboricity is the acyclic version of pseudoarborcity and requires at most $\lceil \mu(G)/2 \rceil$ extra colors, and the Linear Arboricity Conjecture 44 for multigraphs asserts that the analogous

statement is true when it comes to linear arboricity and the degree 2 chromatic index (or "linear pseudoarboricity"). Perhaps the close relationship between pseudoarboricity and the degree 2 chromatic index remarked above could aid in progress on the conjecture.

Actually, we believe that we can strengthen Conjecture 44 with a Goldberg-Seymour-type conjecture for linear arboricity.

Conjecture 45. *For every multigraph G, we have*

$$la(G) \leq \max\left\{\left\lceil \frac{\Delta(G)+1}{2} \right\rceil, a(G)\right\}.$$

This conjecture implies that $la(G) = a(G)$ for every regular multigraph G. This would then imply Conjecture 44 because if G is $\Delta(G)$-regular and the set $S \subseteq V(G)$ is a maximizer for Nash-Williams' Theorem 23, then

$$la(G) = a(G) = \left\lceil \frac{e(S)}{|S|-1} \right\rceil \leq \left\lceil \frac{\Delta(G[S])|S|}{2(|S|-1)} \right\rceil = \left\lceil \frac{\Delta(G[S])}{2} + \frac{\Delta(G[S])}{2(|S|-1)} \right\rceil \leq \left\lceil \frac{\Delta(G)}{2} + \frac{\mu(G)}{2} \right\rceil$$

where the last inequality follows from $\Delta(G[S]) \leq \mu(G)(|S|-1)$. Note that the inequality $pa(G) \leq \chi'_2(G)$ between pseudoarboricity and the degree 2 chromatic index becomes an equality when G is $\Delta(G)$-regular, since in this case maximum average degree is the same as maximum degree. Thus, Conjecture 45 implies that the inequality $a(G) \leq la(G)$ also becomes an equality when G is $\Delta(G)$-regular. Note that this contrasts with the case of the chromatic index, as it is not necessarily true that the inequality $\lceil \rho(G) \rceil \leq \chi'(G)$ becomes an equality when G is $\Delta(G)$-regular (e.g., when G is the Petersen graph); however, it is true when G is bipartite.

Notice that the Conjecture 45 nearly implies the Goldberg-Seymour Conjecture 9. Specifically, note that every linear forest can be decomposed into two matchings, so we derive that

$$\chi'(G) \leq 2 \cdot la(G) \leq \max\left\{2\left\lceil \frac{\Delta(G)+1}{2} \right\rceil, \max_{S \subseteq V(G), |S| \geq 2} 2\left\lceil \frac{e(S)}{|S|-1} \right\rceil\right\},$$

Compare this to the Goldberg-Seymour Conjecture 9, which states that

$$\chi'(G) \leq \max\left\{\Delta(G)+1, \max_{\substack{S \subseteq V(G), \\ |S| \geq 3 \text{ odd}}} \left\lceil \frac{2e(S)}{|S|-1} \right\rceil\right\},$$

which is not too far off. The apparent connection to the Goldberg-Seymour Conjecture might point to the difficulty of proving Conjecture 45, but it could also give insight into the relationship between arboricity and chromatic index of dense multigraphs in future work. We will show that a Goldberg-Seymour formula holds for bounded degree pseudoarboricity in the next section. Also, note that Conjecture 45 implies $a_t(G) = a(G)$ for all $t \geq 2$ when G is $\Delta(G)$-regular. The analogous statement for pseudoarboricity holds based on our previous comments. We already know trivially that $a_t(G) = a(G)$ for any multigraph G when $t \geq \Delta(G)$. This brings up an interesting question toward proving or disproving Conjecture 45, which will end the discussion of the conjecture.

Question 46. *How small can we make t to prove that $a_t(G) = a(G)$ for all $\Delta(G)$-regular multigraphs G? In particular, can we prove a lower bound on t that is independent of G?*

Let us now comment on the Linear Arboricity Conjecture 43 for simple graphs. Analogous to Vizing's Theorem 4 for the chromatic index, it asserts that linear arboricity is determined almost entirely by the maximum degree. And similar to the proper edge-coloring case, it is NP-hard to determine in general whether $la(G) = \lceil \Delta(G)/2 \rceil$ or $la(G) \geq \lceil (\Delta(G)+1)/2 \rceil$ if the two values are not equal. Indeed, even recognizing simple graphs with linear arboricity 2 is NP-complete [74]. The Linear Arboricity Conjecture 43 has been verified for a few classes of graphs, such as complete graphs [85], complete bipartite graphs [4], planar graphs [94], series parallel graphs [95], graphs of girth at least $14\Delta(G)$ [46], and when the maximum degree $\Delta(G)$ is one of $3, 4, 5, 6, 8, 10$ (see [4, 5, 30, 48]). Alon [8] proved that the Conjecture holds asymptotically using the probabilistic method. This asymptotic upper bound has been improved multiple times [11, 33], with the current best bound due to Lang and Postle [65] being $la(G) \leq \Delta/2 + 3\Delta^{1/2}(\log \Delta)^4$ for sufficiently large $\Delta = \Delta(G)$.

For a general upper bound on linear arboricity for simple graphs, note that Vizing's Theorem 4 implies that $la(G) \leq \Delta(G) + 1$ because every matching is itself a linear forest. Guldan [47] improves this bound by using Petersen's Theorem 38 to decompose the graph into smaller subgraphs for which the linear arboricity conjecture is known to hold. We study his result.

Lemma 47. *Let n be a fixed positive integer. If the linear arboricity of every $2k$-regular simple graph is $k+1$ for all $k \leq n$, then for every simple graph G we have*

$$la(G) \leq \left\lceil \frac{n+1}{n} \cdot \frac{\Delta(G)}{2} \right\rceil \qquad \text{if } \Delta(G) \text{ is even,}$$

$$la(G) \leq 1 + \left\lceil \frac{n+1}{n} \cdot \frac{\Delta(G)-1}{2} \right\rceil \qquad \text{if } \Delta(G) \text{ is odd.}$$

Proof. As noted before, G is a subgraph of some $\Delta(G)$-regular simple graph H. First assume that $\Delta(G) = 2m$ is even. By the division algorithm, there is an integer $0 \leq r < n$ such that $2m = 2n\lfloor m/n \rfloor + 2r$. If $r = 0$, then by Petersen's Theorem 38, H can be decomposed into m/n $2n$-factors $H_1, \ldots, H_{m/n}$. By hypothesis, each H_i has linear arboricity $n+1$, so that

$$la(G) \leq la(H) \leq \sum_{i=1}^{m/n} la(H_i) = \frac{m(n+1)}{n} = \left\lceil \frac{n+1}{n} \cdot \frac{\Delta(G)}{2} \right\rceil.$$

If $r \neq 0$, then again by Petersen we can decompose H into $\lfloor m/n \rfloor$ $2n$-factors $H_1, \ldots, H_{\lfloor m/n \rfloor}$ and one $(2m - 2n\lfloor m/n \rfloor)$-factor H_0. Again by the theorem's linear arboricity hypothesis,

$$\begin{aligned} la(G) \leq la(H) &\leq la(H_0) + \sum_{i=1}^{\lfloor m/n \rfloor} la(H_i) = \left(m - n\left\lfloor \frac{m}{n} \right\rfloor + 1\right) + (n+1)\left\lfloor \frac{m}{n} \right\rfloor \\ &= m + \left\lfloor \frac{m}{n} \right\rfloor + 1 = \left\lceil \frac{n+1}{n} \cdot \frac{\Delta(G)}{2} \right\rceil. \end{aligned}$$

Now assume that $\Delta(G) = 2m+1$ is odd. Then the $(2m+1)$-regular graph H has an even number of vertices. Let H' be H plus an arbitrary perfect matching on $V(H)$. Then H' is a $(2m+2)$-regular graph, so by Petersen's Theorem 38, H' has a 2-factor G'. We delete the edges of G' not in H, followed by iteratively deleting one edge from every cycle in G'. The result is a spanning linear forest H_0 in H where every vertex of H_0 has degree at least one. Then H can be decomposed into the linear forest H_0 and the subgraph $H - H_0$ (short for $H - E(H_0)$). Since $H - H_0$ has maximum degree $\Delta(G) - 1 = 2m$, by the even maximum degree case above we have

$$la(G) \leq la(H) \leq la(H_0) + la(H - H_0) = 1 + \left\lceil \frac{n+1}{n} \cdot \frac{\Delta(G)-1}{2} \right\rceil.$$

□

Because the linear arboricity conjecture has been verified for all simple graphs G with even maximum degrees $\Delta(G) = 2, 4, 6, 8, 10$, we derive Guldan's general linear arboricity bound [47].

Corollary 48 (Guldan)**.** *For every simple graph G, we have*

$$la(G) \leq \left\lceil \frac{3\Delta(G)}{5} \right\rceil \qquad \text{if } \Delta(G) \text{ is even,}$$
$$la(G) \leq 1 + \left\lceil \frac{3(\Delta(G)-1)}{5} \right\rceil \qquad \text{if } \Delta(G) \text{ is odd.}$$

Notice that Guldan's bound can be improved if we have even approximate versions of the Linear Arboricity Conjecture 43 for small graphs. Specifically, suppose we know that the linear arboricity of every $2n$-regular graph is at most $n + c(n)$ for some function $c(n) \geq 1$ of n. Then following Guldan's proof of Lemma 47, we would get that for every simple graph G and integer $n \geq 1$,

$$la(G) \leq \left\lceil \frac{n+c(n)}{n} \cdot \frac{\Delta(G)}{2} \right\rceil + c(r) - 1,$$

for some $0 \leq r \leq n-1$. As in the statement of the lemma, there is a small improvement if $\Delta(G)$ is odd. If one can show that $5c(n) < n$ for some integer $n \geq 6$ (i.e., that every $2n$-regular graph has linear arboricity less than $n + n/5$), then one would immediately improve Guldan's bound on linear arboricity in Corollary 48, at least when $\Delta(G)$ is somewhat larger than n. This would be the best bound until one of the asymptotic bounds on $la(G)$ applies.

Let us make one more observation on linear arboricity that was also noted by Guldan in a different paper [45]. As noted before, every linear forest can be decomposed into two matchings. Thus, if we have k-edge-coloring of a multigraph G into linear forests, then we can find a proper $2k$-edge-coloring of G. This proper edge-coloring has the added property that there is a way to pair up colors so that the dichromatic subgraphs they form are all

acyclic. It is natural to ask about proper edge-colorings with the stronger property that for *any* pair of color classes, the subgraph they form is acyclic. That is, no cycle of the graph is two-colored. Such a proper edge-coloring is called an **acyclic edge-coloring**, and we define the **acyclic chromatic index** $\chi'_a(G)$ of a simple graph G to be the minimum number of colors needed in an acyclic edge-coloring of G. (This concept only makes sense for simple graphs unless we allow multigraphs to have dichromatic 2-cycles.) If we arbitrarily pair up all but possibly one of the color classes in an acyclic edge-coloring of G, then we obtain an edge-coloring of G into linear forests. Hence,

$$la(G) \leq \lceil \chi'_a(G)/2 \rceil.$$

A major conjecture of Fiamčík [34] and Alon, Sudakov, and Zaks [12] implies that we could almost derive the Linear Arboricity Conjecture 43 from this easy upper bound.

Conjecture 49 (Acyclic Edge-Coloring Conjecture)**.** *For every simple graph G, we have $\chi'_a(G) \leq \Delta(G) + 2$.*

In other words, the conjecture states that it takes only one more color than given by Vizing's Theorem 4, $\chi'(G) \leq \Delta(G)+1$, to produce an acyclic edge-coloring of a given simple graph G. Unfortunately, the Acyclic Edge-Coloring Conjecture 49 appears even harder to resolve than the Linear Arboricity Conjecture 43. It is not even known asymptotically. The best known general upper bound is $\chi'_a(G) \leq \lceil 3.74(\Delta(G) - 1) \rceil + 1$ due to Giotis, Kirousis, Psaramiligkos, and Thilikos [40], which built on successive improvements that use the Lovász Local Lemma and entropy compression [9, 32, 68, 71]. The conjecture has been shown to hold for graphs of girth at least $c\Delta(G) \log \Delta(G)$ [12], triangle-free planar graphs [83], graphs of small density [92], graphs with $\Delta(G) \leq 3$, and almost all $\Delta(G)$-regular graphs [12]. It is still wide open.

Now let us return to studying the degree t arboricity $a_t(G)$ for more general $t \geq 2$ and multigraphs G. We believe that one could write a Goldberg-Seymour Conjecture extending Conjecture 45 for all $t \geq 2$, of the form

$$a_t(G) \leq \max \left\{ \left\lceil \frac{\Delta(G) + c}{t} \right\rceil, a(G) \right\}$$

for some constant c possibly depending on t. We expect the error term c in the maximum degree expression to grow with t due to the fact that forests with high maximum degree have many leaves, which makes each color class have an uneven degree distribution. In terms purely of the maximum degree $\Delta(G)$ and multiplicity $\mu(G)$, we cannot expect a better general upper than $a_t(G) \leq \lceil (\Delta(G) + \mu(G))/2 \rceil$ because a $\Delta(G)$-regular multigraph G can have arboricity $a(G)$ as high as $\lceil (\Delta(G) + \mu(G))/2 \rceil$. However, the conjecture above asserts that $a_t(G)$ gets arbitrarily close to $\Delta(G)/t$ as $\Delta(G) \to \infty$ so long as $a(G)$ remains bounded; that is, assuming the multigraph G is sparse. Let us affirm this observation using a degeneracy argument of Caro and Roditty [20] (originally written for simple graphs, but extended to multigraphs here). Recall that a multigraph G is k-degenerate if every subgraph of G has a

vertex of degree at most k. By a degeneracy ordering on $V(G)$, we can prove that $e(S) \leq k(|S|-1)$ for all $S \subseteq V(G)$, so that by Nash-Williams' Theorem we have $a(G) \leq k$. Hence, bounded degeneracy implies bounded arboricity.

Proposition 50 (Caro, Roditty)**.** *For every k-degenerate multigraph G and integer $t \geq 2$, we have*

$$a_t(G) \leq \left\lceil \frac{\Delta(G) + ((t-1)k-1)\mu(G)}{t} \right\rceil.$$

Proof. If $k = 1$, then G is a forest and we have $a_t(G) \leq \lceil \Delta(G)/t \rceil$ as mentioned before. Now fix $k \geq 2$ and $t \geq 2$. We prove the upper bound by induction on the number of vertices $v(G)$. The bound clearly holds when $v(G) = 2$. Assume it holds when $v(G) = n$. Let G be a k-degenerate multigraph with $n+1$ vertices, and let b denote the claimed upper bound. By definition, G has a vertex v of degree at most k. Then $G' = G - v$ is a k-degenerate multigraph with n vertices and maximum degree at most $\Delta(G)$, so by the induction hypothesis we have $a_t(G') \leq b$. Let ϕ' be an edge-coloring of G' into b monochromatic forests with maximum degree at most t, using the colors $1, 2, \ldots, b$. We wish to color the edges incident to v. Let $v_1, \ldots, v_m$ be the neighbors of v in G, where $m \leq k$. Observe that $d_{G'}(v_i) \leq \Delta(G) - \mu_G(v, v_i)$ for all $1 \leq i \leq m$. For $1 \leq i \leq m$ and $0 \leq j \leq t$, let L_{ij} be the set of colors that are present at v_i exactly j times with respect to ϕ'. Let $a_{ij} = |L_{ij}|$. By counting colors we have

$$\sum_{j=0}^{t} a_{ij} = b \quad \text{for all } 1 \leq i \leq t.$$

Further, by counting edges we have

$$\sum_{j=0}^{t} j \cdot a_{ij} \leq \Delta(G) - \mu_G(v, v_i) \quad \text{for all } 1 \leq i \leq m.$$

We deduce that for all $1 \leq i \leq m$,

$$\begin{aligned} \sum_{j=0}^{t-1} (t-j) a_{ij} &\geq b \cdot t - \Delta(G) + \mu_G(v, v_i) \\ &\geq [\Delta(G) + ((t-1)k-1)\mu_G(v, v_i)] - \Delta(G) + \mu_G(v, v_i) \\ &\geq (t-1)k \cdot \mu_G(v, v_i). \end{aligned}$$

Divide both sides by $t-1$ (as $t \geq 2$) to find that

$$2a_{i0} + \sum_{j=1}^{t-1} a_{ij} \geq k \cdot \mu_G(v, v_i) \quad \text{for all } 1 \leq i \leq m.$$

Now make copies $1', 2', \ldots, b'$ of the colors used by ϕ', and for each $1 \leq i \leq m$ let L'_{i0} equal L_{i0} where each color is replaced by its copy. Set $L_i = \left(\bigcup_{j=0}^{t-1} L_{ij}\right) \cup L'_{i0}$. By the above inequality, we have $\lfloor |L_i|/\mu_G(v, v_i) \rfloor \geq k$. Thus, we may distribute the colors in L_i to the edges in $E_G(v, v_i)$ in the form of lists so that each list has size at least k. Since $k \geq 2$, this may be done in such a way that copies of the same color lie in the same list. There are at most k edges incident to v, so we may perform a greedy coloring from this list assignment in such a way that each edge gets a different color. Now replace each copy of a color a' by its original color a, and let ϕ be the resulting edge-coloring of G. Then each color has degree at most t at each neighbor v_i of v (since we excluded L_{it} from L_i), and each color has degree at most two at v. Further, if a color has degree two at v, then one of the edges of that color goes to a neighbor v_i of v at which that color had degree zero respect to ϕ'. This implies that the edge-coloring ϕ has no monochromatic cycles, and hence it is a desired edge-coloring of G, finishing the induction. □

Note that it is easier to derive the upper bound $a_t(G) \leq a(G)\lceil \Delta(G)/t \rceil$, using the observation that every forest F has degree t arboricity $a_t(F) = \lceil \Delta(F)/t \rceil$. However, the asymptotics as $\Delta(G) \to \infty$ would be worse than that given by Proposition 50. From the proposition we have

$$\left\lceil \frac{\Delta(G)}{t} \right\rceil \leq a_t(G) \leq \left\lceil \frac{\Delta(G)}{t} + ((t-1)k - 1)\frac{\mu(G)}{t} \right\rceil.$$

Compare this to the degree t chromatic index bounds (Proposition 36):

$$\left\lceil \frac{\Delta(G)}{t} \right\rceil \leq \chi'_t(G) \leq \left\lceil \frac{\Delta(G)}{t} + \frac{\mu(G)}{t} \right\rceil.$$

For multigraphs G of bounded degeneracy (and thus bounded multiplicity $\mu(G)$ and arboricity $a(G)$), $a_t(G)$ gets asymptotically close to the degree t chromatic index $\chi'_t(G) = \Delta(G)/t + O(1)$ as $\Delta(G) \to \infty$, in support of the Goldberg-Seymour-type conjecture for degree t arboricity $a_t(G)$. This result makes intuitive sense since a degree t coloring of a sparse multigraph likely does not require too many extra colors to make it acyclic. In this case, the bound is much better than the general upper bound $a_t(G) \leq \lceil (\Delta(G) + \mu(G))/2 \rceil$ implied by the Linear Arboricity Conjecture 44. Also, note that if we take $k = t = 2$, then Proposition 50 gives us the Linear Arboricity Conjecture 44 for 2-degenerate multigraphs.

Finally, let us take a brief look into the list analogue $a_{t,\ell}(G)$ of degree t arboricity. Analogous to Conjecture 41, we may attempt to generalize the original List Coloring Conjecture 19 in the direction of degree t arboricity.

Conjecture 51. *For every multigraph G and integer $t \geq 1$, we have $a_{t,\ell}(G) = a_t(G)$.*

We also denote $a_{2,\ell}(G)$ by $la_\ell(G)$, referring to it as the **list linear arboricity** of G. The list linear arboricity case of Conjecture 51, that $la_\ell(G) = la(G)$, was originally proposed

by An and Wu [14], who proved in a separate paper [15] that it holds for all planar simple graphs with maximum degree at least 13. For a fixed multigraph G, the conjecture holds for all $t \geq \Delta(G)$ by Seymour's Theorem 27 (since $a_t(G) = a(G)$ and $a_{t,\ell}(G) = a_\ell(G)$ for all $t \geq \Delta(G)$). Kim and Postle [61] proved that Conjecture 51 holds asymptotically for the case $t = 2$ of list linear arboricity $la_\ell(G)$ of simple graphs G. In fact, list linear arboricity is the context in which the best asymptotic bound on the ordinary linear arboricity by Lang and Postle [65] was derived. That is, $la_\ell(G) \leq \Delta/2 + 3\Delta^{1/2}(\log \Delta)^4$ for sufficiently large $\Delta = \Delta(G)$. Other cases of t have not been well-studied.

5.3 Bounded degree pseudoarboricity and list analogues

The connections observed between pseudoarboricity $pa(G)$ and the degree 2 chromatic index $\chi'_2(G)$ after Theorem 42 make sense when we start to consider bounded degree pseudoarboricity $pa_t(G)$. We will be assuming $t \geq 2$. As observed before, $pa_2(G) = \chi'_2(G)$ since every multigraph with maximum degree 2 is a pseudoforest with maximum degree 2. Also, $pa_t(G) = pa(G)$ for all $t \geq \Delta(G)$. Observing that

$$pa(G) \leq pa_t(G) \leq pa_2(G)$$

for every for every multigraph G and integer $t \geq 2$, by Hakimi's Theorem 33 and Theorem 42 we have

$$\max_{S \subseteq V(G), |S| \geq 2} \left\lceil \frac{e(S)}{|S|} \right\rceil \leq pa_t(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil.$$

In particular, since maximum average degree is the same as maximum degree for regular multigraphs, we have

$$pa_t(G) = pa(G) = \lceil \Delta(G)/2 \rceil$$

for every $\Delta(G)$-regular multigraph G and integer $t \geq 2$. This solves the pseudoarboricity analogue of Question 46.

Now we can give the following orientation characterization of degree t pseudoarboricity that generalizes our observations about the connection between pseudoarboricity and the degree 2 chromatic index after Theorem 42.

Proposition 52. *A multigraph G has degree t pseudoarboricity (with $t \geq 2$) at most k if and only if it has an orientation such that every vertex has indegree at most k and outdegree at most $k(t-1)$.*

Proof. For $k = 1$, it is easy to show that a multigraph is a pseudoforest with maximum degree at most t if and only if it has an orientation such that every vertex has indegree at most 1 and outdegree at most $t - 1$, following the ideas of Proposition 31. Now we prove

the proposition for all $k \geq 1$ by following the same approach taken in the proof of Petersen's Theorem 38.

Suppose that a multigraph G has degree t arboricity at most k. Let $F_1, \ldots, F_k$ be a decomposition of G into k pseudoforests with maximum degree at most t. By the case $k = 1$, we can orient each of $F_1, \ldots, F_k$ so that in each of them, every vertex has indegree at most 1 and outdegree at most $t - 1$. Combining these k orientations gives an orientation of G such that every vertex has indegree at most k and outdegree at most $k(t-1)$.

Conversely, suppose that a multigraph G has an orientation D such that every vertex has indegree at most k and outdegree at most $k(t-1)$. Construct the bipartite graph H_1 whose vertex set consists of two copies v_1, v_2 of every vertex v of D, and for every arc from a vertex u to a vertex v of D we put an edge between u_1 and v_2 in H_1. Let X_1, X_2 be the corresponding bipartition of the vertex set of H_1. Then every vertex in X_1 has degree at most $k(t-1)$ and every vertex in X_2 has degree at most k. The goal is to find a k-edge-coloring of H_1 such that each color class has degree at most $t - 1$ at X_1 and degree at most 1 at X_2. To do that, we construct a new bipartite graph H_2 from H_1 by making $t - 1$ copies of every vertex in X_1 and distributing its at most $k(t-1)$ incident edges among these copies in such a way that every vertex in H_2 has degree at most k. By König's Theorem 1, H_2 can be edge-colored into k matchings. Merging the $t - 1$ copies of every vertex in X_1 back to a single vertex, we obtain the desired k-edge-coloring of H_1. Then merging the two copies of a vertex in H_1 back to a single vertex for every vertex of G, we obtain a k-edge-coloring of D such that in each color class, every vertex has indegree at most 1 and outdegree at most $t - 1$. By the case $k = 1$, each color class is a pseudoforest in G of degree at most t, and therefore G has degree t pseudoarboricity at most k. □

Using Proposition 52, we can prove the following formula for $pa_t(G)$ for all integers $t \geq 2$ that resembles the Goldberg-Seymour Conjecture 9. The proof is almost the same as that of Hakimi's Theorem 33, and the "dense spot" will again be a maximal pseudoarboretum.

Theorem 53. *For every multigraph G and integer $t \geq 2$, we have*

$$pa_t(G) = \max\left\{\left\lceil \frac{\Delta(G)}{t} \right\rceil, pa(G)\right\}.$$

Proof. Clearly $pa_t(G) \geq \chi'_t(G) \geq \lceil \Delta(G)/t \rceil$ and $pa_t(G) \geq pa(G)$. To prove that one of these values is an upper bound, let H be a critical subgraph of G with $pa_t(H) = pa_t(G) = k$. If $k \leq \lceil \Delta(H)/t \rceil$, then $pa_t(G) \leq \lceil \Delta(H)/t \rceil \leq \lceil \Delta(G)/t \rceil$ and we are done. Otherwise, we have $\Delta(H) \leq (k-1)t$. Let $e_0 \in E_H(v_0, v_1)$ be any edge of H. Using Proposition 52, let D be an orientation of $H - e_0$ such that every vertex of D has indegree at most $k-1$ and outdegree at most $(k-1)(t-1)$. Since $d_H(v_0), d_H(v_1) \leq (k-1)t$ and e_0 is not oriented, both v_0 and v_1 are "missing" an arc. That is, either their indegree or outdegree is less than the most possible. We say that a vertex v is *deficient in indegree* if $d^-_D(v) < k$, and is *deficient in outdegree* if $d^+_D(v) < (k-1)(t-1)$. It cannot be that v_0 is deficient in indegree and v_1 is deficient in

outdegree, as otherwise we could orient e_0 from v_1 to v_0 and contradict the value of $pa_t(H)$. Likewise, it cannot be that v_0 is deficient in outdegree and v_1 is deficient in indegree.

The rest of the proof follows that of Hakimi's Theorem 33. Suppose that v_0 and v_1 are both deficient in outdegree, and thus both have indegree exactly $k-1$. Let A be the maximal pseudoarboretum with respect to D, that is, the subgraph of G induced by all vertices v for which there is a directed path in D from v to one of v_0 or v_1. By the same path flipping argument as in Proposition 32 on maximal pseudoarboreta, we can conclude that every vertex of A has indegree exactly $k-1$ in $D[V(A)]$. This works because v_0 and v_1 are deficient in outdegree, so they will still have outdegree at most $(k-1)(t-1)$ in D after flipping on the directed path to one of them. Letting $S = V(A)$, the same proof as Hakimi's Theorem 33 shows that $pa_t(G) = k = \lceil e(S)/|S| \rceil$. Since $\lceil e(S)/|S| \rceil \leq pa(G) \leq pa_t(G)$, we get that $pa_t(G) = pa(G) = \lceil e(S)/|S| \rceil$ as required.

If we suppose that v_0 and v_1 are both deficient in indegree, then we can perform the same argument as above with outdegree replacing the role of indegree in the definition of maximal pseudoarboretum. We may similarly derive that $pa_t(G) = k = \lceil e(S)/((t-1)|S|) \rceil$. Again since $\lceil e(S)/((t-1)|S|) \rceil \leq pa(G) \leq pa_t(G)$, we get that $pa_t(G) = pa(G)$ as required. (Of course, this latter case can only arise if $t = 2$ or $pa(G) = 1$.) □

Finally, we can use Proposition 52 to prove that list degree t pseudoarboricity and degree t pseudoarboricity are always the same, generalizing Theorem 34.

Theorem 54. *For every multigraph G and integer $t \geq 2$, we have $pa_{t,\ell}(G) = pa_t(G)$.*

Proof. Suppose G has degree t pseudoarboricity $pa_t(G) = k$, and let L be any k-list assignment for $E(G)$. By Proposition 52, G has an orientation D such that every vertex has indegree at most k and outdegree at most $k(t-1)$. Let H_2 be the same bipartite graph with maximum degree k that was constructed from D in the proof of Proposition 52. Then H_2 inherits the list assignment L from G since the edges of G and H_2 correspond. By Galvin's Theorem 20, H_2 has an L-coloring into matchings. By the same logic as in the proof of Proposition 52, merging all copies of a vertex of G back to a single vertex gives us an L-coloring of G into pseudoforests of maximum degree at most t, as required. □

Chapter 6

Matroid coloring and list coloring

In this chapter, we give an overview of matroid colorings analogous to multigraph edge-colorings, as well as the list versions of them. We then look at a list coloring conjecture for matroids that is similar to the List Coloring Conjecture 19 for multigraphs. This is one attempt to unify certain list coloring results on the chromatic index and on arboricity that we previously discussed.

A **matroid** $\mathcal{M}$ is a pair $(E, \mathcal{I})$ where E is a finite ground set and $\mathcal{I}$ is a family of subsets, so-called **independent sets**, of E that satisfies the following:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $A \subseteq B \subseteq E$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$.

(I3) If $A, B \in \mathcal{I}$ and $|A| < |B|$, then there exists an $e \in B \setminus A$ such that $A \cup \{e\} \in \mathcal{I}$.

Maximal independent sets are called **bases**, and minimal dependent sets are called **circuits**. A consequence of the above axioms is that all bases have the same size. All matroids under consideration will be assumed to be loopless, i.e., all singleton subsets of the ground set are independent. We refer to [73] for further background in matroid theory.

An important example of a matroid that entered our arboricity and pseudoarboricity discussions is the graphic matroid $\mathcal{M}_G$ and bicircular matroid $\mathcal{B}_G$ of a multigraph G:

$$\mathcal{M}_G = (E(G), \mathcal{I} = \{F \subseteq E(G) : F \text{ is a forest in } G\}),$$
$$\mathcal{B}_G = (E(G), \mathcal{I} = \{F \subseteq E(G) : F \text{ is a pseudoforest in } G\}).$$

Other important examples of matroids are the uniform matroids

$$U_n^r = ([n], \mathcal{I} = \{S \subseteq [n] : |S| \leq r\}),$$

(where $r \leq n$) and the partition matroids

$$\mathcal{M} = (E, \mathcal{I} = \{I \subseteq E : |I \cap E_i| \leq 1 \text{ for all } 1 \leq i \leq k\}),$$

(where $E_1, \ldots, E_k$ is a partition of E).

The **rank function** r of a matroid $\mathcal{M} = (E, \mathcal{I})$ is a function on all subsets of E, with $r(X)$ being the size of a largest independent set contained in X, for all $X \subseteq E$. The rank of $\mathcal{M}$ is defined to be $r(E)$, that is, the size of a largest independent set or basis in all of $\mathcal{M}$. The rank function r is characterized by the following properties:

(R1) The outputs of r are nonnegative integers.

(R2) For all $A \subseteq E$ we have $r(A) \leq |A|$.

(R3) For all $A \subseteq E$ and $e \in E$ we have $r(A) \leq r(A \cup \{e\}) \leq r(A) + 1$.

(R4) The function r is **submodular**, i.e., for all $A, B \subseteq E$ we have $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.

As noted in chapter 3, the rank function of a graphic matroid $\mathcal{M}_G$ is given by $r(A) = v(G) - c(A)$ where $c(A)$ is the number of components of A.

For a matroid $\mathcal{M} = (E, \mathcal{I})$, define its **chromatic number** $\chi(M)$ to be the minimum number of colors needed to partition the ground set E into monochromatic independent sets. Thus, for a multigraph G, we have that $\chi(\mathcal{M}_G) = a(G)$ is the arboricity of G, and $\chi(\mathcal{B}_G) = pa(G)$ is the pseudoarboricity of G. We mentioned that our proof of Nash-Williams' Theorem 23 in chapter 3 can be phrased entirely in terms of matroids. This would lead us to a proof of Edmonds' generalization of Nash-Williams' Theorem 23 and Hakimi's Theorem 33 to matroids [26].

Theorem 55 (Edmonds)**.** *For every matroid $\mathcal{M} = (E, \mathcal{I})$, we have*

$$\chi(\mathcal{M}) = \max_{S \subseteq E, |S| \geq 1} \left\lceil \frac{|S|}{r(S)} \right\rceil .$$

As with graph coloring, we may define the **list chromatic number** $\chi_\ell(\mathcal{M})$ of a matroid $\mathcal{M} = (E, \mathcal{I})$ as the minimum integer k such that for every list assignment L for E with k colors in each list, there exists an L-coloring of $\mathcal{M}$ in which each color class forms an independent set. We easily see that $\chi_\ell(\mathcal{M}) \geq \chi(\mathcal{M})$, and that for every multigraph G, $\chi_\ell(\mathcal{M}_G) = a_\ell(G)$ (list arboricity) and $\chi_\ell(\mathcal{B}_G) = pa_\ell(G)$ (list pseudoarboricity). As with our arboricity proof, we can rephrase our proof of Seymour's Theorem 27 to prove its natural matroid generalization [79], which was the setting of Seymour's original proof. The proof of Lasoń's Theorem 30, in which the lists had variable sizes, likewise extends to matroids [66].

Theorem 56 (Seymour)**.** *For every matroid $\mathcal{M}$, we have $\chi_\ell(\mathcal{M}) = \chi(\mathcal{M})$.*

On the other hand, recall that Galvin's Theorem 20 states that for any bipartite multigraph G we have $\chi'_\ell(G) = \chi'(G)$. Matchings in a bipartite multigraph $G = (X_1 \cup X_2, E)$

form the common independent sets of two partition matroids, namely

$$\mathcal{M}_1' = (E, \mathcal{I}_1 = \{A \subseteq E : |A \cap \delta(v)| \leq 1 \text{ for all } v \in X_1\}),$$
$$\mathcal{M}_2' = (E, \mathcal{I}_2 = \{A \subseteq E : |A \cap \delta(v)| \leq 1 \text{ for all } v \in X_2\}),$$

where $\delta(v)$ is the set of all edges incident to v. From this perspective, Galvin's theorem is a theorem about the list-version of the problem of partitioning a common ground set of two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ into the fewest possible common independent sets.

This provides us motivation to define the **joint-chromatic number** $\chi(\mathcal{M}_1, \mathcal{M}_2)$ of two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ on the same ground set E to be the minimum number of colors needed to partition E into monochromatic sets that are independent in both $\mathcal{M}_1$ and $\mathcal{M}_2$. It is immediate that $\chi(\mathcal{M}_1, \mathcal{M}_2) \geq \max\{\chi(\mathcal{M}_1), \chi(\mathcal{M}_2)\}$, and that $\chi(\mathcal{M}_1', \mathcal{M}_2') = \chi'(G)$ where $\mathcal{M}_1', \mathcal{M}_2'$ are the partition matroids defined above. We may also naturally define the **list joint-chromatic number** $\chi_\ell(\mathcal{M}_1, \mathcal{M}_2)$, where the coloring must be done with respect to some list assignment for the ground set E. We see that $\chi_\ell(\mathcal{M}_1', \mathcal{M}_2') = \chi_\ell'(G)$. Hence, Galvin's theorem states that $\chi_\ell(\mathcal{M}_1', \mathcal{M}_2') = \chi(\mathcal{M}_1', \mathcal{M}_2')$. All partition matroids can be represented in the form of $\mathcal{M}_1', \mathcal{M}_2'$ above, so in fact we have that $\chi_\ell(\mathcal{M}_1, \mathcal{M}_2) = \chi(\mathcal{M}_1, \mathcal{M}_2)$ for any two partition matroids $\mathcal{M}_1, \mathcal{M}_2$ on the same ground set. Moreover, Seymour's Theorem 56 can be restated as $\chi_\ell(\mathcal{M}, \mathcal{M}) = \chi(\mathcal{M}, \mathcal{M}) = \chi(\mathcal{M})$ for any matroid $\mathcal{M}$. As a common generalization, it is natural to conjecture the following matroid analogue of the List Coloring Conjecture 19 [63].

Conjecture 57. *For any two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ on the same ground set, we have $\chi_\ell(\mathcal{M}_1, \mathcal{M}_2) = \chi(\mathcal{M}_1, \mathcal{M}_2)$.*

Unfortunately, this conjecture has been verified for only very few classes of matroids $\mathcal{M}_1$ and $\mathcal{M}_2$. In addition to the cases of two partition matroids and the same matroids, Király and Pap [63] verified the conjecture for two transversal matroids, for two matroids of rank 2, and if the common bases are the arborescences of the disjoint union of two arborescences rooted at the same vertex. Bérczi, Schwarcz, and Yamaguchi [17] proved non-trivial approximations of this conjecture for various combinations of graphic matroids, paving matroids, and gammoids. The following example of Király [62] shows that an analogous conjecture for the case of three matroids does not hold. Let $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, and let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ be the partition matroids defined by the circuit sets $\mathcal{C}_1 = \{\{e_1, e_4\}, \{e_2, e_5\}, \{e_3, e_6\}\}, \mathcal{C}_2 = \{\{e_1, e_5\}, \{e_2, e_6\}, \{e_3, e_4\}\}, \mathcal{C}_3 = \{\{e_1, e_6\}, \{e_2, e_4\}, \{e_3, e_5\}\}$. Note that $\{e_1, e_2, e_3\}, \{e_4, e_5, e_6\}$ is a partition of E into two common bases of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$. However, for the list assignment L given by $L(e_1) = L(e_4) = \{1, 2\}$, $L(e_2) = L(e_5) = \{1, 3\}$, $L(e_3) = L(e_6) = \{2, 3\}$, there is no L-coloring that is proper for each of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$.

For some final remarks, we note that the joint-chromatic number $\chi(\mathcal{M}_1, \mathcal{M}_2)$ is itself hard to determine and is a source of many interesting results and conjectures. As before, we have the trivial lower bound $\chi(\mathcal{M}_1, \mathcal{M}_2) \geq \max\{\chi(\mathcal{M}_1), \chi(\mathcal{M}_2)\}$, and Edmonds' Theorem 11 gives a formula for both $\chi(\mathcal{M}_1)$ and $\chi(\mathcal{M}_2)$. König's Theorem 1 on the edge-chromatic number of bipartite multigraphs implies that $\chi(\mathcal{M}_1, \mathcal{M}_2) = \max\{\chi(\mathcal{M}_1), \chi(\mathcal{M}_2)\}$ if $\mathcal{M}_1$,

$\mathcal{M}_2$ are two partition matroids. This equality may not hold if one of $\mathcal{M}_1, \mathcal{M}_2$ is not a partition matroid. For example (see [78]), let $\mathcal{M}_1$ be the graphic matroid of the complete graph K_4, and let $\mathcal{M}_2$ be the partition matroid on the same ground set whose parts are the three pairs of non-adjacent edges. Then $\max\{\chi(\mathcal{M}_1), \chi(\mathcal{M}_2)\} = 2$ while $\chi(\mathcal{M}_1, \mathcal{M}_2) = 3$. Nevertheless, using nice topological arguments, Aharoni and Berger [2] proved that $\chi(\mathcal{M}_1, \mathcal{M}_2)$ and $\max\{\chi(\mathcal{M}_1), \chi(\mathcal{M}_2)\}$ are not too far apart.

Theorem 58 (Aharoni, Berger)**.** *For any two matroids $\mathcal{M}_1, \mathcal{M}_2$ on the same ground set, we have that $\chi(\mathcal{M}_1, \mathcal{M}_2) \leq 2\max\{\chi(\mathcal{M}_1), \chi(\mathcal{M}_2)\}$.*

As a way to generalize their theorem, Aharoni and Berger conjectured the following stronger statement. They proved it in the case that one of $\chi(\mathcal{M}_1)$ or $\chi(\mathcal{M}_2)$ is an integer multiple of the other.

Conjecture 59. *For any two matroids $\mathcal{M}_1, \mathcal{M}_2$ on the same ground set, we have that $\chi(\mathcal{M}_1, \mathcal{M}_2) \leq \chi(\mathcal{M}_1) + \chi(\mathcal{M}_2)$.*

In addition to this, a famous conjecture of Rota [2, 93] can be stated in terms of the joint-chromatic number as follows.

Conjecture 60 (Rota's basis conjecture)**.** *Let $\mathcal{M}_1$ be a matroid of rank k, and let $\mathcal{M}_2$ be a partition matroid on the same ground set with k parts, each of which is a basis of $\mathcal{M}_1$. Then $\chi(\mathcal{M}_1, \mathcal{M}_2) = k$.*

Aharoni and Berger's Theorem 58 implies that under the conditions of Rota's basis conjecture, $\chi(\mathcal{M}_1, \mathcal{M}_2) \leq 2k$. Conjecture 60 has been verfied for all paving matroids [39], for all matroids of rank at most three [21], and recently, asymptotically for all matroids [76]. Work on the joint-chromatic number of matroids is also relevant for Ryser's hypergraph conjecture (see [1, 2]).

Chapter 7

Star arboricity and list star arboricity

We study one more version of arboricity known as star arboricity. A **star forest** is a forest in which each component is a star $K_{1,\ell}$ for some $\ell \geq 1$. For a multigraph G, its **star arboricity** $sa(G)$ is the minimum number of colors needed for an edge-coloring into monochromatic star forests. Since every star forest is a forest, we immediately have that $sa(G) \geq a(G)$. Observe that every forest has star arboricity at most two: on each component, do a breadth-first search starting at any vertex, and alternate the colors of the edges going up from one level to the next. Because we can partition the edges of G into $a(G)$ monochromatic forests, each with star arboricity at most two, we get that

$$a(G) \leq sa(G) \leq 2a(G).$$

Star arboricity was introduced by Akiyama and Kano in [6], where they showed that the star arboricity of the complete graph K_n is $\lceil n/2 \rceil + 1$. This was generalized by Aoki in [16] to determine the star arboricity of complete multipartite graphs with partition classes of equal size. These graphs are r-regular for some $r \geq 1$, and he showed that they have star arboricity either $\lceil r/2 \rceil + 1$ or $\lceil r/2 \rceil + 2$. The star arboricity $sa(G)$ for these graphs G are close to their ordinary arboricity lower bound $a(G) = \lceil (r+1)/2 \rceil$. On the other hand, Alon, McDiarmid, and Reed [10] showed that there are infinitely many graphs whose star arboricity is exactly the upper bound $2a(G)$.

Before we establish this result of Alon, McDiarmid, and Reed, let us associate the star arboricity of a multigraph with certain orientations. Given a star forest edge-coloring ϕ of a multigraph G, we may orient each edge e of G away from the center of the star component in which it lies. We obtain an orientation D of G such that every vertex has indegree at most k, and every vertex that is the center of some star component has indegree at most $k-1$. Let us call D the **star forest orientation** of G with respect to ϕ. (Note that it may not be possible to recover a star forest edge-coloring from an orientation of this sort.)

Theorem 61 (Alon, McDiarmid, Reed)**.** *For every integer $k \geq 1$, there exists a simple graph G with arboricity $a(G) = k$ and star arboricity $sa(G) = 2k$.*

Proof. Fix $k \geq 1$. Let G be the bipartite simple graph defined as follows (see Figure 7.1):

(i) $V(G) = A \cup B \cup C$ (disjoint union), where $|A| = k$, $|B| = (k-1)\binom{2k-1}{k} + 2k^2 - k + 1$, $|C| = (2k^2 - k + 1)\binom{|B|}{k}$, and each of A, B, C is an independent set,

(ii) Every vertex in A is adjacent to every vertex in B and no vertex in C.

(iii) Partition C into $\binom{|B|}{k}$ subsets of size $2k^2 - k + 1$. There is a bijection between these subsets of C and the subsets of B of size k. A vertex in C is adjacent precisely to those vertices of B in the corresponding k-subset.

The number of edges of G is $e(G) = |A| \cdot |B| + k \cdot |C| = k(|B| + |C|) = k(v(G) - k)$. First we show that $a(G) = k$. Observe that

$$a(G) \geq \left\lceil \frac{e(G)}{v(G) - 1} \right\rceil = \left\lceil \frac{k(v(G) - k)}{v(G) - 1} \right\rceil = k,$$

where the last equality comes from the fact that $v(G) > k^2$. To show that $a(G) \leq k$, we orient the edges of G so that all edges between A and B are oriented toward B, and that all edges between B and C are oriented toward C. In this orientation, the maximum indegree of a vertex is k. Moreover, this orientation is acyclic, so every subgraph contains a vertex of indegree 0. Hence, we have $e(S) \leq k(|S| - 1)$ for all $S \subseteq V(G)$. It follows from Nash-Williams' Theorem 23 that $a(G) \leq k$.

Now we show that $sa(G) \geq 2k$. Suppose for contradiction that ϕ is an edge-coloring of G into $2k-1$ monochromatic star forests $F_1, \ldots, F_{2k-1}$. Let D be the star forest orientation of G with respect to ϕ. Each vertex of D has indegree at most $2k - 1$, so the sum of indegrees of A is at most $2k^2 - k$. This implies that all but at most $2k^2 - k$ vertices of B have indegree k from A. Let B' be the set of vertices in B with indegree at least k. For each $x \in B'$, we can find a k-subset $S(x)$ of $\{F_1, \ldots, F_{2k-1}\}$ such that x is not the center of any star component of $S(x)$. Since $|B'| \geq |B| - (2k^2 - k) > \binom{2k-1}{k}(k-1)$, we can find a k-subset $X = \{x_1, \ldots, x_k\}$ of B' such that $S(x_1) = \ldots = S(x_k)$. Put $S = S(x_1)$. Let C' be the set of $2k^2 - k + 1$ vertices of C corresponding to X. As before, since each vertex of X has indegree at most $2k - 1$, at most $2k^2 - k$ arcs are oriented from C' to X. Because $|C'| = 2k^2 - k + 1$, there is a vertex $y \in C'$ such that (x_i, y) is an arc of D for each $x_i \in X$. For each $i \in \{1, \ldots, k\}$, let H_i be the star forest with respect to ϕ that contains the edge $e_i = x_i y$. By the definition of the orientation D, all the H_i are distinct and do not lie in S. Hence, there are at least $|S| + |\{H_1, \ldots, H_k\}| = 2k$ star forests with respect to ϕ, a contradiction. Therefore, $sa(G) \geq 2k$ as required. □

Of course, the gap between $a(G)$ and $2a(G)$ is quite large, similar to the gap between $\Delta(G)$ and $2\Delta(G) - 1$ when it comes to the chromatic index $\chi'(G)$. Unsurprisingly at this point, it is NP-hard to determine star arboricity in general, and it is even NP-complete to recognize whether a simple graph has star arboricity two [53]. Still, it would be nice to find a

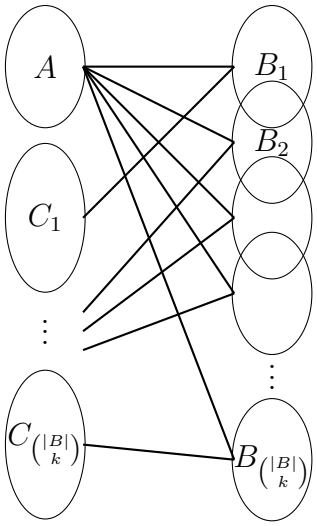

Figure 7.1: The bipartite graph G of Alon, McDiarmid, and Reed. The B_i's are k-subsets of B. The C_i's form the partition of C. Solid lines represent complete bipartite graphs between the connected parts.

graph parameter that more closely determines the star arboricity $sa(G)$ than the arboricity $a(G)$ by itself. One approach is to involve the maximum degree $\Delta(G)$. When it comes to the maximum degree $\Delta = \Delta(G)$ by itself, Algor and Alon [7] proved that for every Δ-regular simple graph G,

$$\Delta/2 < sa(G) \leq \Delta/2 + O(\Delta^{2/3}(\log \Delta)^{1/3}),$$

(the lower bound being simply edge counting), so the star arboricity is asymptotically $\Delta/2$ as $\Delta \to \infty$. At the same time, the authors showed that there are Δ-regular simple graphs G for which $sa(G) \geq \Delta/2 + \Omega(\log \Delta) = a(G) + \Omega(\log \Delta)$, so the error term in the upper bound cannot be eliminated. On the other hand, Alon, McDiarmid, and Reed [10] later proved that $sa(G) \leq a(G) + O(\log \Delta)$ as $\Delta \to \infty$ for every simple graph G with maximum degree Δ, which is a matching upper bound. Harris, Su, and Vu [55] recently improved this upper bound to

$$sa(G) \leq a(G) + O(\log a(G) + \sqrt{\log \Delta(G)}),$$

and this is best possible as a function of $a(G)$ and $\Delta(G)$ separately. These asymptotic bounds were proven using probabilistic methods. The relationship between star arboricity and maximum degree/maximum density even for simple graphs appears somewhat complicated based on these known results, but little has been studied here.

For planar simple graphs, it is easy to show via Nash-Williams' Theorem 23 that every planar simple graph has arboricity at most 3, and therefore has star arboricity at most 6. Algor and Alon [7] constructed a planar simple graph with star arboricity 5, and they asked whether 5 or 6 is the true upper bound. Hakimi, Mitchem, and Schmeichel [53] later proved

that the answer is 5, relying on a computationally difficult result of Borodin [19] on acyclic vertex colorings. Star arboricity has also been connected to incidence colorings (see [44]).

One other parameter to consider in relation to star arboricity is degeneracy, which is related to both maximum degree and arboricity. Recall that a multigraph G is k-degenerate if every subgraph of G contains a vertex of degree at most k, and that we can define a degeneracy ordering $v_1, v_2, \ldots, v_n$ on $V(G)$ by recursively setting v_i to be a vertex of minimum degree in $G - \{v_{i+1}, \ldots, v_n\}$, for $i \in \{n, n-1, \ldots, 1\}$. This degeneracy ordering lets us show that $e(S) \leq k(|S| - 1)$ for all $S \subseteq V(G)$. By Nash-Williams' Theorem 23, we have that $a(G) \leq k$ and thus $sa(G) \leq 2k$ for every k-degenerate multigraph G.

Proposition 62. *For every k-degenerate multigraph G, we have $sa(G) \leq 2k$.*

The bipartite simple graph constructed in Theorem 61 is in fact k-degenerate, with a degeneracy ordering given by the vertices in A, then in B, and then in C. Thus, Theorem 61 shows that it is possible for a k-degenerate multigraph to have star arboricity $2k$, for every integer $k \geq 1$. On the other hand, by making assumptions on the degeneracy structure of G, one can sometimes prove a better upper bound than $2k$. This idea was used in [53] to prove that every outerplanar simple graph has star arboricity at most 3.

As usual, we can define a list version of star arboricity. We define the **list star arboricity** $sa_\ell(G)$ of G to be the minimum integer k such that for every k-list assignment L for $E(G)$ there is a star forest L-coloring of G. It is immediate that $sa_\ell(G) \geq sa(G)$. We may conjecture that the analogue of the List Coloring Conjecture 19 holds for list star arboricity.

Conjecture 63. *For every multigraph G, we have $sa_\ell(G) = sa(G)$.*

For an easy upper bound on $sa_\ell(G)$, we clearly have that $sa_\ell(G) \leq \chi'_\ell(G) \leq 2\Delta(G) - 1$ because every matching is a star forest. To improve this, let us prove the following generalization of Proposition 62 (see [55]).

Proposition 64. *For every k-degenerate multigraph G, we have $sa_\ell(G) \leq 2k$.*

Proof. Let G be a k-degenerate multigraph, and let L be any list assignment for $E(G)$ with $2k$ colors in each list. Let $v_1, v_2, \ldots, v_n$ be a degeneracy ordering of the vertices of G. We start by coloring all the edges incident to v_1 arbitrarily from their respective lists, making sure that parallel edges get different colors (note that $\mu(G) \leq k$). Now iterating through $i \in \{2, \ldots, n\}$, we consider all the uncolored edges incident to v_i. For each such edge $e \in E_G(v_i, v_j)$ where $j > i$, we delete from its list $L(e)$ every color that was given to an adjacent edge. By the degeneracy ordering, e has at most k adjacent colored edges at its end-vertex v_i, and at most $k - \mu_G(v_i, v_j)$ adjacent colored edges at its other end-vertex v_j, since all edges between v_i and v_j are uncolored. Thus, we delete at most $2k - \mu_G(v_i, v_j)$ colors from the list $L(e)$, leaving us with a list of at least $\mu_G(v_i, v_j)$ colors at $e \in E_G(v_i, v_j)$. Hence, we can L-color all the uncolored edges incident to v_i while ensuring that all parallel edges get different colors. Continuing for all i, it is easy to check that this list coloring procedure leads to star forest L-coloring of G. □

As a corollary, we get that the k-degenerate bipartite simple graph G from Theorem 61 has list star arboricity $sa_\ell(G) = sa(G) = 2k$, in support of Conjecture 63. Like ordinary star arboricity, it is possible to improve this upper bound $2k$ if we make assumptions on the degeneracy structure of G.

Finally, let us write an upper bound for $sa_\ell(G)$ in terms of the arboricity $a(G)$. Assume that G has at least one edge so that $a(G) \geq 1$. We have $2a(G) \geq 2e(H)/(v(H)-1) > d(H)$ for every subgraph H of G with $e(H) \geq 1$, where $d(H)$ is the average degree of H. This implies that every subgraph H of G contains a vertex of degree at most $2a(G)-1$, which is to say, G is $(2a(G)-1)$-degenerate. By Proposition 64 we have

$$a(G) \leq sa_\ell(G) \leq 4a(G) - 2$$

for every multigraph G. It would be worthwhile to improve the constant factors in this as well as in the degeneracy upper bound for some classes of multigraphs, even asymptotically. In this direction, Harris, Su, and Vu [55] proved that

$$sa_\ell(G) \leq a(G) + O(\log \Delta(G)),$$

for every simple graph G as $\Delta(G) \to \infty$, as was shown for ordinary star arboricity. There is more work to be done on multigraphs.

Chapter 8

Conclusion

In this essay we looked into how techniques similar to those developed to approach the Goldberg-Seymour Conjecture 9,

$$\chi'(G) \leq \max\left\{\Delta(G)+1, \max_{\substack{S\subseteq V(G),\\ |S|\geq 3 \text{ odd}}} \left\lceil \frac{2e(S)}{|S|-1} \right\rceil \right\},$$

on the chromatic index (e.g., alternating paths, critical subgraphs) can be used to study other edge-coloring problems such as arboricity $a(G)$ and pseudoarboricity $pa(G)$. We saw that we can construct a maximal arboretum and a maximal pseudoarboretum that form the "dense spots" asserted by the maximum density formulas of Nash-Williams' Theorem 23 and Hakimi's Theorem 33. This is similar in spirit to the role of Tashkinov trees in the study of the Goldberg-Seymour Conjecture 9. Due to the similar dependence on maximum degree and maximum density for many of these edge-coloring problems, we sought to "interpolate" among these problems by studying bounded degree versions. These problems turned out to be interesting in their own right and surprisingly related to the Goldberg-Seymour Conjecture.

We saw that we could prove exact formulas for degree t subgraph colorings when t is even (Theorem 39),

$$\chi'_t(G) = \left\lceil \frac{\Delta(G)}{t} \right\rceil,$$

as well as for degree t pseudoarboricity for all integers $t \geq 2$ (Theorem 53),

$$pa_t(G) = \max\left\{ \left\lceil \frac{\Delta(G)}{t} \right\rceil, pa(G) \right\}.$$

On the other hand, for degree t subgraph colorings when t is odd we could only conjecture a generalization of the Goldberg-Seymour Conjecture (Conjecture 40),

$$\chi'_t(G) \leq \max\left\{ \left\lceil \frac{\Delta(G)+1}{t} \right\rceil, \max_{S\subseteq V(G), |S|\geq 2} \left\lceil \frac{e(S)}{\lfloor t|S|/2 \rfloor} \right\rceil \right\}.$$

We also conjectured that a Goldberg-Seymour bound occurs for degree t arboricity $a_t(G)$ for all $t \geq 2$. We paid particular attention to the case $t = 2$ of linear arboricity $a_2(G) = la(G)$, where we conjectured that the classical Linear Arboricity Conjecture 43 can be extended to multigraphs in a strong way (Conjecture 45),

$$a_2(G) = la(G) \leq \max\left\{\left\lceil \frac{\Delta(G)+1}{2} \right\rceil, a(G)\right\}.$$

This conjecture would bring us quite close to the ordinary Goldberg-Seymour Conjecture 9, noting that $2 \cdot pa_2(G) \leq \chi'(G) \leq 2 \cdot a_2(G)$ because merging two color classes in a proper edge-coloring results in an edge-coloring into paths and even cycles. In future work, we would like to explore further connections among the chromatic index, arboricity, and pseudoarboricity as suggested by Conjecture 45.

The omnipresence of Goldberg-Seymour bounds and formulas hints that these edge-coloring problems could be related in deeper ways than we have been able to uncover in this exposition. One consistent pattern with the exact results we were able to prove (chromatic index for bipartite multigraphs, degree t chromatic index for even t, and degree t pseudoarboricity for general t) is that in these cases the associated fractional maximum density parameter meets an associated maximum degree parameter when the multigraph G is $\Delta(G)$-regular (basically, when the multigraph is made as dense as possible without increasing its maximum degree). This is not the case with the unknown cases (degree t chromatic index for odd t, degree t arboricity for general t), and this has to do with $|S| - 1$ being in the denominator of the maximum density parameter rather than $|S|$. This makes it all the more remarkable that such a result as Nash-Williams' Theorem 23 holds, but it also suggests that progress can still be made. Somehow, edge-coloring problems whose associated maximum density parameter satisfies this sort of minimax property have a nice orientation structure that enable a Goldberg-Seymour formula to be found. These orientation properties also enabled us to prove that the natural analogues of the classical List Coloring Conjecture 19 hold for these edge-coloring problems, via Galvin's Theorem 20. In future work, we would like to further explore these kinds of density patterns in edge-coloring.

There are many other related edge-coloring problems out there worth exploring, such as joint coloring matroids and star arboricity as we've briefly written about. There remain a lot of unknowns in edge-coloring, but with the connections drawn here and continued work it is hopeful that many new interesting results could be discovered.

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