Discrete Diffusion on Graphs and Real Hyperplane Arrangements

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Overview

1 Discrete diffusion on graphs.
- The model.
- The "standard" chip-firing model.
- Physical analogies.

Overview

1. Discrete diffusion on graphs.
   - The model.
   - The "standard" chip-firing model.
   - Physical analogies.
2. Real hyperplane arrangements.
   - Equilibrium steady state.
   - Non-equilibrium steady state.
   - Questions.

Thanks to

Danielle Cox (MSVU)
Martin van Bommel (StFXU)
Stephen Finbow (StFXU)

for the

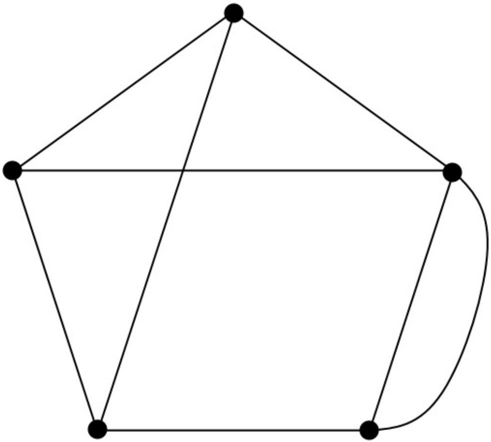
East Coast Combinatorics Conference
(Antigonish, 2019)

where Danielle's cohort were talking about this.

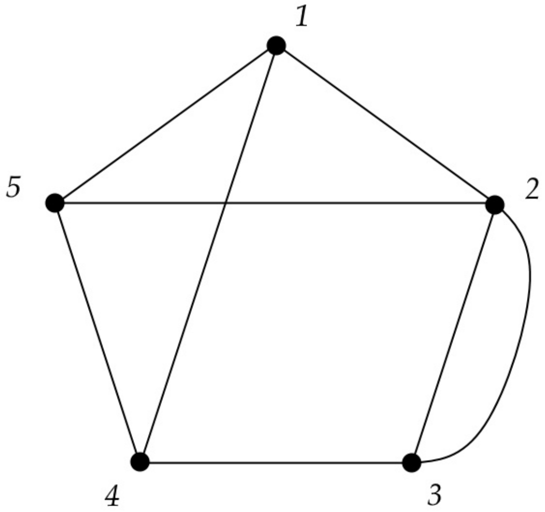
the model

Duffy, Lidbetter, Messinger, Nowakowski (2016)

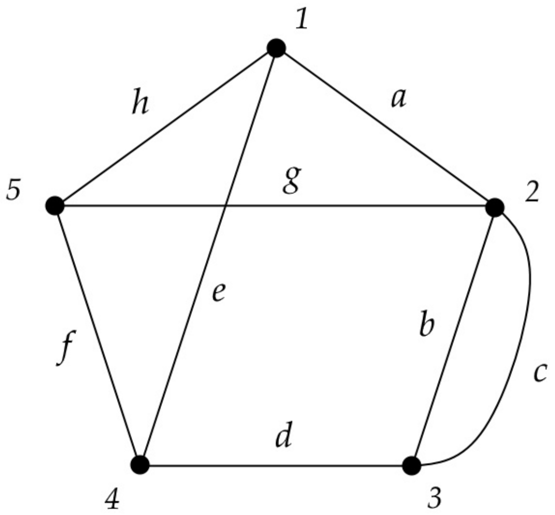
a multigraph $G = (V, E)$

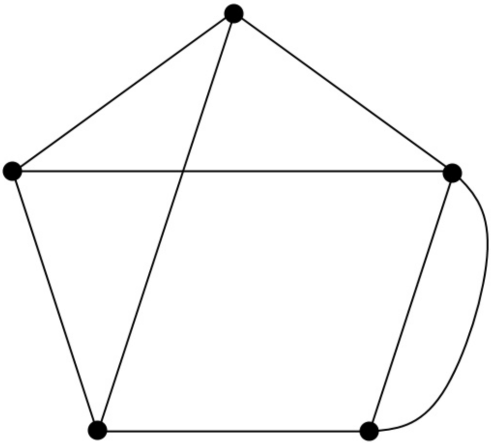


finite, undirected

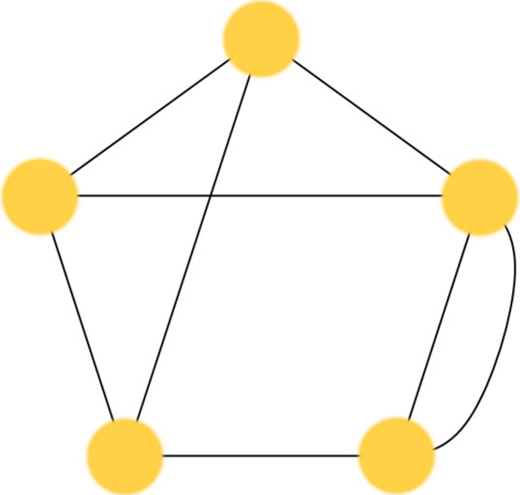


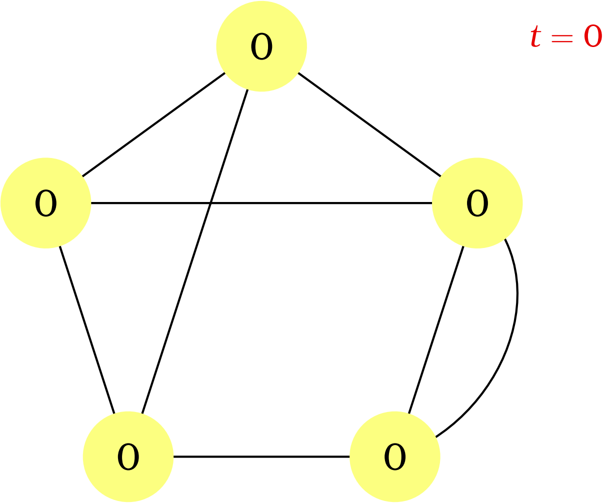
connected, loopless

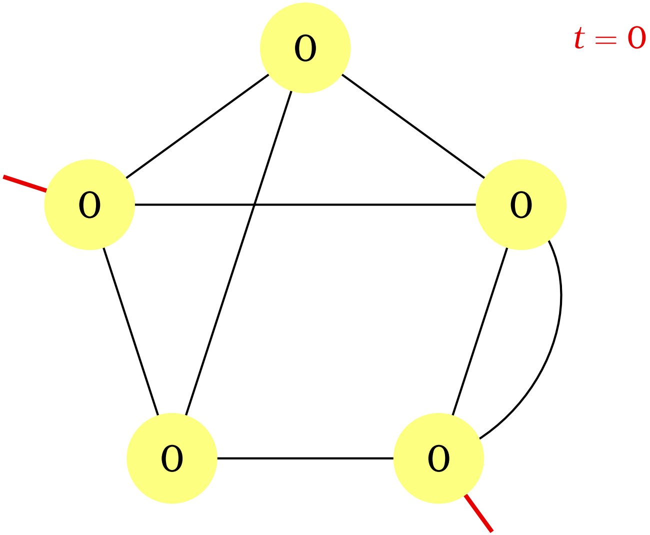


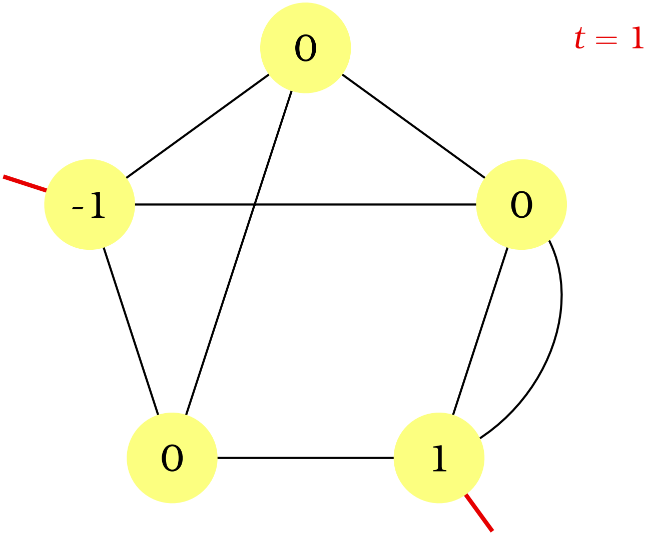


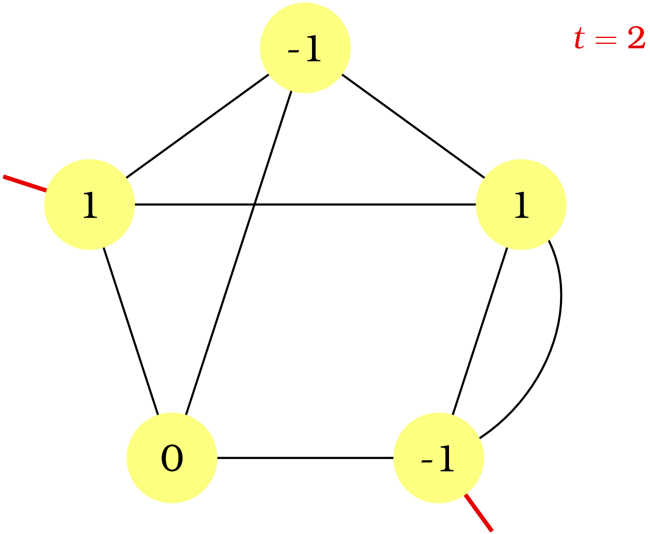
a real $c(v) \in \mathbb{R}$ at each vertex $v \in V$




0
0
0
0
0
$t = 0$


0
0
0
0
0
t = 0


$t = 1$
0
-1
0
0
1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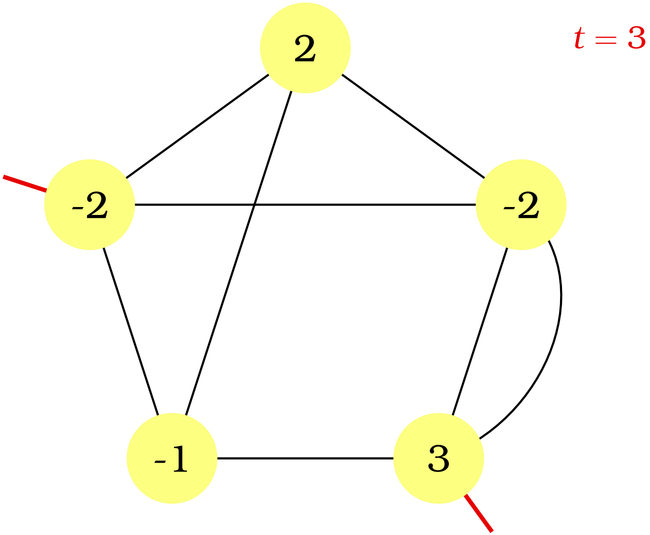
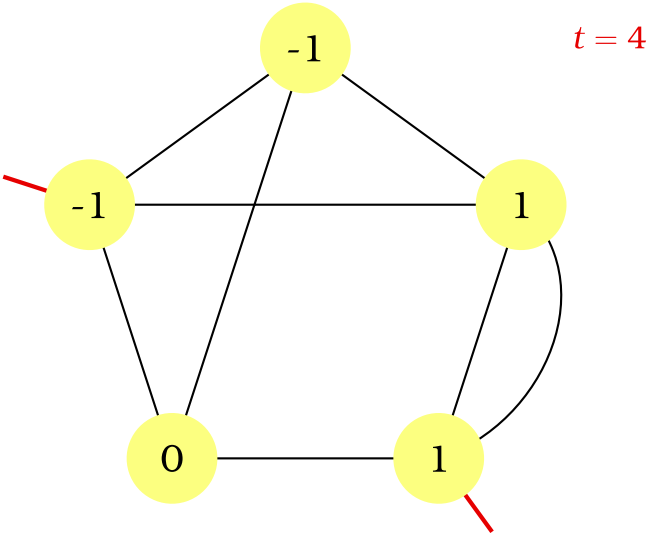
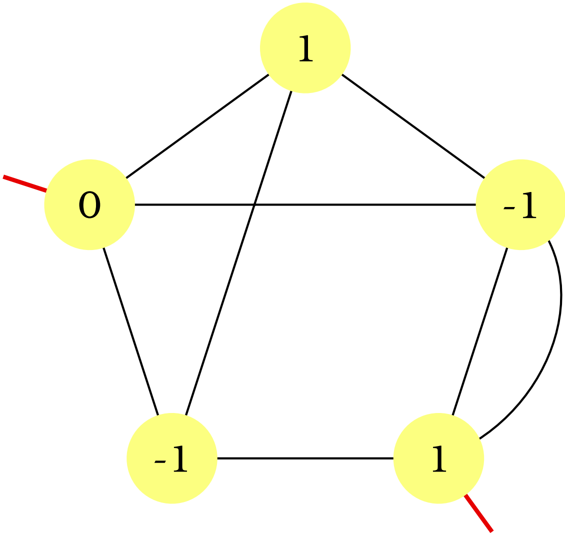
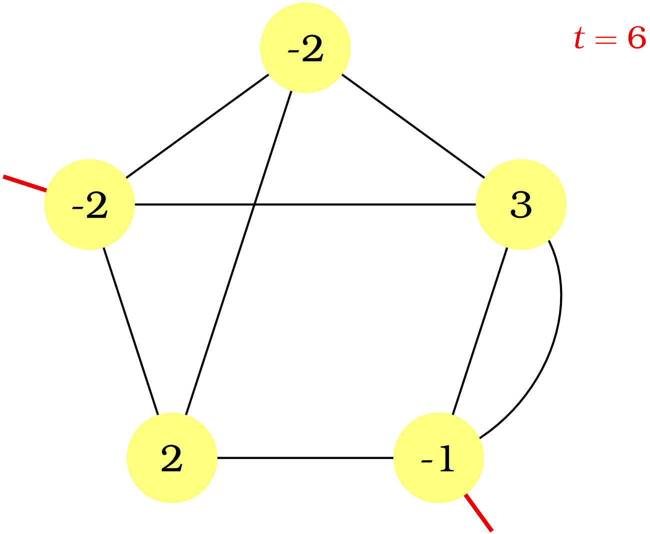
-1
1
1
0
-1


2
-2
-2
-1
3
$t = 3$

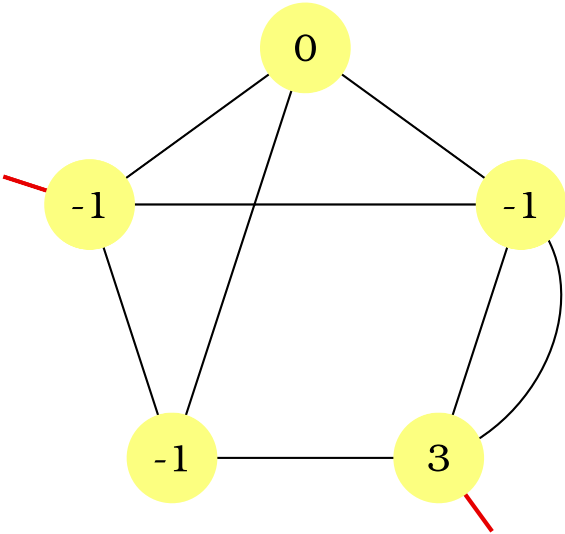

-1
-1
1
0
1
$t = 4$

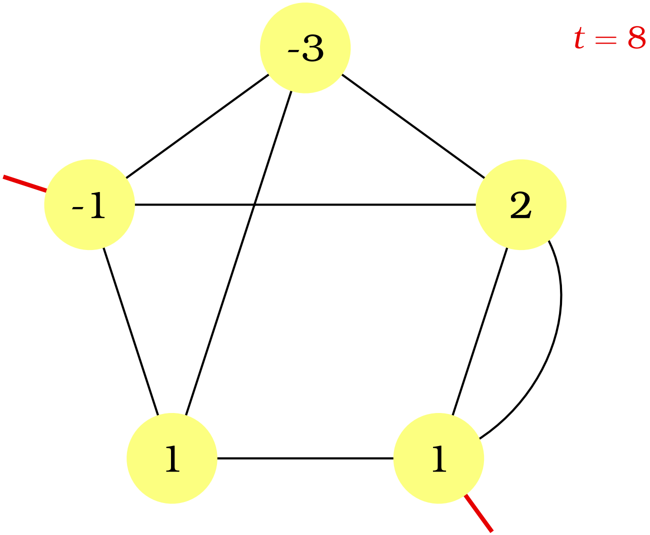


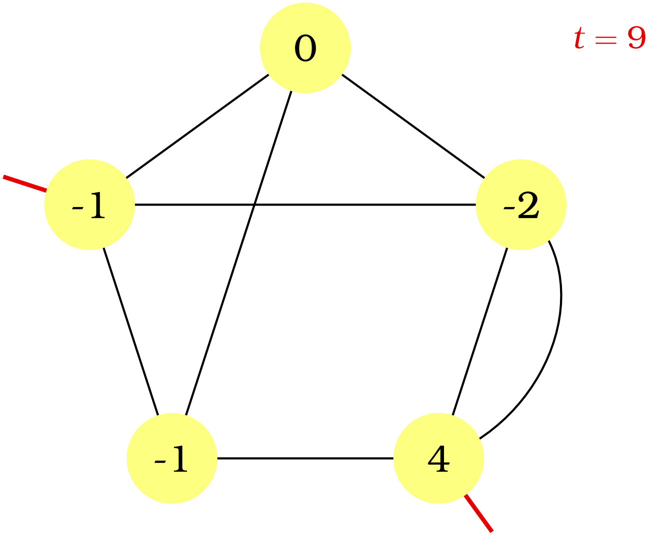
$t = 5$


-2
-2
3
2
-1
$t = 6$

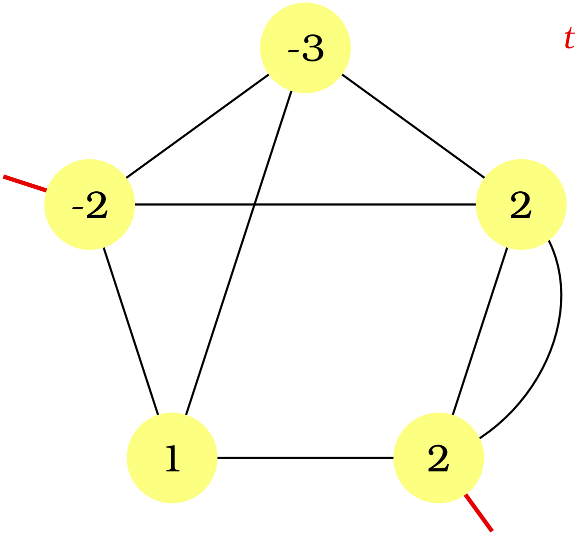
$t = 7$

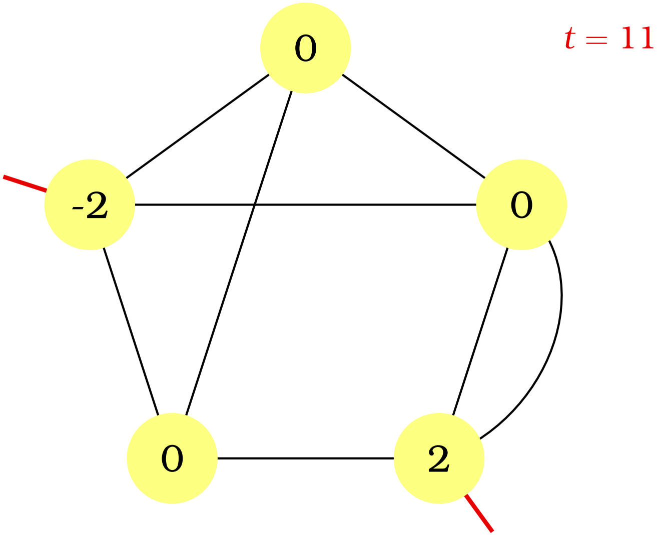



-3
-1
2
1
1
$t = 8$

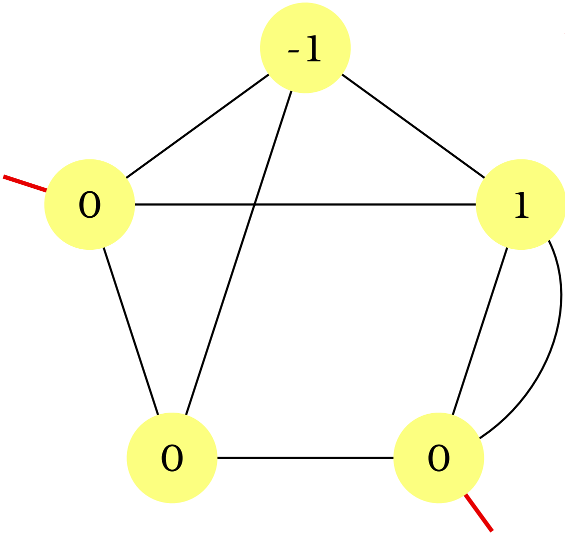

0
-1
-2
-1
4
$t = 9$

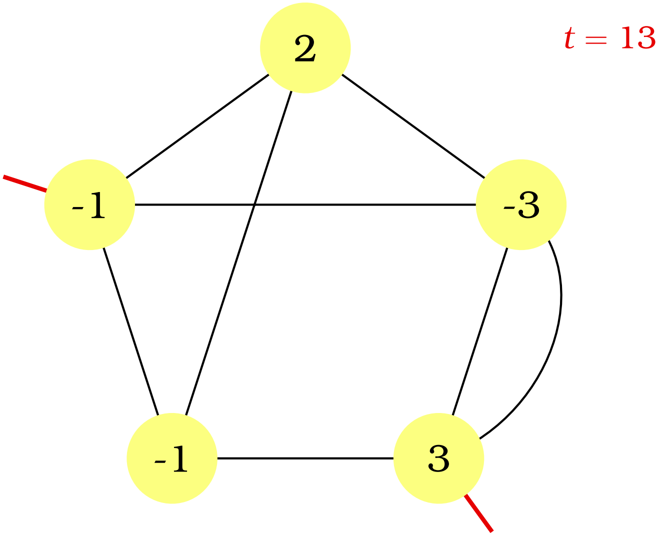
$t = 10$

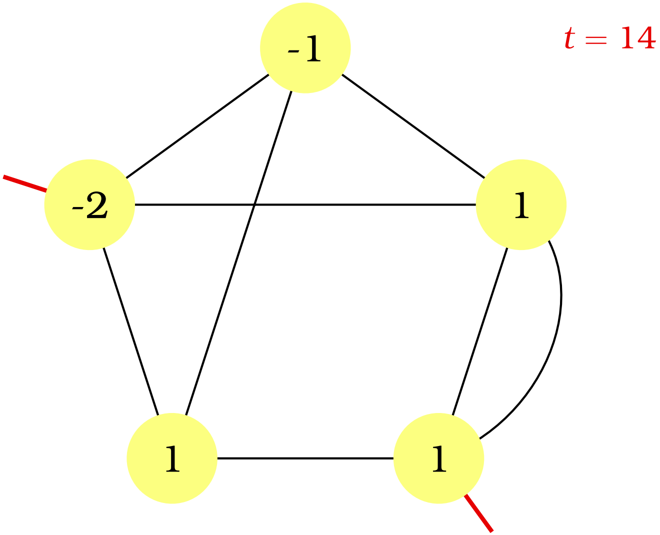


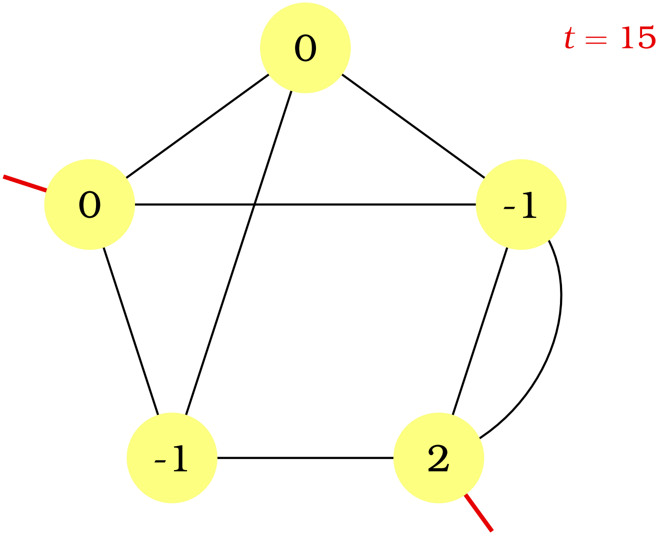

$t = 11$
0
-2
0
0
2

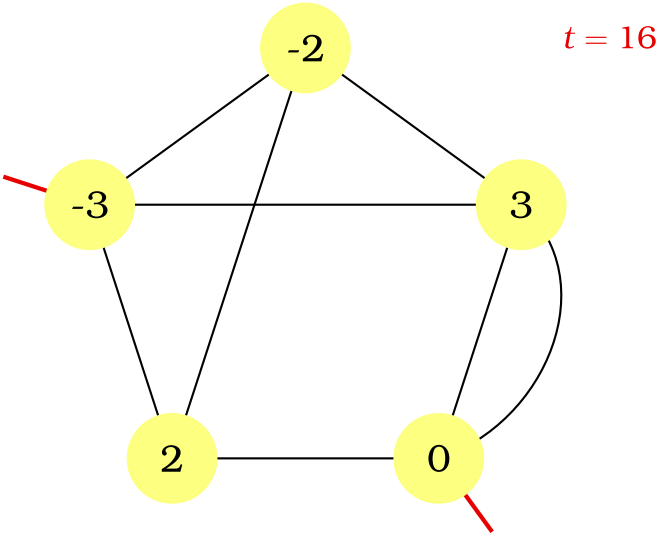
$t = 12$

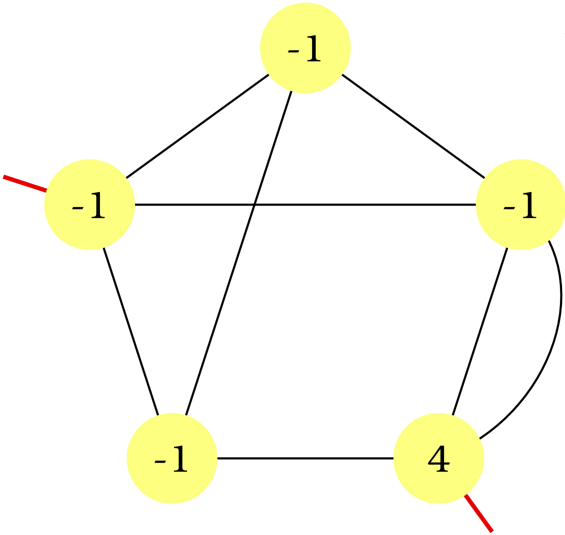



2
-1
-3
-1
3
$t = 13$

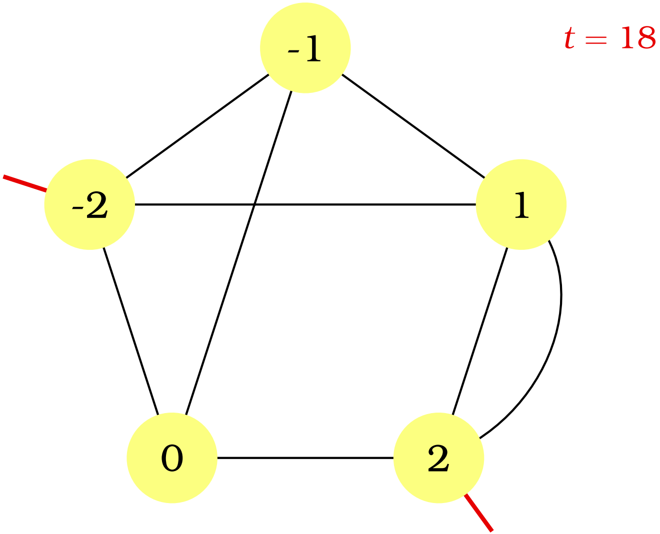

-1
-2
1
1
1
$t = 14$


0
0
-1
-1
2
$t = 15$

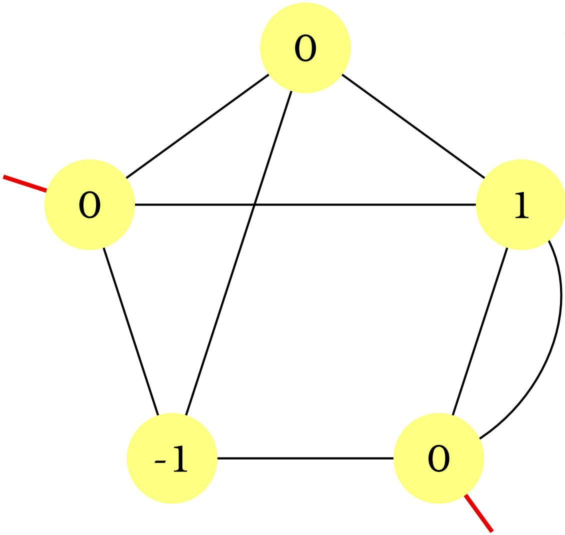

-2
-3
3
2
0
$t = 16$

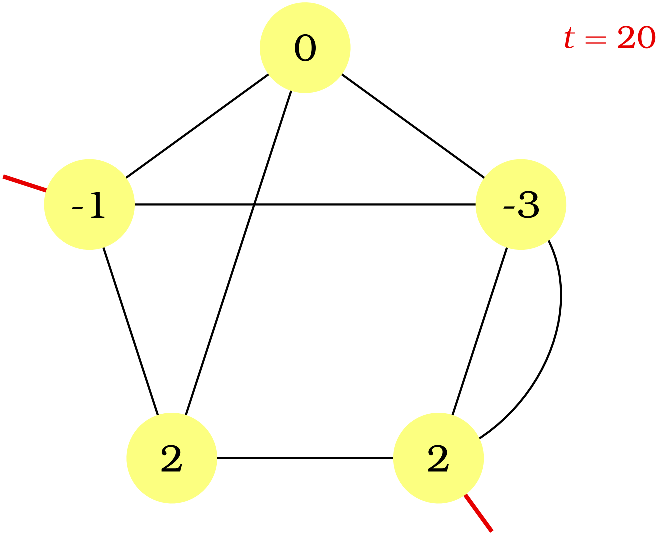


$t = 17$

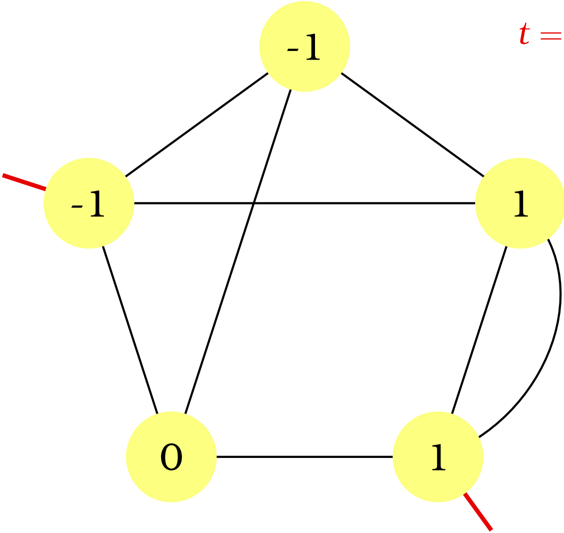

-1
-2
1
0
2
$t = 18$

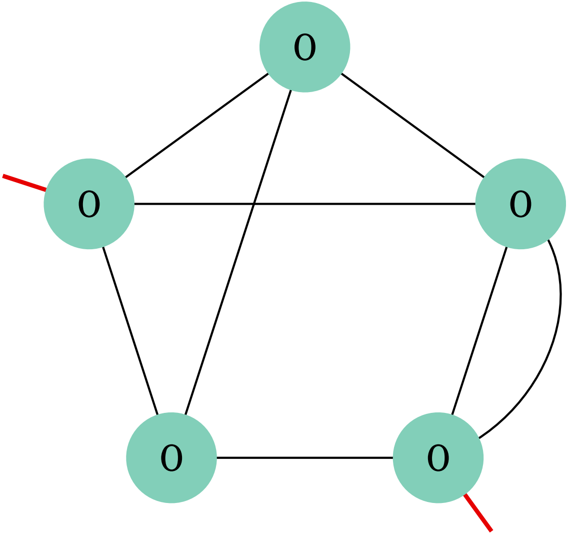
$t = 19$



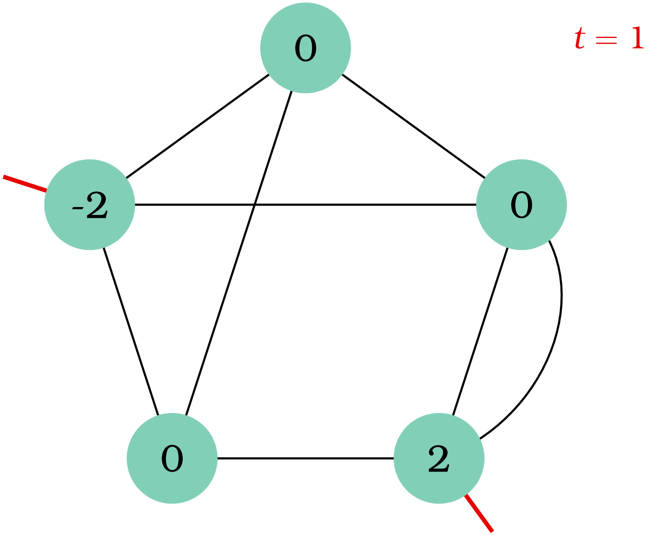

$t = 20$
0
-1
-3
2
2

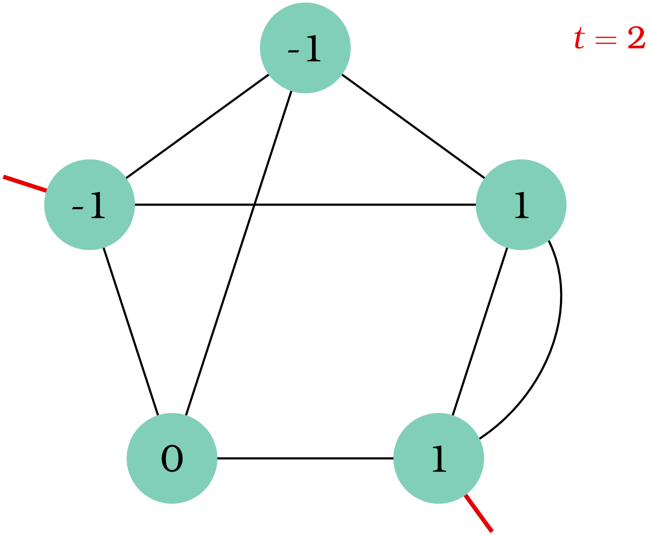
$t = 21, t = 4$

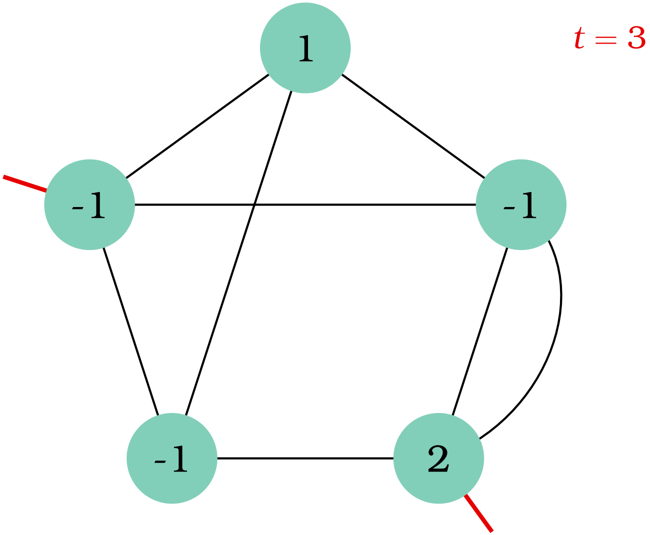


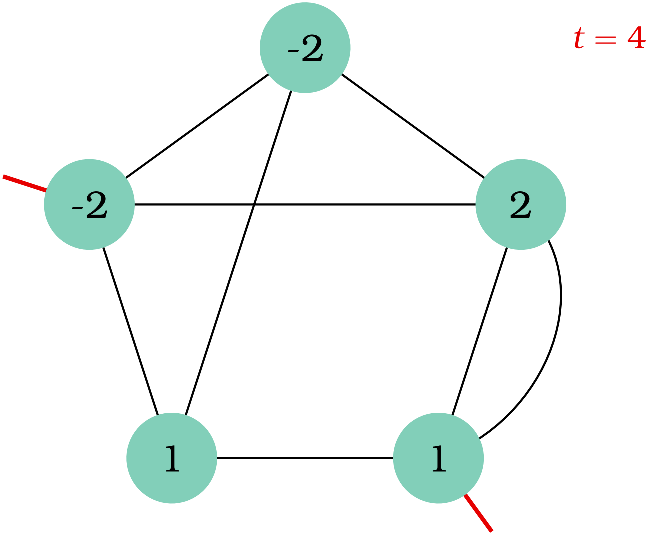


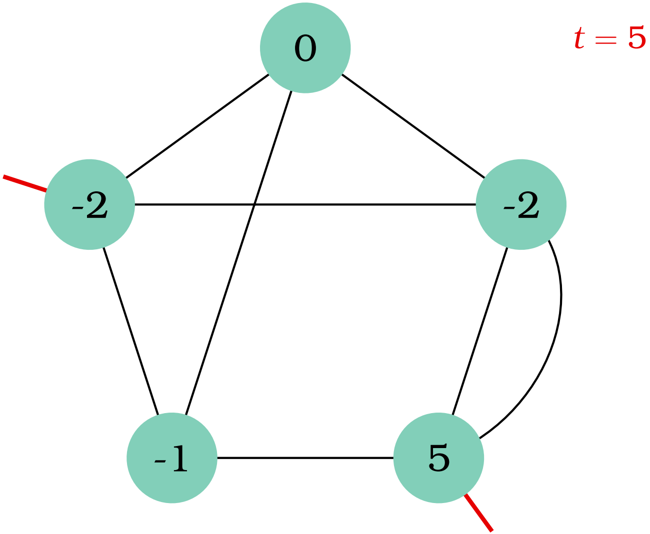
$t = 0$

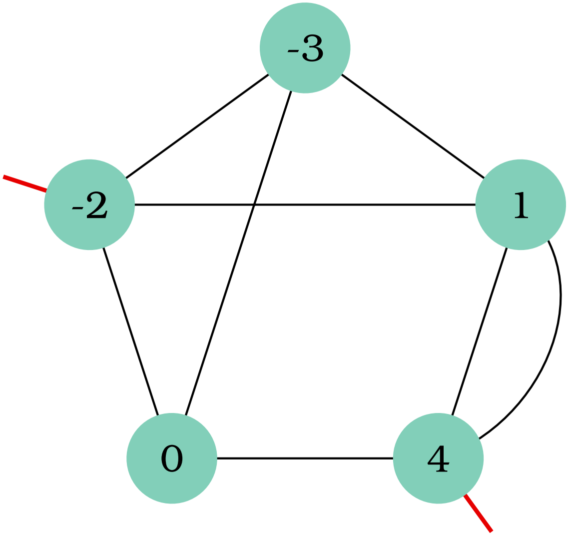

0
-2
0
0
2
$t = 1$


$t = 2$
-1
-1
1
0
1

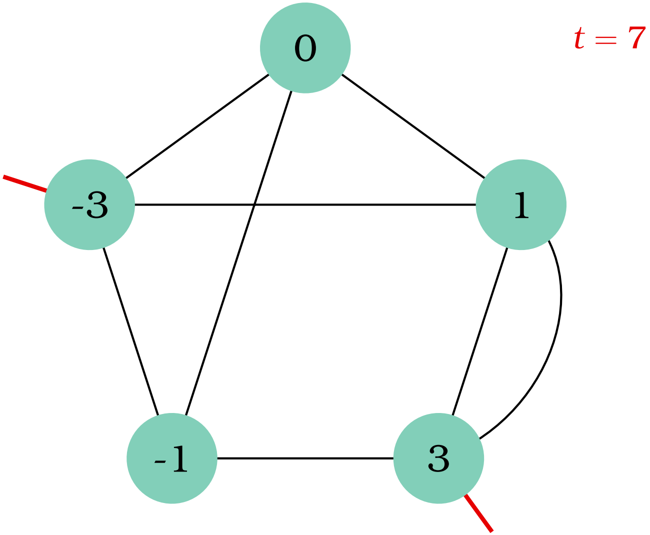

1
-1
-1
-1
2
$t = 3$

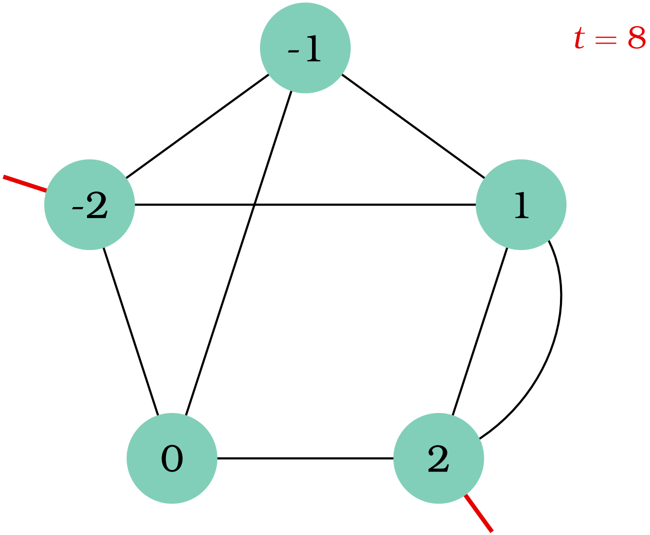

-2
-2
2
1
1
$t = 4$


0
-2
-2
-1
5
$t = 5$

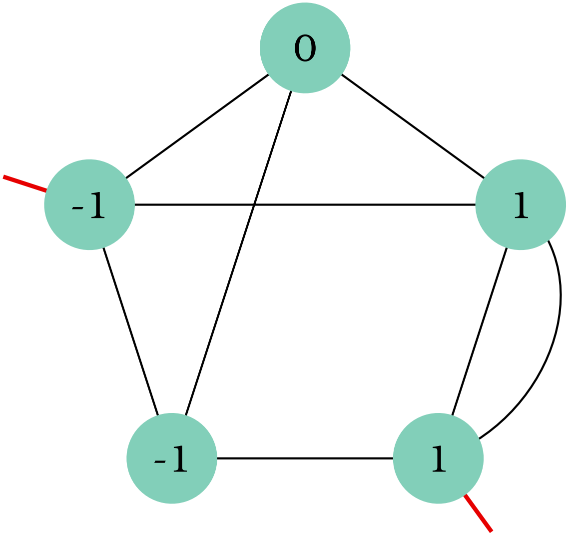


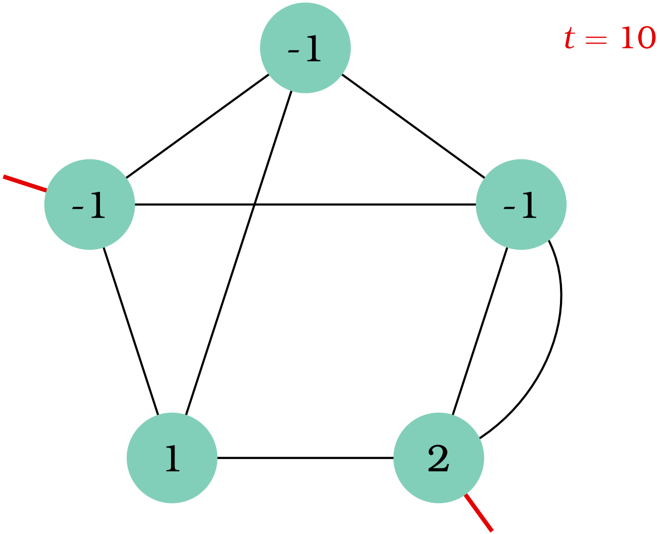
$t = 6$

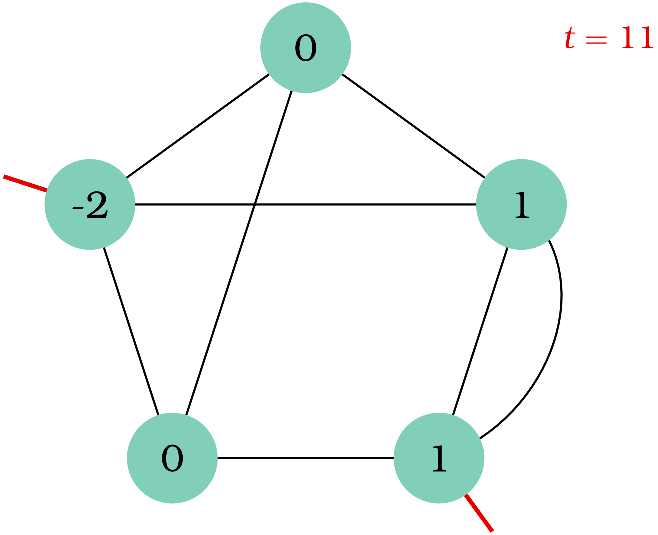

0
-3
1
-1
3
t = 7


-1
-2
1
0
2
$t = 8$

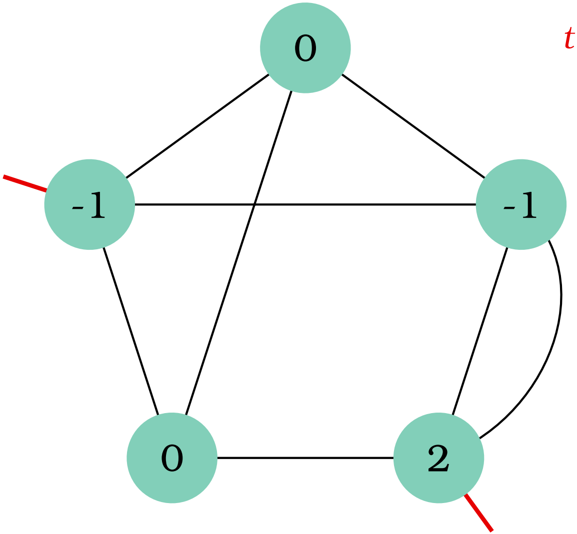
$t = 9$




$t = 10$
-1
-1
-1
1
2


0
-2
1
0
1
$t = 11$

$t = 12$



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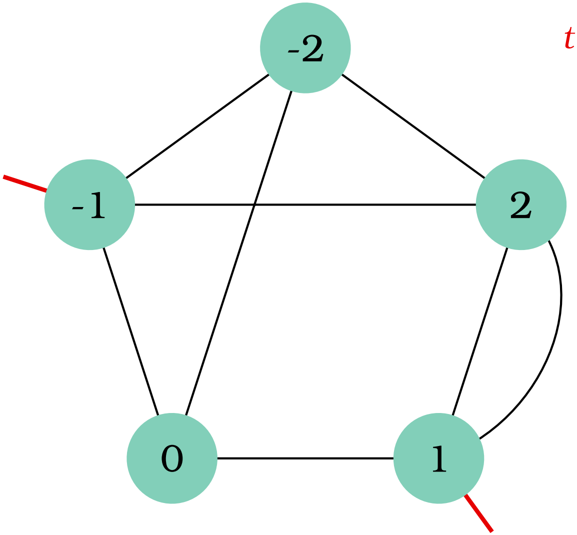
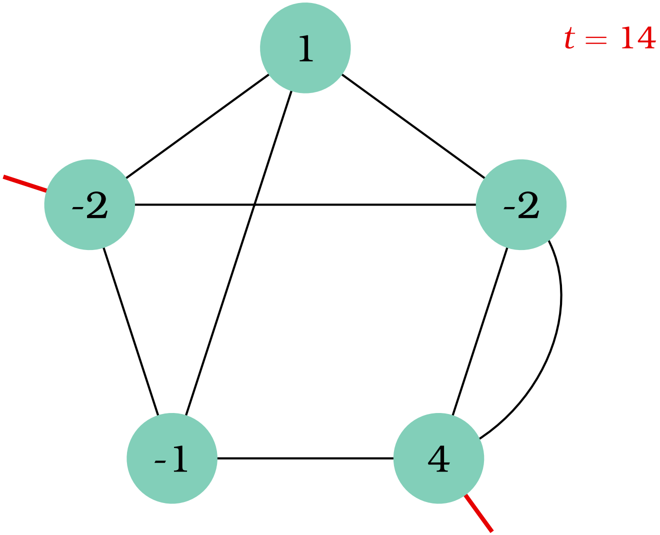
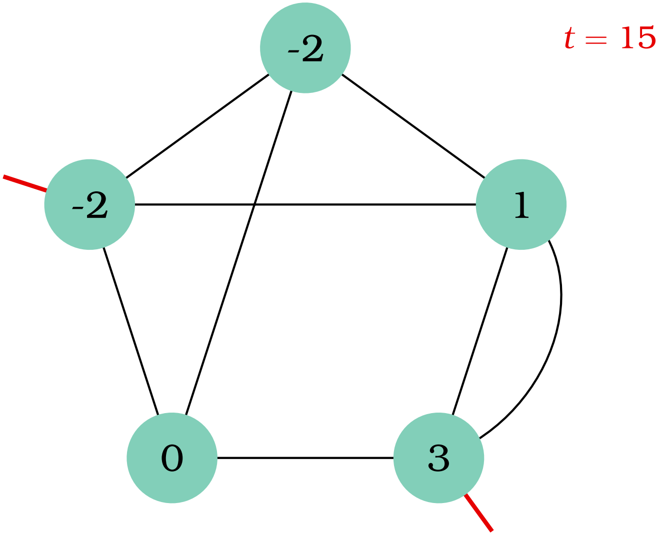
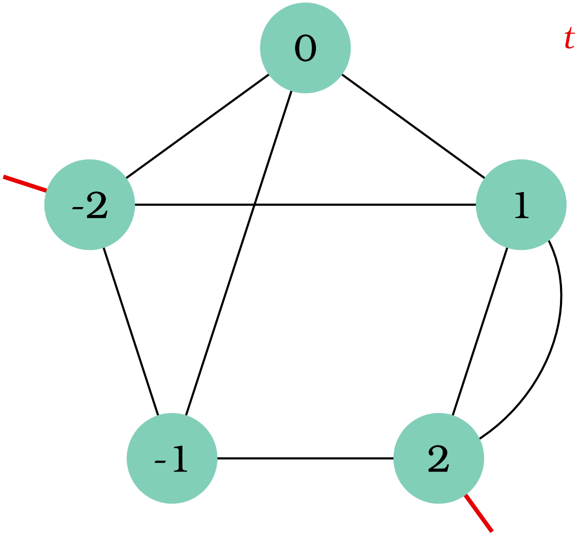



$t = 14$
1
-2
-2
-1
4


-2
-2
1
0
3
$t = 15$



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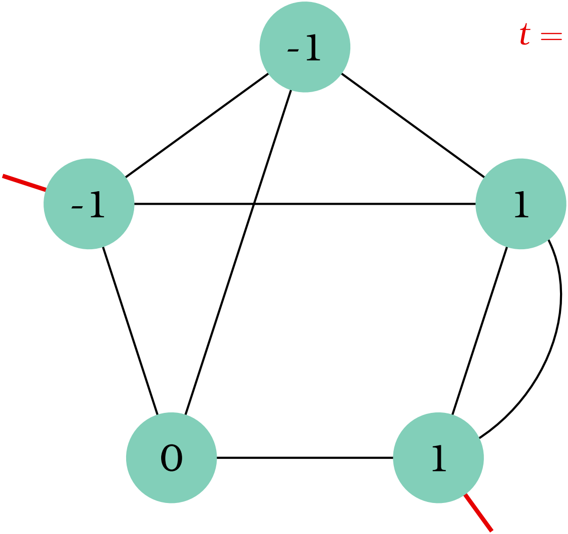
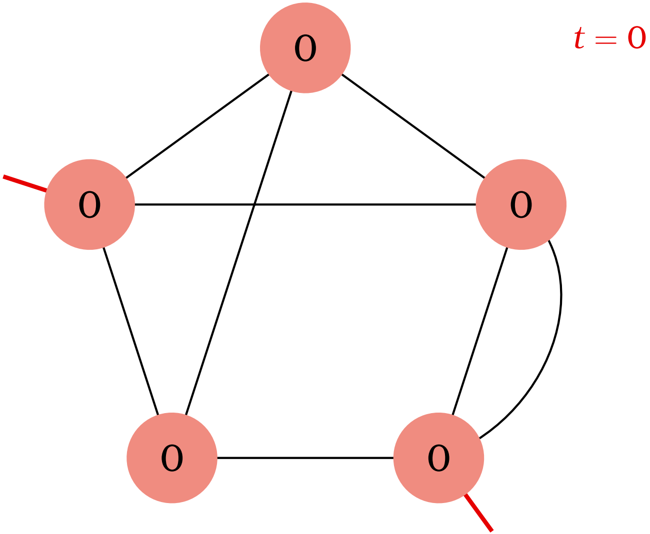
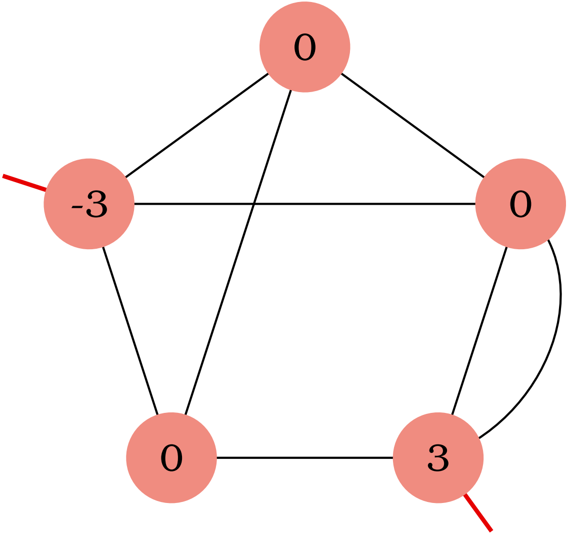
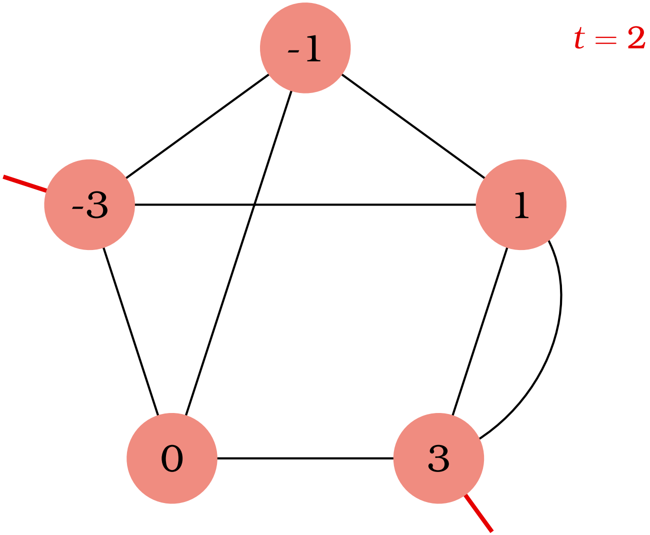


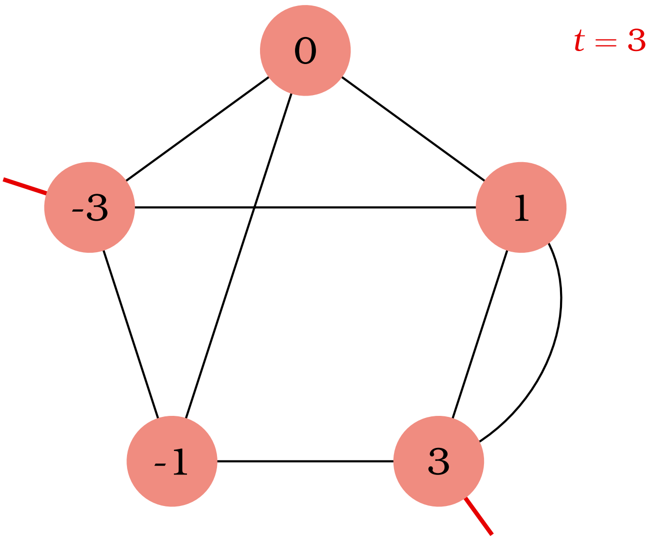
$t = 17, t = 2$

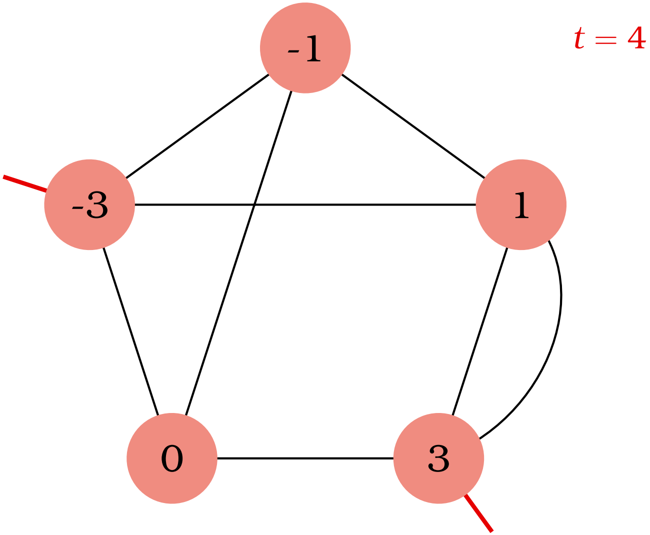

0
0
0
0
0
$t = 0$

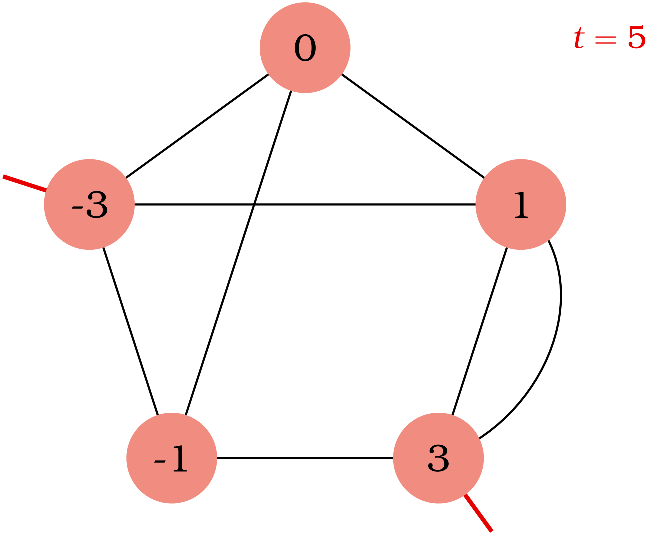
$t = 1$




-1
-3
1
0
3
$t = 2$


$t = 3$
0
-3
1
-1
3


-1
-3
1
0
3
$t = 4$


0
-3
1
-1
3
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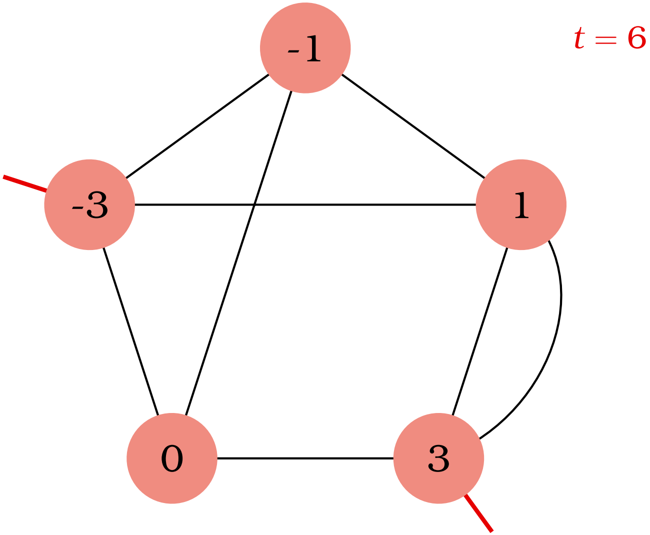
-1
-3
1
0
3
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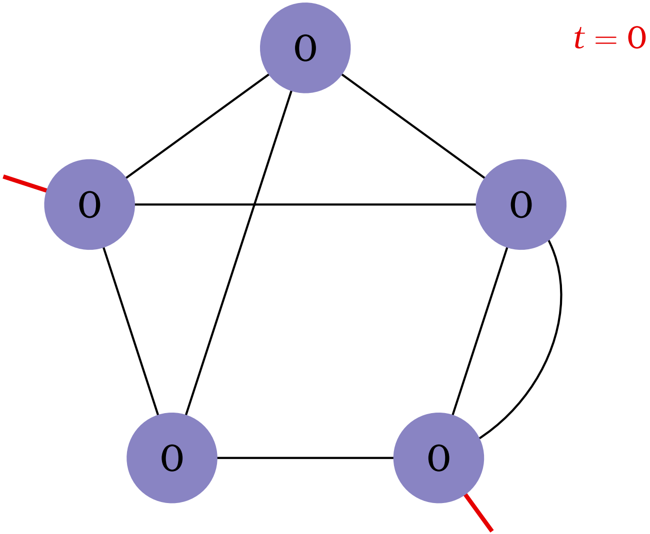
0
0
0
0
0
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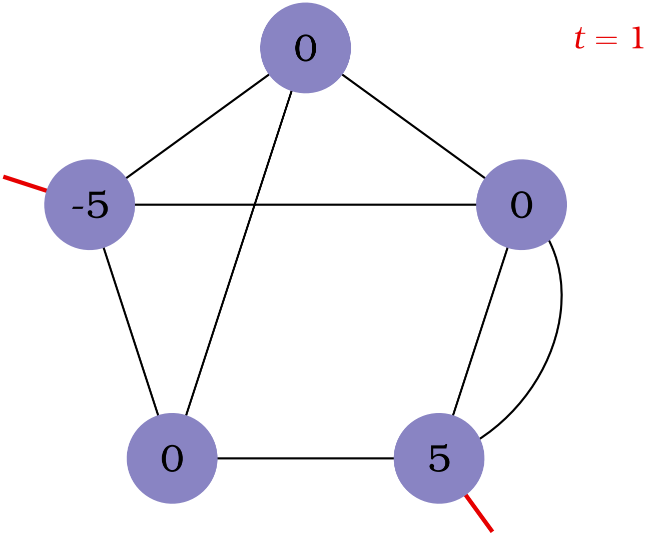
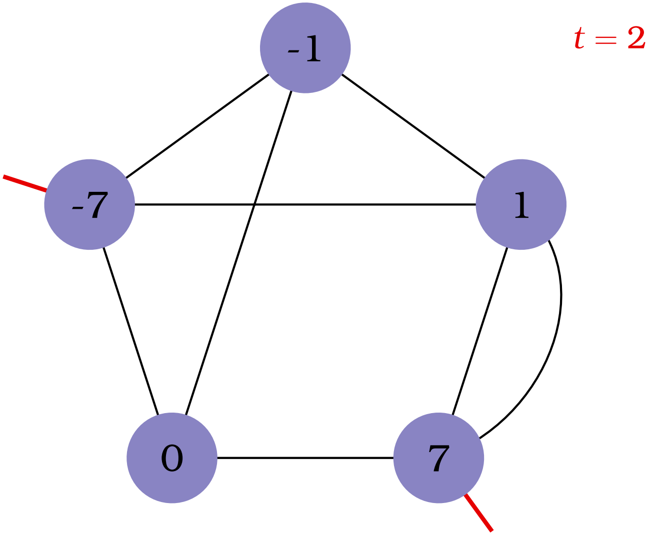
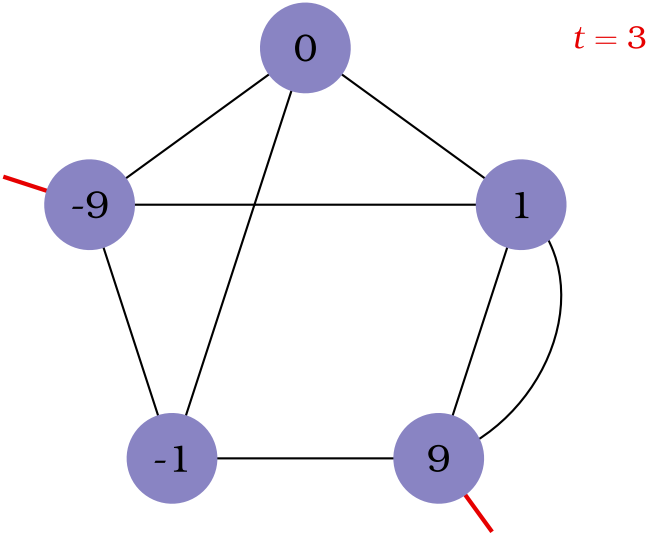

0
-5
0
0
5
t = 1


-1
-7
1
0
7
$t = 2$


0
-9
1
-1
9
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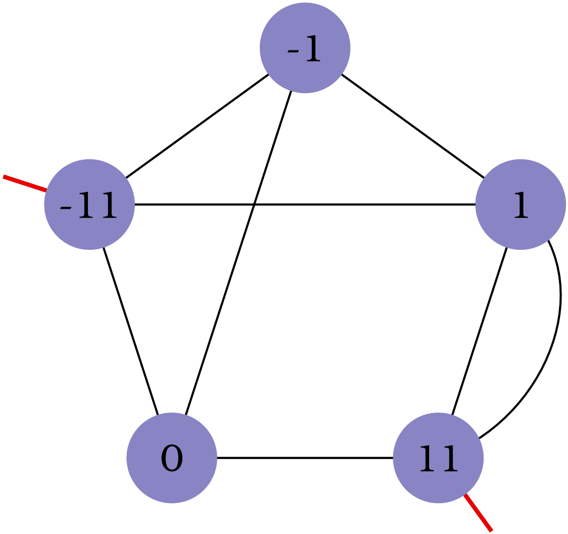

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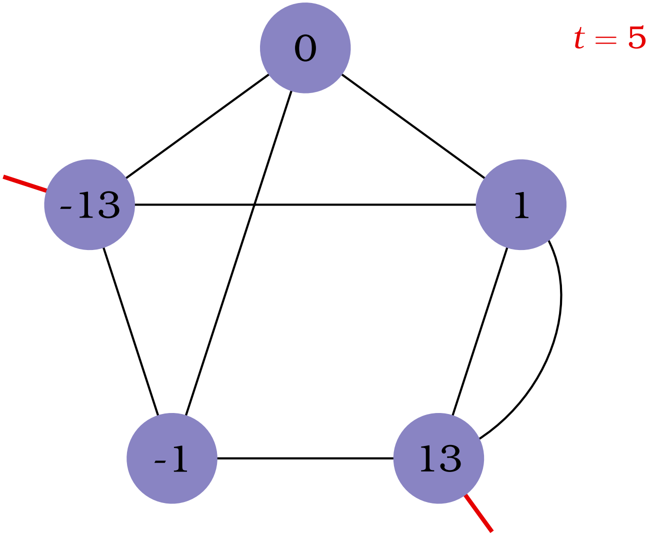
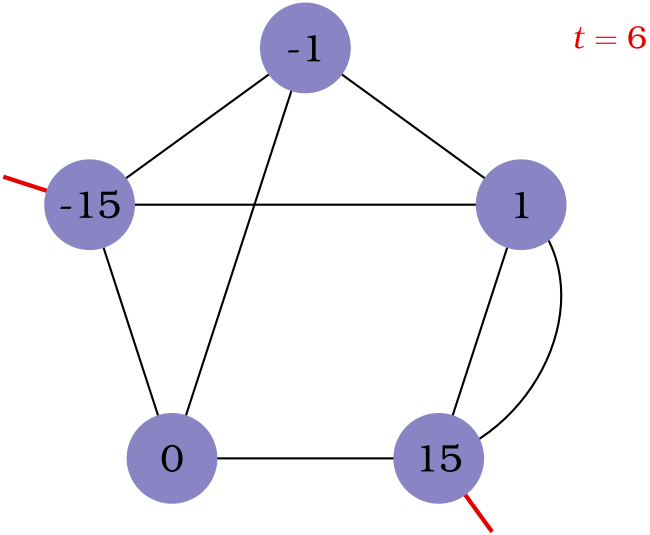

0
-13
1
-1
13
$t = 5$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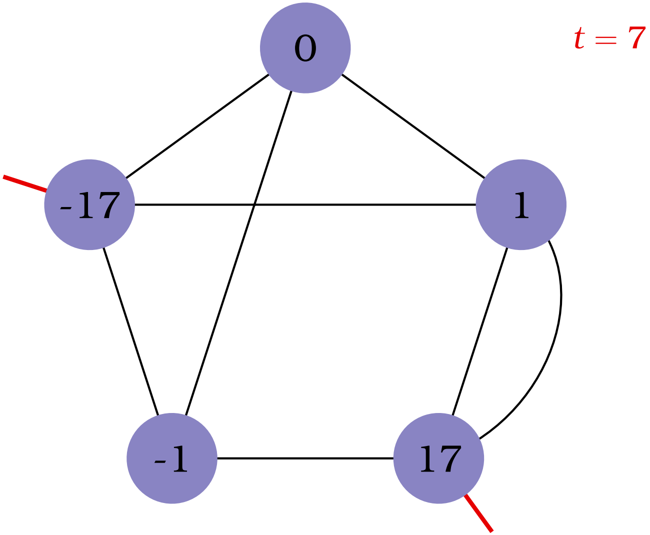
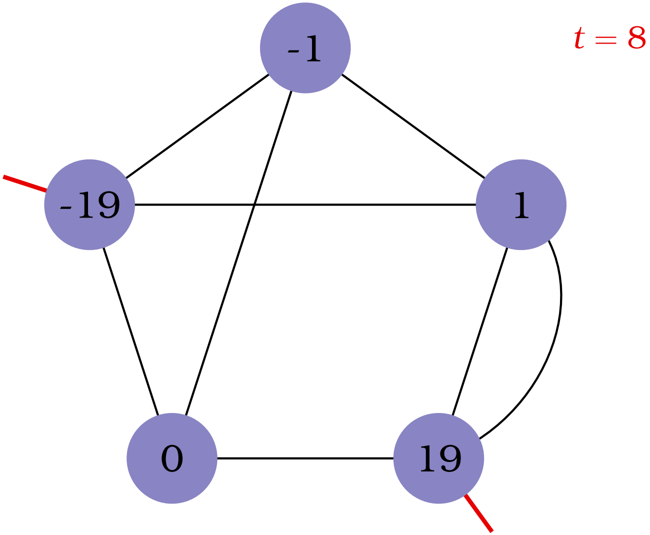
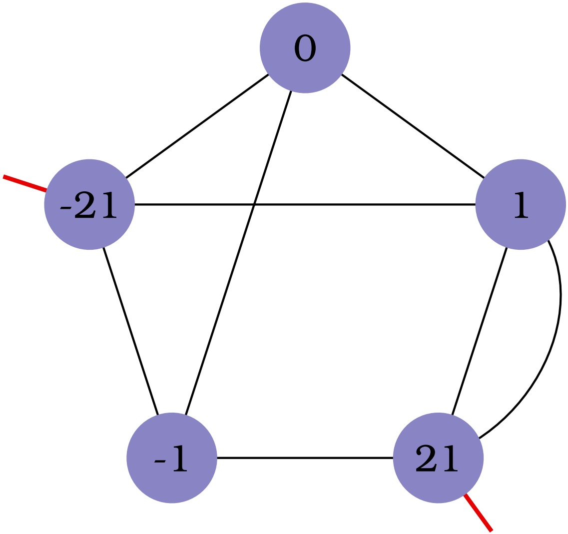
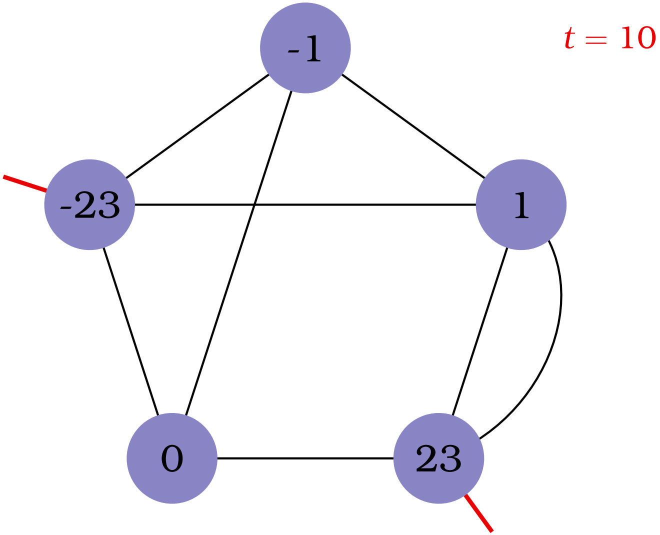
-1
-15
1
0
15

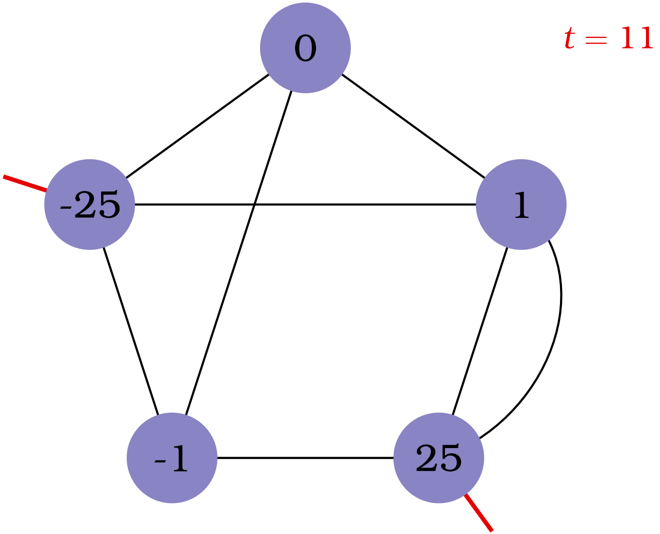

0
-17
1
-1
17
$t = 7$

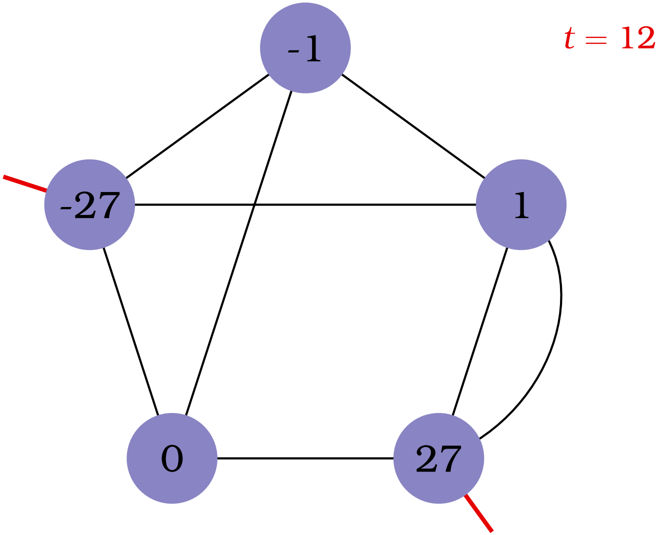

-1
-19
1
0
19
$t = 8$



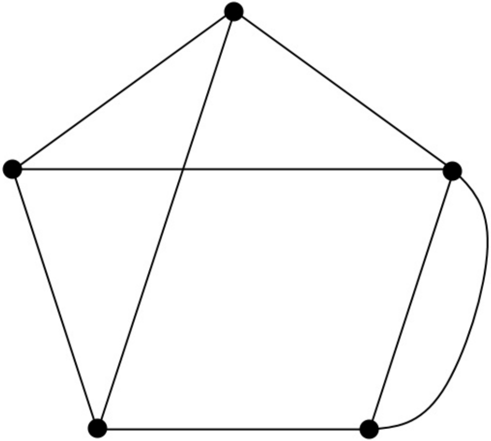
$t = 9$


$t = 10$
-1
-23
1
0
23

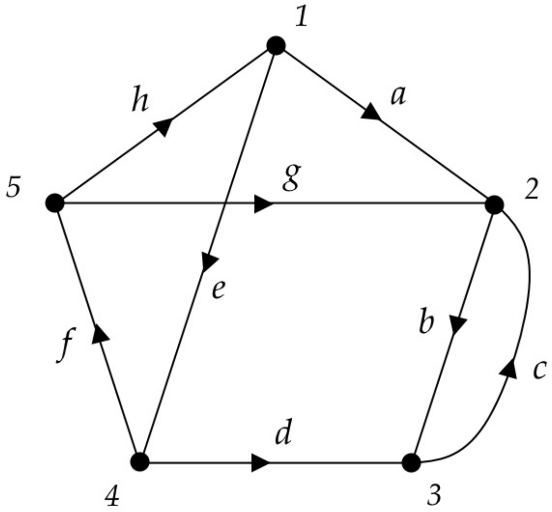

0
-25
1
-1
25
t = 11


$t = 12$
-1
-27
1
0
27

let's describe that with algebra



orient the edges arbitrarily



the V-by-E signed incidence matrix

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix}$$

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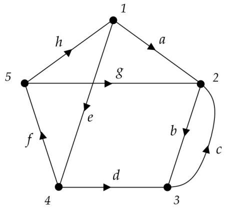

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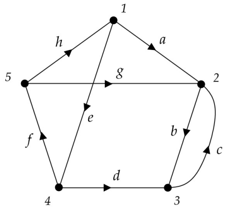

$$D_{ve} = \begin{cases} 1 & \text{if } e \text{ points in to } v, \\ -1 & \text{if } e \text{ points out of } v, \\ 0 & \text{otherwise.} \end{cases}$$

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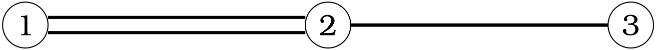
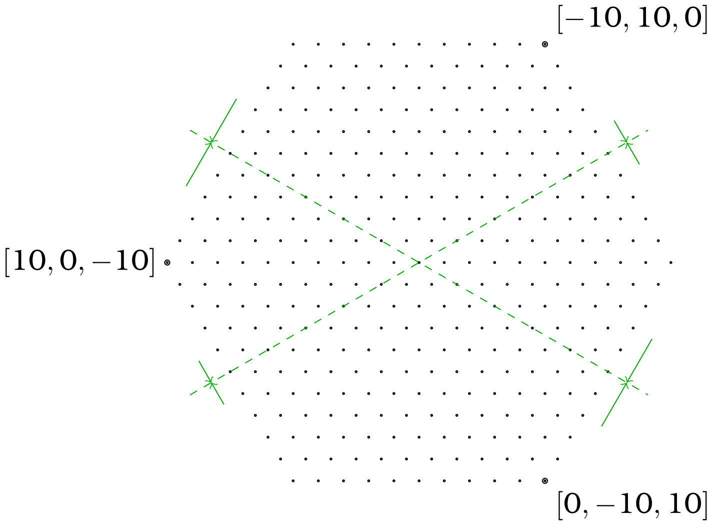
- Compare: the heat equation for $\mathbf{c} : \Omega \times \mathbb{R} \to \mathbb{R}$

$$\frac{\partial}{\partial t}\,\mathbf{c}(\mathbf{x}, t) \;=\; \nabla^2\,\mathbf{c}(\mathbf{x}, t) \;+\; \mathbf{h}.$$

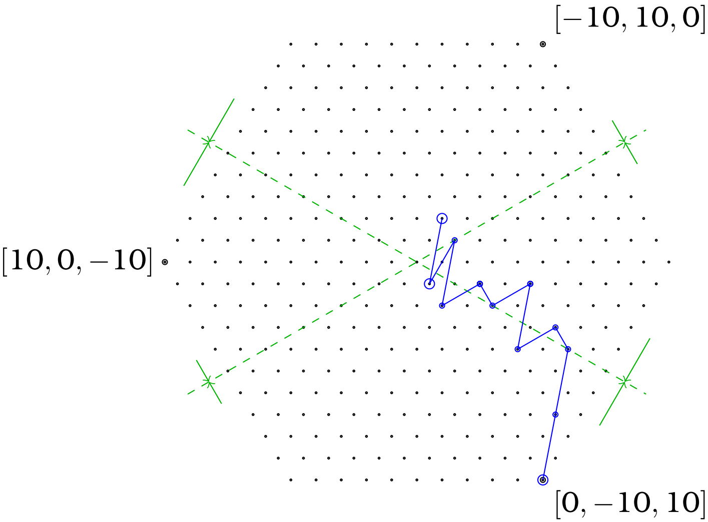
an example



in $\mathbb{Z}^V \cap \ker(\mathbf{1}^\intercal)$



the case $\mathbf{h} = \mathbf{0}$



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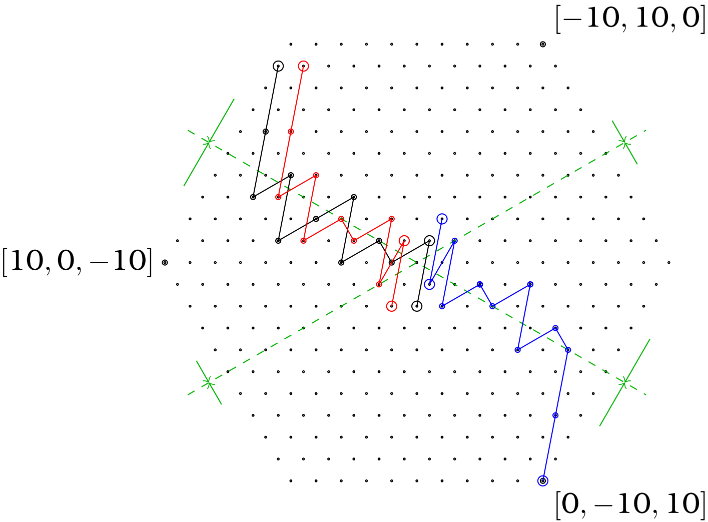

the critical group

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- Well-studied since the 1980s.

physical analogies

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- *conductances* $Y = \mathrm{diag}(y_e : \; e \in E)$ on edges.

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- Together these yield $DYD^{\intercal}\varphi = \mathbf{h}$.
- Given D, Y, and $\mathbf{h}$,
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- Relaxes exponentially quickly to
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- Let D be the n-by-m matrix with $\boldsymbol{\ell}_j$ as the j-th column, for $j = 1, ..., m$.

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  but the general case reduces to that case.

equilibrium steady state

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Lemma

Assume $\mathbf{h} = \mathbf{0}$. Then for all $t \in \mathbb{N}$,

$$\mathbf{c}_{t+1}^{\intercal}\mathbf{c}_t = \mathrm{Pot}_{\mathcal{L}}(\mathbf{c}_t).$$

Long-Narayanan (2017) updated by W.

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Theorem (at present)

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- (If all of $\mathbf{c}_0, \ell_1, ..., \ell_m$ are in a lattice in $\mathcal{V}$, then the sequence $(\mathrm{Pot}_{\mathcal{L}}(\mathbf{c}_t))$ attains its limit.)

visualizing the LN potential

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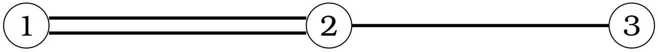
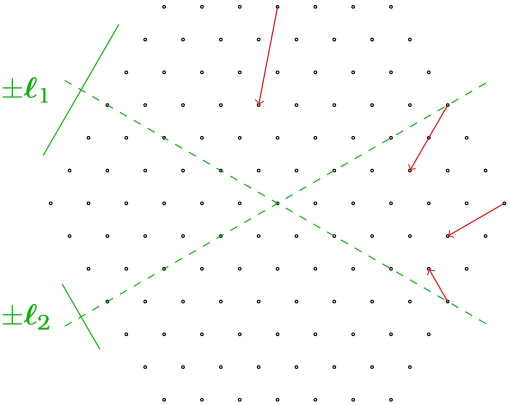
- For $\alpha \in \mathcal{A}$, let $\mathbf{s}_\alpha = (-1/2) D\alpha$ be the *site* of Δ_α. (Note: it is possible that $\mathbf{s}_\alpha \notin \Delta_\alpha$.)

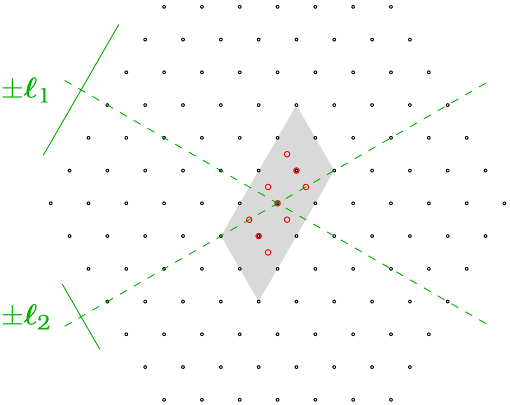
an example



Δ_α and $D\alpha$ for $\alpha \in \mathcal{A}$



$-\mathrm{conv}D\mathcal{A}$ and $\mathbf{s}_\alpha$ for $\alpha \in \mathcal{A}$



visualizing the LN potential

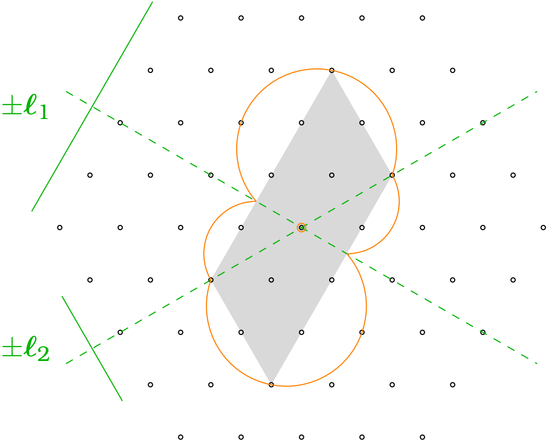
Proposition

For $\mathbf{v} \in \Delta_\alpha$,

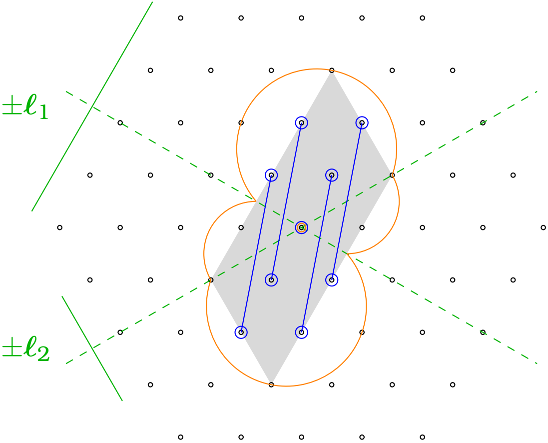
$$\mathrm{Pot}_{\mathcal{L}}(\mathbf{v}) = (\mathbf{v} - \mathbf{s}_\alpha)^\intercal (\mathbf{v} - \mathbf{s}_\alpha) - \mathbf{s}_\alpha^\intercal \mathbf{s}_\alpha.$$

On Δ_α, that is the squared distance to $\mathbf{s}_\alpha$ minus the squared length of $\mathbf{s}_\alpha$.

the zero set of $\mathrm{Pot}_{\mathcal{L}}$



the steady states



non-equilibrium steady state

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- (So $\mathbf{b}$ is the average "buildup/backlog" per time step.)
- (Conjecture: this happens for input confined to a lattice.)

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- Notice that $-\mathbf{1} \leq \mathbf{j} \leq \mathbf{1}$ coordinatewise in $\mathbb{R}^m$.

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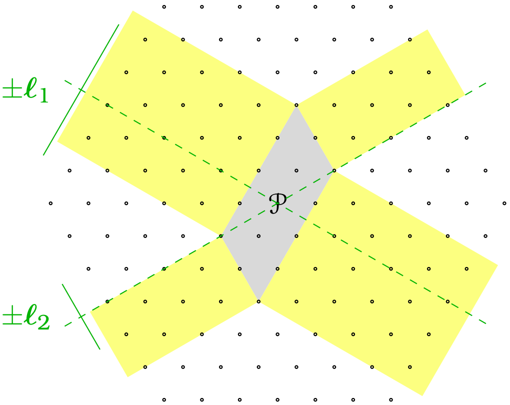
- In the graph case, these are the vectors in $\mathbb{R}^V$ that result from flows that are bounded by 1 on each edge.

$\mathbf{h} - \mathbf{b}$ is in $\mathcal{P}$



current projects/conjectures

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- Generalize $\mathrm{Pot}_\mathcal{L}$ accordingly.
- Eventual periodicity is not required.

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- Issues regarding irrationality and non-periodic steady states.

Thank You!