The Efficient Frontier for Partially Correlated, Bounded Assets

by

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An essay
presented to the University of Waterloo
in fulfillment of the
essay requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, April 2006

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Acknowledgement

Many people have played direct and indirect roles in the stage of my life that has lead to this dissertation, and I would like to acknowledge them here.

I would like to express my gratitude to my supervisor and my best friend, Professor Michael J. Best for his infinite patience, wise advice, and un- conditional support on my research in Portfolio Optimization.

I am deeply appreciative of the valuable comments and suggestions provided by Professor Chek Beng Chua.

I wish to express my thanks to my parents for all the care they take of me.

I also would like to thank all those who made my time in Waterloo the greatest experience in my life.

Last and most of all, I would like to thank my husband, John and my dear daughter, Yangzi for their unwavering support and encouragement. I am forever grateful to them.

Abstract

In general, computation of the efficient portfolios for a portfolio optimization problem requires the inversion of the covariance matrix Σ, which is computationally quite expensive. Best and Hlouskova[1] showed that a portfolio optimization problem can be solved in closed form in the special case that assets are uncorrelated. However, the assumption of uncorrelated assets is unduly restrictive. The aim of this essay is to generalize a closed form solution of the mean-variance portfolio selection problem to partially correlated and bounded assets. Although we only discuss a triple-branch covariance matrix (See Definition 2.1), the explicit representation is sufficient to analyze the expected return and the variance of the efficient portfolios. It shows insight into the nature of the efficient frontier in the presence of inequality constraints. In this essay, we give a closed form solution for a portfolio optimization problem having lower bounds for two cases; a universe of only risky assets and a universe of risky assets plus an additional risk free asset. In each of theses cases, the assets are partially correlated in the sense that covariance matrix is a triple-branch matrix.

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Chapter 1

Introduction

The Mean-Variance (M-V) portfolio selection problem and the related Capital Asset Pricing Model (CAPM) have been studied by many researchers under a variety of assumptions. Brennan [4] addressed the issue of borrowing and lending rates. Turnbull [10] also considered this along with personal taxation, uncertain inflation and non-market assets. Levy [7] dealt with the problem of short sales as did Schnabel [5].

In general, portfolio optimization problems with inequality constraints must be solved with a QP (quadratic programming) algorithm [2]. It is often impossible to obtain a closed form solution. Best and Hlouskova [1] developed a closed form solution for the mean-variance portfolio selection problem for uncorrelated and bounded assets when an additional technical assumption was satisfied. However, the assumption of uncorrelated assets is unduly restrictive.

The contribution of this essay is as follows. In this essay, we discuss a special covariance matrix. The form of the matrix is that the diagonal elements and the last row and column are nonzero, while the others are zero. This matrix is called triple-branch matrix. Although we only discuss a special covariance matrix, the matrix can reflect the partial correlation between asset x_i and asset x_n. When the covariance matrix is positive definite, and some additional technical assumptions are satisfied, we derive a closed form solution for the efficient portfolios. Obtaining an explicit representation of efficient asset holdings subject to bound constraints with the triple-branch covariance matrix does give insight into the efficient frontier in the presence of inequality constraints.

In Chapter 2, we derive a closed form solution for a portfolio optimization problem with just a budget constraint. This illustrates the nature of the triple-branch covariance matrix. However, such a model will generally have extremely large long and short positions which are unrealistic. Therefore, in Chapter 3, we deal with a model to preclude short selling, i.e., we require that all components of the asset holdings must be non-negative.

In Chapter 4, we deal with a variation of the model of Chapter 3. In addition to the n risky assets, we suppose that there is an additional asset which is risk free. The risk free asset has a zero variance and a zero covariance with the risky asset. An example of a risk free asset is Treasury bills.

In Chapter 5, we will find that the uncorrelated and bounded assets discussed by Best and Hlouskova [1] are a special case of the Partially Correlated, Bounded Assets model presented by the essay.

Finally, we give the conclusions.

Chapter 2

Assets with an Equality Constraint

In this chapter, We will discuss the portfolio optimization problem without inequality constraints

$$\min\{-t\mu' x + \frac{1}{2} x' \Sigma x \mid \quad l' x = 1\}, \tag{2.1}$$

where Σ is a triple-branch matrix, l is an n-vector of ones, μ is an n-vector of expected returns, x is an n-vector of asset holdings to be determined and t is a non-negative scalar parameter denoting the investor's aversion to risk.

2.1 Some Notations and Definitions

The type of covariance matrix we will be using is formulated in the following definition.

Definition 2.1. *A (n, n) matrix Σ is called triple-branch matrix, if*

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & b_1 \\ 0 & \sigma_2 & \cdots & b_2 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{n-1} \\ b_1 & b_2 & \cdots & \sigma_n \end{bmatrix}$$

where $\sigma_1, \cdots, \sigma_n$ and $b_1, \cdots, b_{n-1}$ can be non-zero.

Besides being used in financial models, in modern control theory the parameter matrix of an nonlinear adjustment system is often in this triple-branch form [6].

Prime(') denotes matrix transposition. Any non-primed vector is a column vector.

For a more concise formulation of a optimal solution, we define the following constants.

$$\theta_{1k} = (\frac{1}{\sigma_k} + ... + \frac{1}{\sigma_{n-1}}), \tag{2.2}$$

$$\theta_{2k} = (\frac{\mu_k}{\sigma_k} + ... + \frac{\mu_{n-1}}{\sigma_{n-1}}), \tag{2.3}$$

$$\theta_{3k} = (\frac{b_k^2}{\sigma_k} + ... + \frac{b_{n-1}^2}{\sigma_{n-1}}), \tag{2.4}$$

$$\theta_{4k} = (\frac{b_k}{\sigma_k} + ... + \frac{b_{n-1}}{\sigma_{n-1}} - 1)/(\sigma_n - \theta_{3k}), \tag{2.5}$$

$$\theta_{5k} = (\frac{\mu_k b_k}{\sigma_k} + ... + \frac{\mu_{n-1} b_{n-1}}{\sigma_{n-1}} - \mu_n)/(\sigma_n - \theta_{3k}), \tag{2.6}$$

$$\theta_{6k} = (\frac{\mu_k^2}{\sigma_k} + ... + \frac{\mu_{n-1}^2}{\sigma_{n-1}}), \tag{2.7}$$

$$Q_{1k} = \frac{1}{\theta_{1k} + \theta_{4k}^2(\sigma_n - \theta_{3k})}, \tag{2.8}$$

$$Q_{2k} = Q_{1k}(\theta_{2k} + \theta_{4k}\theta_{5k}(\sigma_n - \theta_{3k})), \tag{2.9}$$

for $k = 1, ..., n-1$.

2.2 A Closed Form Solution for Assets with an Equality Constraint

The solution of (2.1) can be formulated concisely in terms of the constants from (2.2) to (2.9).

Assumption 2.1.

1) $\sigma_i > 0, \quad i = 1, \cdots n, \quad \sigma_n - \theta_{31} > o.$

Assumption 2.1 guarantees the matrix Σ positive definite. Because Σ is positive definite, it is also invertible and furthermore the optimality conditions (Karush-Kuhn-Tucker conditions) are both necessary and sufficient for optimality.

The optimality conditions for (2.1) are

$$t\mu - \Sigma x = ul, \quad l'x = 1. \tag{2.10}$$

Based on these conditions, (2.1) has the optimal solution [3]

$$x = h_0 + th_1 = \frac{\Sigma^{-1}l}{l'\Sigma^{-1}l} + t(\Sigma^{-1}\mu - \frac{l'\Sigma^{-1}\mu}{l'\Sigma^{-1}l}\Sigma^{-1}l), \tag{2.11}$$

$$u = \frac{tl'\Sigma^{-1}\mu - 1}{l'\Sigma^{-1}l}. \tag{2.12}$$

The variance and expected return of the efficient portfolios are

$$\sigma_p^2 = h_0'\Sigma h_0 + 2th_1'\Sigma h_0 + t^2 h_1'\Sigma h_1, \tag{2.13}$$

and

$$\mu_p = \mu' h_0 + t\mu' h_1, \tag{2.14}$$

respectively.

Lemma 2.1. *Let A be an m by m non-singular matrix with*

$$A^{-1} = \begin{bmatrix} a^{11} & \cdots & \cdots & a^{1m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a^{m1} & \cdots & \cdots & a^{mm} \end{bmatrix},$$

and let $\bar{A}$ be a matrix which is the same as A except that the k^{th} column of A has been replaced by the vector

$$\beta = [\beta_1, \ \cdots, \ \cdots, \ \beta_m]'.$$

Suppose $y = A^{-1}\beta$, thus the coefficients $\bar{a}^{ij}$ of $(\bar{A})^{-1}$ satisfy

$$\bar{a}^{ij} = \begin{cases} a^{ij} - (y_i/y_k)a^{kj}, & i \neq k, \\ a^{kj}/y_k, & i = k. \end{cases}$$

Proof is given in [9]. □

Similarly, there is

Lemma 2.2. *Suppose that $\bar{A}$ is a matrix which is the same as A except that the k^{th} row of A has been replaced by the row vector*

$$\beta = [\beta_1, \ \cdots, \ \cdots, \ \beta_m].$$

Let $y = \beta A^{-1}$, *then the coefficients* $\bar{a}^{ij}$ *of* $(\bar{A})^{-1}$ *satisfy*

$$\bar{a}^{ij} = \begin{cases} a^{ij} - (y_j/y_k)a^{ik}, & j \neq k, \\ a^{ik}/y_k, & j = k. \end{cases}$$

Theorem 2.1. *Let Assumption 2.1 be satisfied. Then*

$$\Sigma^{-1} = \begin{cases} \frac{1}{\sigma_i} + \frac{b_i^2}{\sigma_i^2 \sigma_n y_n}, & i = j \neq n, \\ \frac{b_i b_j}{\sigma_i \sigma_j \sigma_n y_n}, & i \neq j, i \neq n, j \neq n, \\ -\frac{b_j}{\sigma_j \sigma_n y_n}, & i \neq j, i = n, \\ -\frac{b_i}{\sigma_i \sigma_n y_n}, & i \neq j, j = n, \\ \frac{1}{\sigma_n y_n}, & i = j = n, \end{cases} \tag{2.15}$$

where

$$y_n = 1 - \frac{\theta_{31}}{\sigma_n}. \tag{2.16}$$

Proof: Let

$$A_1 = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix},$$

$$\bar{A}_1 = \begin{bmatrix} \sigma_1 & 0 & \cdots & b_1 \\ 0 & \sigma_2 & \cdots & b_2 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{n-2} \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix},$$

$$\beta_1 = [b_1,\ \cdots,\ b_{n-1},\ \sigma_n]',$$

and

$$y_1 = A_1^{-1}\beta_1 = [\frac{b_1}{\sigma_1},\ \cdots,\ \frac{b_{n-1}}{\sigma_{n-1}},\ 1]'.$$

Lemma 2.1 implies that the coefficients $\bar{a_1}^{ij}$ of $(\bar{A}_1)^{-1}$ satisfy

$$\bar{a_1}^{ij} = \begin{cases} \frac{1}{\sigma_i}, & i = j, \\ \frac{-b_i}{\sigma_i\sigma_n}, & i \neq j, j = n, \\ 0, & otherwise. \end{cases} \tag{2.17}$$

Therefore Σ is a matrix whose n^{th} row of $\bar{A}_1$ is replaced by the row vector

$$\beta_1 = [b_1,\ \cdots,\ b_{n-1},\ \sigma_n].$$

Suppose

$$y = \beta\bar{A_1}^{-1} = [\frac{b_1}{\sigma_1},\ \ldots,\ \frac{b_{n-1}}{\sigma_{n-1}},\ 1 - \frac{\theta_{31}}{\sigma_n}], \tag{2.18}$$

then Lemma 2.2 with (2.17) and (2.18) implies that the coefficients a^{ij} of Σ^{-1} satisfy

$$a^{ij} = \begin{cases} \frac{1}{\sigma_i} + \frac{b_i^2}{\sigma_i^2\sigma_n y_n}, & i = j \neq n, \\ \frac{b_i b_j}{\sigma_i\sigma_j\sigma_n y_n}, & i \neq j, i \neq n, j \neq n, \\ -\frac{b_j}{\sigma_j\sigma_n y_n}, & i \neq j, i = n, \\ -\frac{b_i}{\sigma_i\sigma_n y_n}, & i \neq j, j = n, \\ \frac{1}{\sigma_n y_n}, & i = j = n, \end{cases} \tag{2.19}$$

which is the desired result. □

Using (2.11) and Theorem 2.1, we can determine the efficient portfolios as follows.

First observe

$$l'\Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} - \frac{b_1}{\sigma_1\sigma_n y_n} + \frac{b_1}{\sigma_1\sigma_n y_n}\sum_{i=1}^{n-1}\frac{b_i}{\sigma_i} \\ \cdots \\ \frac{1}{\sigma_i} - \frac{b_i}{\sigma_i\sigma_n y_n} + \frac{b_i}{\sigma_i\sigma_n y_n}\sum_{i=1}^{n-1}\frac{b_i}{\sigma_i} \\ \cdots \\ \frac{1}{\sigma_n y_n} - \frac{1}{\sigma_n y_n}\sum_{i=1}^{n-1}\frac{b_i}{\sigma_i} \end{bmatrix}' = \begin{bmatrix} \frac{1}{\sigma_1} + \frac{b_1\theta_{41}}{\sigma_1} \\ \cdots \\ \frac{1}{\sigma_i} + \frac{b_i\theta_{41}}{\sigma_i} \\ \cdots \\ -\theta_{41} \end{bmatrix}', \tag{2.20}$$

and

$$\Sigma^{-1}\mu = \begin{bmatrix} \frac{\mu_1}{\sigma_1} - \frac{b_1\mu_n}{\sigma_1\sigma_n y_n} + \frac{b_1}{\sigma_1\sigma_n y_n}\sum_{i=1}^{n-1}\frac{b_i\mu_i}{\sigma_i} \\ \cdots \\ \frac{\mu_i}{\sigma_i} - \frac{b_i\mu_n}{\sigma_i\sigma_n y_n} + \frac{b_i}{\sigma_i\sigma_n y_n}\sum_{i=1}^{n-1}\frac{b_i\mu_i}{\sigma_i} \\ \cdots \\ \frac{\mu_n}{\sigma_n y_n} - \frac{1}{\sigma_n y_n}\sum_{i=1}^{n-1}\frac{b_i\mu_i}{\sigma_i} \end{bmatrix} = \begin{bmatrix} \frac{\mu_1}{\sigma_1} + \frac{b_1\theta_{51}}{\sigma_1} \\ \cdots \\ \frac{\mu_i}{\sigma_i} + \frac{b_i\theta_{51}}{\sigma_i} \\ \cdots \\ -\theta_{51} \end{bmatrix}. \tag{2.21}$$

From (2.20) and (2.21),

$$l'\Sigma^{-1}l = \sum_{i=1}^{n-1}\frac{1}{\sigma_i} + \frac{1}{\sigma_n y_n} - \frac{2}{\sigma_n y_n}\sum_{i=1}^{n-1}\frac{b_i}{\sigma_i} + \frac{1}{\sigma_n y_n}(\sum_{i=1}^{n-1}\frac{b_i}{\sigma_i})^2$$

$$= \theta_{11} + \theta_{41}^2(\sigma_n - \theta_{31}),$$

$$l'\Sigma^{-1}\mu = \sum_{i=1}^{n-1}\frac{\mu_i}{\sigma_i} + \frac{\mu_n}{\sigma_n y_n} - \frac{1}{\sigma_n y_n}\sum_{i=1}^{n-1}\frac{b_i\mu_i}{\sigma_i}$$

$$-\frac{\mu_n}{\sigma_n y_n}\sum_{i=1}^{n-1}\frac{b_i}{\sigma_i}+\frac{1}{\sigma_n y_n}(\sum_{i=1}^{n-1}\frac{b_i}{\sigma_i})(\sum_{i=1}^{n-1}\frac{b_i\mu_i}{\sigma_i})$$
$$=\theta_{21}+\theta_{41}\theta_{51}(\sigma_n-\theta_{31}).$$

Therefore, we obtain the optimal solution for (2.1) as

$$x_i=\frac{\frac{1}{\sigma_i}+\frac{b_i}{\sigma_i}\theta_{41}}{\theta_{11}+\theta_{41}^2(\sigma_n-\theta_{31})}$$
$$+t[\frac{\mu_i}{\sigma_i}+\frac{b_i}{\sigma_i}\theta_{51}-\frac{\theta_{21}+\theta_{41}\theta_{51}(\sigma_n-\theta_{31})}{\theta_{11}+\theta_{41}^2(\sigma_n-\theta_{31})}(\frac{1}{\sigma_i}+\frac{b_i}{\sigma_i}\theta_{41})]$$
$$=[Q_{11}(1+b_i\theta_{41})+t(\mu_i+b_i\theta_{51}-Q_{21}(1+b_i\theta_{41}))]/\sigma_i, \tag{2.22}$$

$i=1,2,\cdots,n-1,$

$$x_n=\frac{-\theta_{41}}{\theta_{11}+\theta_{41}^2(\sigma_n-\theta_{31})}+t[-\theta_{51}+\frac{\theta_{21}+\theta_{41}\theta_{51}(\sigma_n-\theta_{31})}{\theta_{11}+\theta_{41}^2(\sigma_n-\theta_{31})}\theta_{41}]$$
$$=-\theta_{41}Q_{11}+t[-\theta_{51}+Q_{21}\theta_{41}], \tag{2.23}$$

$$u=-Q_{11}+tQ_{21}. \tag{2.24}$$

The expected return is

$$\mu_p=\mu' x(t)=Q_{11}\sum_{i=1}^{n-1}\frac{\mu_i}{\sigma_i}[(1+b_i\theta_{41})+t(\mu_i\theta_{51}-Q_{21}(1+b_i\theta_{41})]$$
$$-Q_{11}\mu_n\theta_{41}+t\mu_n(-\theta_{51}+Q_{21}\theta_{41})$$
$$=Q_{21}+t(\theta_{61}+\theta_{51}^2(\sigma_n-\theta_{31})-\frac{Q_{21}^2}{Q_{11}}).$$

The variance is

$$\sigma_p^2=h_0'\Sigma h_0+\mu' h_1 t^2,$$

where

$$h_0'\Sigma h_0=Q_{11}^2\theta_{11}-Q_{11}^2\theta_{31}\theta_{41}^2+\sigma_n Q_{11}^2\theta_{41}^2=Q_{11}, \tag{2.25}$$

and

$$\mu' h_1 t^2 = (\theta_{61} + \theta_{51}^2(\sigma_n - \theta_{31}) - Q_{21}\theta_{21} - Q_{21}\theta_{41}\theta_{51}(\sigma_n - \theta_{31}))t^2$$

$$= (\theta_{61} + \theta_{51}^2(\sigma_n - \theta_{31}) - \frac{Q_{21}^2}{Q_{11}})t^2.$$

Hence

$$\sigma_p^2 = Q_{11} + (\theta_{61} + \theta_{51}^2(\sigma_n - \theta_{31}) - \frac{Q_{21}^2}{Q_{11}})t^2$$

From the above discussions, we obtain the following result.

Theorem 2.2. *Let Assumption 2.1 be satisfied and $\sum$ be triple-branch matrix, then the optimal solution for (2.1) is*

$$x_i = [Q_{11}(1 + b_i\theta_{41}) + t(\mu_i + b_i\theta_{51} - Q_{21}(1 + b_i\theta_{41}))]/\sigma_i,$$

$$i = 1, 2, \cdots, n-1,$$

$$x_n = -\theta_{41}Q_{11} + t(-\theta_{51} + Q_{21}\theta_{41}),$$

and the multiplier for the budget constraint is

$$u = -Q_{11} + tQ_{21},$$

where θ_{41}, θ_{51}, Q_{11}, and Q_{21} are defined from (2.5) to (2.9).

Chapter 3

Assets with Lower Bounds

3.1 Several Important Results

In this section, we extend the results of Chapter 2 by augmenting the model problem (2.1) with non-negativity constraints.

One model problem for the section is thus

$$\min\{-t\mu' x + \frac{1}{2} x' \Sigma x \mid \quad l' x = 1, \quad x \geq 0\}. \tag{3.1}$$

We require the following

Assumption 3.1.

1) $\sigma_i > 0, \quad i = 1, \cdots n, \quad \sigma_n - \theta_{31} > o,$

2) $-\theta_{4k} > 0, \quad 1 + b_i \theta_{4k} > 0, \quad k = 1, \cdots n-1, \; i = k, \cdots n-1,$

3) $b_i \geq b_{i+1}, \quad i = 1, \cdots n-2,$

4) $\mu_i < \mu_{i+1}, \quad i = 1, \cdots n-1,$

5) $\theta_{5k} \le \mu_n \theta_{4k}, \quad k = 1, \cdots, n-1.$

For $k = 0, 1, \cdots, n$, define

$$t_k = \begin{cases} 0, & k = 0, \\ \frac{Q_{1k}(1+b_k\theta_{4k})}{Q_{2k}(1+b_k\theta_{4k})-(\mu_k+b_k\theta_{5k})}, & k = 1, \cdots, n-1, \\ \infty, & k = n. \end{cases} \tag{3.2}$$

For $k = 1, \cdots, n-1$, define

$$\begin{cases} x_k = x_k(t) = ((x_k)_1, (x_k)_2, \cdots, (x_k)_n)' \quad where, & \\ (x_k)_i = 0, & i = 1, \cdots, k-1, \\ (x_k)_i = (x_k(t))_i & \\ = [Q_{1k}(1 + b_i\theta_{4k}) + t(\mu_i + b_i\theta_{5k} - Q_{2k}(1 + b_i\theta_{4k}))]/\sigma_i, & i = k, \cdots, n-1, \\ (x_k)_n = -\theta_{4k}Q_{1k} + t[-\theta_{5k} + Q_{2k}\theta_{4k}], & \end{cases} \tag{3.3}$$

$$\begin{cases} (x_n)_i = 0, \quad i = 1, \cdots, n-1, \\ (x_n)_n = 1, \end{cases} \tag{3.4}$$

$$u_k = u_k(t) = -Q_{1k} + tQ_{2k}, \tag{3.5}$$

$$u_n(t) = -\sigma_n + t\mu_n, \tag{3.6}$$

$$\begin{cases} v_k = v_k(t) = ((v_k)_1, (v_k)_2, \cdots, (v_k)_n)' \quad where, & \\ (v_k)_i = (v_k(t))_i & \\ = -Q_{1k}(1 + b_i\theta_{4k}) + t[Q_{2k}(1 + b_i\theta_{4k}) - b_i\theta_{5k} - \mu_i] & i = 1, \cdots, k-1, \\ (v_k)_i = 0, & i = k, \cdots, n, \end{cases} \tag{3.7}$$

$$
\begin{cases}
(v_n)_i = (v_n(t))_i \\
= (-\sigma_n + b_i) + t(\mu_n - \mu_i) \quad i = 1, \cdots, n-1, \\
(v_n)_n = 0.
\end{cases}
\tag{3.8}
$$

Lemma 3.1. *For $k = 1, \cdots, n-2$, we have*

$$
\frac{Q_{1k}(1 + b_k\theta_{4k})}{Q_{2k}(1 + b_k\theta_{4k}) - (\mu_k + b_k\theta_{5k})} = \frac{Q_{1,k+1}(1 + b_k\theta_{4,k+1})}{Q_{2,k+1}(1 + b_k\theta_{4,k+1}) - (\mu_k + b_k\theta_{5,k+1})}. \tag{3.9}
$$

Proof: From (2.8) and (2.9), we obtain

$$
\frac{Q_{1k}(1 + b_k\theta_{4k})}{Q_{2k}(1 + b_k\theta_{4k}) - (\mu_k + b_k\theta_{5k})}
$$

$$
= \frac{1 + b_k\theta_{4k}}{(1 + b_k\theta_{4k})(\theta_{2k} + \theta_{4k}\theta_{5k}(\sigma_n - \theta_{3k})) - (\mu_k + b_k\theta_{5k})(\theta_{1k} + \theta_{4k}^2)(\sigma_n - \theta_{3k})}
$$

$$
= \frac{1 + b_k\theta_{4k}}{(\theta_{2k} - \mu_k\theta_{1k}) + (\sigma_n - \theta_{3k})(\theta_{4k}\theta_{5k} - \mu_k\theta_{4k}^2) + b_k(\theta_{4k}\theta_{2k} - \theta_{5k}\theta_{1k})}.
$$

Since

$$
\theta_{2k} - \mu_k\theta_{1k} = \theta_{2,k+1} - \mu_k\theta_{1(k+1)},
$$

$$
(\sigma_n - \theta_{3k})(\theta_{4k}\theta_{5k} - \mu_k\theta_{4k}^2)
$$

$$
= \frac{1}{(\sigma_n - \theta_{3k})}[(\frac{b_k}{\sigma_k} + \theta_{4,k+1}(\sigma_n - \theta_{3(k+1)}))(\frac{b_k\mu_k}{\sigma_k} + \theta_{5,k+1}(\sigma_n - \theta_{3(k+1)}))
$$

$$
-\mu_k(\frac{b_k}{\sigma_k} + \theta_{4,k+1}(\sigma_n - \theta_{3(k+1)}))^2]
$$

$$
= \frac{(\sigma_n - \theta_{3(k+1)})^2}{(\sigma_n - \theta_{3k})}(\theta_{4,k+1}\theta_{5,k+1} - \mu_k\theta_{4,k+1}^2)
$$

$$
+\frac{(\sigma_n - \theta_{3(k+1)})}{(\sigma_n - \theta_{3k})}\frac{b_k}{\sigma_k}\theta_{5,k+1} - \frac{(\sigma_n - \theta_{3(k+1)})}{(\sigma_n - \theta_{3k})}\frac{\mu_k b_k}{\sigma_k}\theta_{4,k+1},
$$

and

$$
\begin{aligned}
& b_k(\theta_{4k}\theta_{2k} - \theta_{5k}\theta_{1k}) \\
&= \frac{b_k}{(\sigma_n - \theta_{3k})}[(\frac{b_k}{\sigma_k} + \theta_{4,k+1}(\sigma_n - \theta_{3(k+1)}))(\frac{\mu_k}{\sigma_k} + \theta_{2,k+1}) \\
&-(\frac{\mu_k b_k}{\sigma_k} + \theta_{5,k+1}(\sigma_n - \theta_{3(k+1)}))(\frac{1}{\sigma_k} + \theta_{1(k+1)})] \\
&= \frac{b_k}{(\sigma_n - \theta_{3k})}(\theta_{4,k+1}\theta_{2,k+1} - \theta_{5,k+1}\theta_{1(k+1)})(\sigma_n - \theta_{3(k+1)}) \\
&+\frac{b_k}{(\sigma_n - \theta_{3k})}(\frac{b_k}{\sigma_k}\theta_{2,k+1} - \frac{\mu_k b_k}{\sigma_k}\theta_{1(k+1)}) \\
&+\frac{b_k(\sigma_n - \theta_{3(k+1)})}{(\sigma_n - \theta_{3k})}(\frac{\mu_k}{\sigma_k}\theta_{4,k+1} - \frac{1}{\sigma_k}\theta_{5,k+1}).
\end{aligned}
$$

Then, for ease of notation, define

$$\Theta_1 = \theta_{2,k+1} - \mu_k\theta_{1(k+1)},$$

$$\Theta_2 = \theta_{4,k+1}\theta_{5,k+1} - \mu_k\theta_{4,k+1}^2,$$

$$\Theta_3 = \theta_{4,k+1}\theta_{2,k+1} - \theta_{5,k+1}\theta_{1(k+1)}.$$

Hence

$$
\begin{aligned}
& \frac{Q_{1k}(1 + b_k\theta_{4k})}{Q_{2k}(1 + b_k\theta_{4k}) - (\mu_k + b_k\theta_{5k})} \\
&= \frac{1 + b_k\theta_{4,k+1}}{\Theta_1 + (\sigma_n - \theta_{3(k+1)})\Theta_2 + b_k\Theta_3} \\
&= \frac{Q_{1,k+1}(1 + b_k\theta_{4,k+1})}{Q_{2,k+1}(1 + b_k\theta_{4,k+1}) - (\mu_k + b_k\theta_{5,k+1})}.
\end{aligned}
$$

□

Lemma 3.2. *Let Assumption 3.1 be satisfied. Then for* $k = 1, \cdots, n-1$,

$$-\theta_{5k} + Q_{2k}\theta_{4k} \geq 0. \tag{3.10}$$

Proof: (3.10) can be easily obtained from Assumption 3.1. Since Assumption 3.1 (1) and (4) imply that

$$\frac{\theta_{2k}}{\theta_{1k}} \leq \mu_n, \tag{3.11}$$

then, this inequality with Assumption 3.1 (2) and (5) implies that

$$-\theta_{5k}\theta_{1k} + \theta_{4k}\theta_{2k} \geq 0, \tag{3.12}$$

which implies that for $k = 1, \cdots, n-1$,

$$-\theta_{5k} + Q_{2k}\theta_{4k} = \frac{-\theta_{5k}\theta_{1k} + \theta_{4k}\theta_{2k}}{\theta_{1k} + \theta_{4k}^2(\sigma_n - \theta_{3k})} \geq 0.$$

□

Lemma 3.3. *Let Assumption 3.1 be satisfied. Then for* $k = 1, \cdots, n-1, \quad i = 1, \cdots, n-1,$

$$\mu_i + b_i\theta_{5k} - Q_{2k}(1 + b_i\theta_{4k}) < \mu_{i+1} + b_{i+1}\theta_{5k} - Q_{2k}(1 + b_{i+1}\theta_{4k}). \tag{3.13}$$

Proof: From Assumption 3.1(3) and Lemma 3.2, we have
for $k = 1, \cdots, n-1, \quad i = k, \cdots, n-1,$

$$(b_i - b_{i+1})(\theta_{5k} - Q_{2k}\theta_{4k}) \leq 0. \tag{3.14}$$

This with Assumption 3.1(4) implies that

$$\mu_i + b_i\theta_{5k} - Q_{2k}(1 + b_i\theta_{4k}) < \mu_{i+1} + b_{i+1}\theta_{5k} - Q_{2k}(1 + b_{i+1}\theta_{4k}). \tag{3.15}$$

□

Lemma 3.4. *For* $k = 1, \cdots n,$

$$l'x_k = 1.$$

Proof: Since for $k = 1, \cdots n-1$,

$$l'x_k = \sum_{i=k}^{n-1} (x_k)_i + (x_k)_n$$

$$= \sum_{i=k}^{n-1} \frac{Q_{1k}(1 + b_i\theta_{4k})}{\sigma_i} - \theta_{4k}Q_{1k}$$

$$+t[\sum_{i=k}^{n-1} \frac{\mu_i + b_i\theta_{5k} - Q_{2k}(1 + b_i\theta_{4k})}{\sigma_i} - \theta_{5k} + Q_{2k}\theta_{4k}],$$

and

$$\sum_{i=k}^{n-1} \frac{Q_{1k}(1 + b_i\theta_{4k})}{\sigma_i} - \theta_{4k}Q_{1k}$$

$$= Q_{1k}(\sum_{i=k}^{n-1} \frac{1}{\sigma_i} + \theta_{4k} \sum_{i=k}^{n-1} \frac{b_i}{\sigma_i}) - \theta_{4k}Q_{1k}$$

$$= Q_{1k}(\theta_{1k} + \theta_{4k}(\theta_{4k}(\sigma_n - \theta_{3k}) + 1)) - \theta_{4k}Q_{1k}$$

$$= Q_{1k}(\theta_{1k} + \theta_{4k}^2(\sigma_n - \theta_{3k}))$$

$$= 1,$$

and

$$\sum_{i=k}^{n-1} \frac{\mu_i + b_i\theta_{5k} - Q_{2k}(1 + b_i\theta_{4k})}{\sigma_i} - \theta_{5k} + Q_{2k}\theta_{4k}$$

$$= \sum_{i=k}^{n-1} \frac{\mu_i}{\sigma_i} + \theta_{5k} \sum_{i=k}^{n-1} \frac{b_i}{\sigma_i} - Q_{2k}(\sum_{i=k}^{n-1} \frac{1}{\sigma_i} + \theta_{4k} \sum_{i=k}^{n-1} \frac{b_i}{\sigma_i}) - \theta_{5k} + Q_{2k}\theta_{4k}$$

$$= \theta_{2k} + \theta_{5k}(\theta_{4k}(\sigma_n - \theta_{3k}) + 1) - Q_{2k}(\theta_{1k} + \theta_{4k}(\theta_{4k}(\sigma_n - \theta_{3k}) + 1)) - \theta_{5k} + Q_{2k}\theta_{4k}$$

$$= \theta_{2k} + \theta_{4k}\theta_{5k}(\sigma_n - \theta_{3k}) - Q_{2k}(\theta_{1k} + \theta_{4k}^2(\sigma_n - \theta_{3k}))$$

$$= \theta_{2k} + \theta_{4k}\theta_{5k}(\sigma_n - \theta_{3k}) - (\theta_{2k} + \theta_{4k}\theta_{5k}(\sigma_n - \theta_{3k}))$$

$$= 0,$$

then for $k = 1, \cdots n-1$,

$$l' x_k = 1.$$

For $k = n$, from (3.4), we can directly obtain $l' x_n = 1$.

So, $l' x_k = 1$, for $k = 1, \cdots n$. □

Lemma 3.5. *Let Assumption 3.1 be satisfied and $t_0, \cdots, t_n$ be defined by (3.2). Then for $k = 1, \cdots, n$, $t_{k-1} < t_k$.*

Proof: From Lemma 3.4, we have that $l' x_k = 1$ is satisfied for any value of the parameter t. This implies that the sum of the coefficients of t in x_k equals to zero, i.e.,

$$\sum_{i=k}^{n-1} \frac{\mu_i + b_i \theta_{5k} - Q_{2k}(1 + b_i \theta_{4k})}{\sigma_i} + (-\theta_{5k} + Q_{2k}\theta_{4k}) = 0. \tag{3.16}$$

From Lemma 3.2, we have that

$$-\theta_{5k} + Q_{2k}\theta_{4k} \geq 0.$$

Then

$$\sum_{i=k}^{n-1} \frac{\mu_i + b_i \theta_{5k} - Q_{2k}(1 + b_i \theta_{4k})}{\sigma_i} \leq 0.$$

Lemma 3.2 and Lemma 3.3 imply the existence of an integer ρ_k with $k \leq \rho_k \leq n-1$, such that

$$\mu_i + b_i \theta_{5k} - Q_{2k}(1 + b_i \theta_{4k}) < 0, \quad i = k, \cdots, \rho_k \quad \textit{and} \tag{3.17}$$

$$\mu_i + b_i \theta_{5k} - Q_{2k}(1 + b_i \theta_{4k}) \geq 0, \quad i = \rho_k + 1, \cdots, n-1. \tag{3.18}$$

Hence for $k = 1, \cdots, n-1$,

$$\mu_k + b_1\theta_{5k} - Q_{2k}(1 + b_k\theta_{4k}) < 0. \tag{3.19}$$

(3.19) with Assumption 3.1 (2) implies that

$$t_1 = \frac{Q_{11}(1 + b_1\theta_{41})}{Q_{21}(1 + b_1\theta_{41}) - (\mu_1 + b_1\theta_{51})} > 0 = t_0.$$

For $k = 2, \cdots, n-2$, Lemma 3.1 implies that

$$t_k = \frac{Q_{1k}(1 + b_k\theta_{4k})}{Q_{2k}(1 + b_k\theta_{4k}) - (\mu_k + b_k\theta_{5k})}$$

$$= \frac{Q_{1,k+1}(1 + b_k\theta_{4,k+1})}{Q_{2,k+1}(1 + b_k\theta_{4,k+1}) - (\mu_k + b_k\theta_{5,k+1})}.$$

From Lemma 3.2 and Assumption 3.1 (3), we obtain

$$b_k(Q_{2,k+1}\theta_{4,k+1} - \theta_{5,k+1}) \geq b_{k+1}(Q_{2,k+1}\theta_{4,k+1} - \theta_{5,k+1}).$$

This inequality with Assumption 3.1 (4) implies

$$Q_{2,k+1}(1 + b_k\theta_{4,k+1}) - (\mu_k + b_k\theta_{5,k+1})$$

$$> Q_{2,k+1}(1 + b_{k+1}\theta_{4,k+1}) - (\mu_{k+1} + b_{k+1}\theta_{5,k+1}) > 0. \tag{3.20}$$

(3.20) with Assumption 3.1 (2) and (3) implies that

$$\frac{Q_{1,k+1}(1 + b_k\theta_{4,k+1})}{Q_{2,k+1}(1 + b_k\theta_{4,k+1}) - (\mu_k + b_k\theta_{5,k+1})}$$

$$< \frac{Q_{1,k+1}(1 + b_{(k+1)}\theta_{4,k+1})}{Q_{2,k+1}(1 + b_{(k+1)}\theta_{4,k+1}) - (\mu_{(k+1)} + b_{(k+1)}\theta_{5,k+1})},$$

i.e.

$$t_k < t_{k+1}.$$

For $k = n-1$, $\quad t_k < t_{k+1}$ holds trivially and this completes the proof of Lemma 3.5.

□

Lemma 3.6. *For* $k = 1, \cdots, n, \quad i = 1, \cdots, k-1,$

$$(v_k(t))_i \geq 0, \quad for \qquad t \quad with \quad t \geq t_{k-1}. \tag{3.21}$$

Proof: From (3.2) to (3.7), for $k = 1, \cdots, n-1, \quad i = 1, \cdots, k-1,$

$$\begin{aligned} &(v_k(t_{k-1}))_{k-1} \\ &= -Q_{1k}(1 + b_{k-1}\theta_{4k}) + \frac{Q_{1,k-1}(1 + b_{k-1}\theta_{4,k-1})(Q_{2k}(1 + b_{k-1}\theta_{4k}) - b_{k-1}\theta_{5k} - \mu_{k-1})}{Q_{2,k-1}(1 + b_{k-1}\theta_{4,k-1}) - (\mu_{k-1} + b_{k-1}\theta_{5,k-1})}. \end{aligned}$$

Re-arranging and then applying Lemma 3.1 gives

$$(v_k(t_{k-1}))_{k-1} = 0. \tag{3.22}$$

Using (3.7), Assumptions 3.1 (2),(3), and Lemma 3.3, we have

$$(v_k(t))_{i-1} \geq (v_k(t))_i, \quad 2 \leq i \leq k-1, \quad t \geq 0. \tag{3.23}$$

So

$$(v_k(t))_{i-1} \geq (v_k(t))_{k-1}. \tag{3.24}$$

Using Lemma 3.3 and (3.19)

$$Q_{2k} + b_{k-1}(Q_{2k}\theta_{4k} - \theta_{5k}) - \mu_{k-1} > Q_{2k} + b_k(Q_{2k}\theta_{4k} - \theta_{5k}) - \mu_k > 0. \tag{3.25}$$

By definition of v_k , $(v_k(t))_{k-1}$ are strictly increasing functions of t. This with (3.22) and (3.24) implies that (3.21) is satisfied for all t with $t \geq t_{k-1}$ when $k = 1, \cdots, n-1, \quad i = 1, \cdots, k-1$.

In order to show (3.21), it remains to prove that for $i = 1, \cdots, n-1$,

$$(v_n(t))_i \geq 0, \quad for \qquad t \quad with \quad t \geq t_{n-1}. \tag{3.26}$$

We first prove that

$$t_{n-1} = \frac{\sigma_n - b_{n-1}}{\mu_n - \mu_{n-1}}. \tag{3.27}$$

Since

$$t_{n-1} = \frac{Q_{1,n-1}(1 + b_{n-1}\theta_{4,n-1})}{Q_{2,n-1}(1 + b_{n-1}\theta_{4,n-1}) - (\mu_{n-1} + b_{n-1}\theta_{5,n-1})}, \tag{3.28}$$

where

$$\begin{aligned} & 1 + b_{n-1}\theta_{4,n-1} \\ & = 1 + \frac{b_{n-1}(b_{n-1} - \sigma_{n-1})}{\sigma_{n-1}\sigma_n - b_{n-1}^2} \\ & = \frac{\sigma_{n-1}(\sigma_n - b_{n-1})}{\sigma_{n-1}\sigma_n - b_{n-1}^2}, \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} & \frac{Q_{2,n-1}(1 + b_{n-1}\theta_{4,n-1})}{Q_{1,n-1}} \\ & = (\theta_{2,n-1} + \theta_{4,n-1}\theta_{5,n-1}(\sigma_n - \theta_{3,n-1}))(1 + b_{n-1}\theta_{4,n-1}) \\ & = \frac{\mu_{n-1}(\sigma_{n-1}\sigma_n - b_{n-1}^2) + (b_{n-1} - \sigma_{n-1})(\mu_{n-1}b_{n-1} - \mu_n\sigma_{n-1})}{(\sigma_{n-1}\sigma_n - b_{n-1}^2)^2}(\sigma_n - b_{n-1}) \\ & = \frac{\mu_{n-1}(\sigma_n - b_{n-1})(\sigma_{n-1}\sigma_n - b_{n-1}^2)}{(\sigma_{n-1}\sigma_n - b_{n-1}^2)^2} + \frac{b_{n-1}(\sigma_n - b_{n-1})(\mu_{n-1}b_{n-1} - \mu_n\sigma_{n-1})}{(\sigma_{n-1}\sigma_n - b_{n-1}^2)^2} \\ & - \frac{\sigma_{n-1}(\sigma_n - b_{n-1})(\mu_{n-1}b_{n-1} - \mu_n\sigma_{n-1})}{(\sigma_{n-1}\sigma_n - b_{n-1}^2)^2}, \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} & \frac{\mu_{n-1} + b_{n-1}\theta_{5,n-1}}{Q_{1,n-1}} \\ & = \frac{(\mu_{n-1}(\sigma_{n-1}\sigma_n - b_{n-1}^2) + b_{n-1}(\mu_{n-1}b_{n-1} - \mu_n\sigma_{n-1}))((\sigma_n - b_{n-1}) + (\sigma_{n-1} - b_{n-1}))}{(\sigma_{n-1}\sigma_n - b_{n-1}^2)^2} \end{aligned}$$

$$= \frac{\mu_{n-1}(\sigma_n - b_{n-1})(\sigma_{n-1}\sigma_n - b_{n-1}^2)}{(\sigma_{n-1}\sigma_n - b_{n-1}^2)^2} + \frac{b_{n-1}(\sigma_n - b_{n-1})(\mu_{n-1}b_{n-1} - \mu_n\sigma_{n-1})}{(\sigma_{n-1}\sigma_n - b_{n-1}^2)^2}$$
$$+ \frac{(\mu_{n-1}(\sigma_{n-1}\sigma_n - b_{n-1}^2) + b_{n-1}(\mu_{n-1}b_{n-1} - \mu_n\sigma_{n-1}))(\sigma_{n-1} - b_{n-1})}{(\sigma_{n-1}\sigma_n - b_{n-1}^2)^2}, \tag{3.31}$$

then from (3.30) and (3.31), we obtain that

$$\frac{Q_{2,n-1}(1 + b_{n-1}\theta_{4,n-1})}{Q_{1,n-1}} - \frac{\mu_{n-1} + b_{n-1}\theta_{5,n-1}}{Q_{1,n-1}}$$
$$= \frac{\mu_n\sigma_{n-1} - \sigma_{n-1}\mu_{n-1}}{\sigma_{n-1}\sigma_n - b_{n-1}^2}.$$

Substitution this and (3.29) into (3.28), we proved (3.27). Now we will prove (3.26). From (3.8) and Assumptions 3.1 (3),(4), we have that

$$(v_n(t))_{i-1} \geq (v_n(t))_i, \quad 2 \leq i \leq n-1, \quad t \geq 0. \tag{3.32}$$

Since

$$(v_n(t_{n-1}))_{n-1} = -\sigma_n + b_{n-1} + t_{n-1}(\mu_n - \mu_{n-1}),$$

then this with (3.27), we obtain that

$$(v_n(t_{n-1}))_{n-1} = 0. \tag{3.33}$$

By definition of $v_n(t)$, $(v_n(t))_{n-1}$ is strictly increasing function of t. This with (3.32) and (3.33) implies that (3.26) is satisfied. □

3.2 A Closed Form Solution for Assets with Lower Bounds

Theorem 3.1. *Let Assumption 3.1 be satisfied and $t_0, \cdots, t_n$ be defined by (3.2). Then for $k = 1, \cdots, n$,*

(a) $t_{k-1} < t_k$,

(b) $x(t) = x_k(t)$, for all $t \in [t_{k-1}, t_k]$, is optimal for (3.1) with $x_k(t)$ being given by (3.3) and (3.4),

(c) the multiplier for the budget constraint is given by $u(t) = u_k(t)$ for all $t \in [t_{k-1}, t_k]$, where $u_k(t)$ is given by (3.5) and (3.6),

(d) the multiplier for the lower bounds are given by $v(t) = v_k(t)$ for all $t \in [t_{k-1}, t_k]$, where $v_k(t)$ is given by (3.7) and (3.8).

Proof: Lemma 3.5 completes the proof of part(a).

With Assumption 3.1 (1), the KKT conditions are both necessary and sufficient for optimal problem (3.1)(See [8]). These conditions are

$$\begin{cases} x \geq 0, & l'x = 1, \\ t\mu - \Sigma x = ul - v, & v \geq 0, \\ v'x = 0. \end{cases} \tag{3.34}$$

From Lemma 3.4, we know that $l'x_k = 1$, for $k = 1, \cdots n$.

In addition, $t\mu - \Sigma x_k = u_k l - v_k$ also follows directly (3.3)— (3.7).

Furthermore, the definitions of x_k, v_k imply that $v_k' x_k = 0$.

In order to show (3.34), it remains to prove that for all t with $t_{k-1} \leq t \leq t_k$, the following inequalities are satisfied.

$$(x_k)_i \geq 0, \quad i = k, \cdots, n \tag{3.35}$$

and

$$(v_k)_i \geq 0, \quad i = 1, \cdots, k-1. \tag{3.36}$$

When $k = n$, the coefficient of t vanishes. Thus, (3.35) holds for $k = n$. Now let k be such that $1 \leq k \leq n-1$. From (3.10), (3.18), Assumption 3.1 (1) and (2), it follows that $(x_k)_i \geq 0$ for $i = \rho_k + 1, \cdots, n$ and

$$t \geq 0. \tag{3.37}$$

In order to prove $(x_k)_i$ also satisfying the lower bounds for i with $k \leq i \leq \rho_k$, it follows from (3.3) and (3.17) that t must satisfy

$$t \leq min\{\frac{Q_{1k}(1 + b_i\theta_{4k})}{Q_{2k}(1 + b_i\theta_{4k}) - (\mu_i + b_i\theta_{5k})} \mid i = k, \cdots, \rho_k\}. \tag{3.38}$$

Assumption 3.1 (1), (2) and (3) imply that

$$Q_{1k}(1 + b_k\theta_{4k}) \leq Q_{1k}(1 + b_i\theta_{4k}) \quad i = k+1, \cdots, n-1 \tag{3.39}$$

and from Lemma 3.3, we have

$$\mu_k + b_k\theta_{5k} - Q_{2k}(1 + b_k\theta_{4k}) < \mu_i + b_i\theta_{5k} - Q_{2k}(1 + b_i\theta_{4k}),$$

$$i = k+1, \cdots, n-1. \tag{3.40}$$

From (3.17), $\mu_i + b_i\theta_{5k} - Q_{2k}(1 + b_i\theta_{4k}) < 0, \quad i = k, \cdots, \rho_k$ and from Assumption 3.1 (1) and (2), $Q_{1k}(1 + b_i\theta_{4k}) > 0$ for $i = 1, \cdots, n-1$.

It now follows from (3.39) and (3.40) that

$$\frac{Q_{1k}(1 + b_k\theta_{4k})}{Q_{2k}(1 + b_k\theta_{4k}) - (\mu_k + b_k\theta_{5k})} < \frac{Q_{1k}(1 + b_i\theta_{4k})}{Q_{2k}(1 + b_i\theta_{4k}) - (\mu_i + b_i\theta_{5k})},$$

$$i = k+1, \cdots, \rho_k. \tag{3.41}$$

Inequality (3.41) implies that

$$min\{\frac{Q_{1k}(1+b_i\theta_{4k})}{Q_{2k}(1+b_i\theta_{4k})-(\mu_i+b_i\theta_{5k})} \mid i=k,\cdots,\rho_k\}$$

$$=\frac{Q_{1k}(1+b_k\theta_{4k})}{Q_{2k}(1+b_k\theta_{4k})-(\mu_k+b_k\theta_{5k})}=t_k.$$

(3.37) and (3.38) imply that

$$(x_k)_i \geq 0, \quad i=k,\cdots,n \tag{3.42}$$

for all t with $0 \leq t \leq t_k$.

From Lemma 3.6, we have

$$(v_k(t))_i \geq 0, \quad t \geq t_{k-1}. \tag{3.43}$$

Thus, (3.42) together with (3.43) imply that (3.35) and (3.36) are satisfied simultaneously for $t_{k-1} \leq t \leq t_k$ which completes the proof of the theorem. □

The principal result for (3.1) with Assumption 3.1 is as follow. For $t=0$, all assets are held positively with values given below. As t is increased, eventually asset 1 is reduced to zero at $t=t_1$. Asset 1 will remain at zero for all $t \geq t_1$. Assets $2,\cdots,n$ will be held positively for $t \geq t_1$ until asset 2 is reduced to zero when $t=t_2$. Now both assets 1 and 2 remain at zero for all $t \geq t_2$ and assets $3,\cdots,n$ are held positively until asset 3 is reduced to zero when $t=t_3$. The process continues with the assets dropping out and staying out in the order of their indices. This monotonicity result is a consequence of Assumption 3.1. This result is illustrated in Figure 3.1.

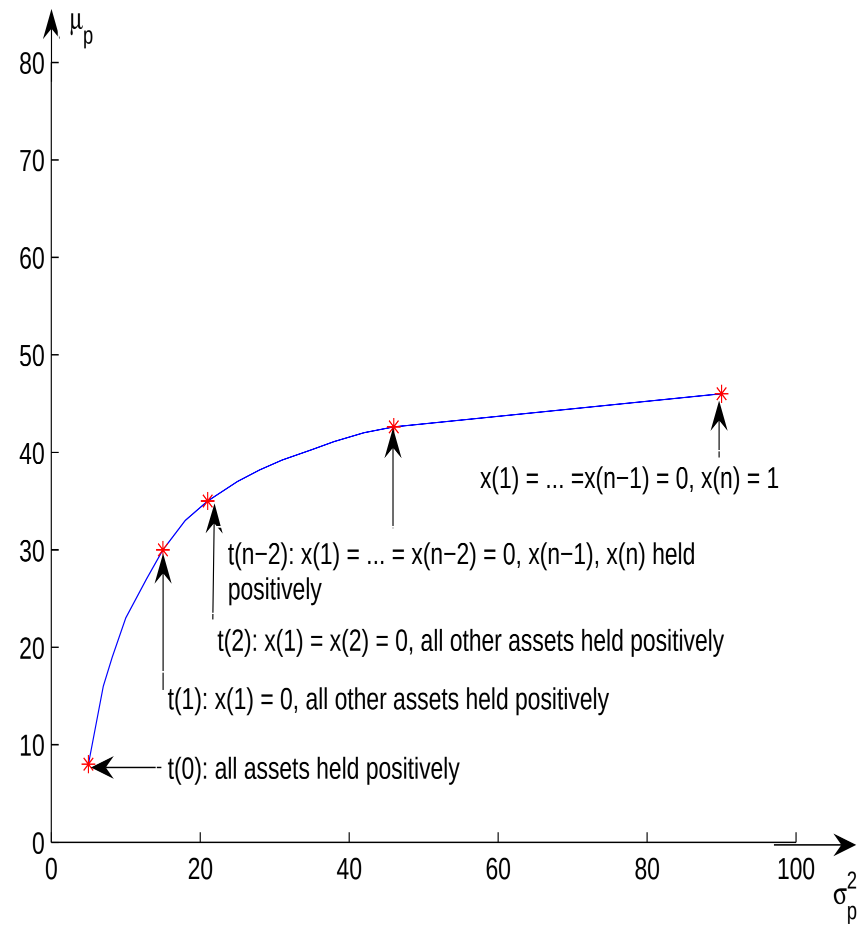

Figure 3.1: Efficient Frontier for Risk Assets with No Short Sales

3.3 Example

Example 3.1. *Find all points on the efficient frontier for (3.1) for the problem with* $n = 3$, $\mu = (1, 2, 3)'$ *and*

$$\Sigma = \begin{bmatrix} 1 & 0 & 0.2 \\ 0 & 4 & 0.1 \\ 0.2 & 0.1 & 3 \end{bmatrix}.$$

The results of applying Theorem 3.1 are summarized in Table 3.1. Observe that in the first interval, $x_1(t) > 0$ *for* $t_0 \leq t < t_1$, $(x_1(t))_1$ *is decreasing in* t *and is reduced to 0 for* $t = t_1$. *In interval 2,* $(x_2)(t)_1 = 0$, $(x_2(t))_2$ *is decreasing in* t *and is reduced to 0 when* $t = t_2$. *Furthermore, the multiplier for the non-negativity constraint on asset 1, namely* $(v_2(t))_1$, *is increasing in* t. *In interval 3, everything is placed in asset 3 and the multipliers for the non-negativity constraints on asset 1 and asset 2, namely* $(v_3(t))_1$ *and* $(v_3(t))_2$, *respectively, are increasing functions of* t. *This results are summarized in Table 3.1.*

Comparing Table 3.1 with Table 3.2, we find that the solution of example 3.1 agrees with results obtained by QP algorithm when $t = 0, t = 1, t = 1.0284, t = 2.9$.

Table 3.1: Optimal Solution for Example 3.1

Interval 1:	$t_0 = 0 \leq t \leq 1.0284 = t_1$	
$Q_{11} = 0.6882$	$Q_{21} = 1.5282$	u_1 = -0.6882 + 1.5282t
index	optimal portfolio $x_1(t)$	dual variables $v_1(t)$
1	0.6521 - 0.6341t	0.0000 + 0.0000t
2	0.1675 + 0.1047t	0.0000 + 0.0000t
3	0.1803 + 0.5294t	0.0000 + 0.0000t
Interval 2:	$t_1 = 1.0284 \leq t \leq 2.9000 = t_2$	
$Q_{12} = 1.7632$	$Q_{22} = 2.5735$	u_2 = -1.7632+2.5735t
index	optimal portfolio $x_2(t)$	dual variables $v_2(t)$
1	0.0000 + 0.0000t	-1.6485 + 0.6029t
2	0.4265 - 0.1471t	0.0000 + 0.0000t
3	0.5735 + 0.1471t	0.0000 + 0.0000t
Interval 3:	$t_2 = 2.9000 \leq t \leq \infty = t_3$	
$\sigma_3 = 3$	$\mu_3 = 3$	u_3 = -3 + 3t
index	optimal portfolio $x_3(t)$	dual variables $v_3(t)$
1	0.0000 + 0.0000t	-2.8000 + 2.0000t
2	0.0000 + 0.0000t	-2.9000 + 1.0000t
3	1.0000 + 0.0000t	0.0000 + 0.0000t

Table 3.2: Results of Example 3.1 Using QP Algorithm

t	Efficient Portfolios		
	1	2	3
0.0000	0.6521	0.1675	0.1803
1.0000	0.0180	0.2723	0.7097
1.0284	0.0000	0.2752	0.7248
2.0000	0.0000	0.1324	0.8676
2.9000	0.0000	0.0000	1.0000

Chapter 4

Assets with Lower Bounds and with a Risk Free Asset

4.1 Several Important Results

In this section, we will look at a problem closely related to that of the previous section. We will consider the following problem with lower bounds

$$\min\{-t(\mu_0 x_0 + \mu' x) + \frac{1}{2} x' \Sigma x \mid \quad l' x + x_0 = 1, \quad x \geq 0, \quad x_0 \geq 0\}, \tag{4.1}$$

where x_0 denotes the proportion of wealth invested in the risk free asset, μ_0 denotes the expected return of asset x_0 and the covariance matrix Σ is the triple-branch matrix as in Chapter 2.

Assumption 4.1.

1) $\sigma_i > 0, \quad i = 1, \cdots n, \quad \sigma_n - \theta_{31} > 0,$

2) $-\theta_{4k} > 0, \quad 1 + b_i\theta_{4k} > 0, \quad k = 1, \cdots n-1, \; i = k, \cdots n-1,$

3) $b_i \geq b_{i+1}, \quad i = 1, \cdots n-2,$

4) $\mu_i < \mu_{i+1}, \quad i = 0, \cdots n-1,$

5) $\theta_{5i} \leq \mu_n\theta_{4i}, \quad i = 1, \cdots, n-1,$

6) $\mu_0(1 + b_1\theta_{41}) < \mu_1 + b_1\theta_{51}.$

First we will discuss the problem

$$\min\{-t(\mu_0 x_0 + \mu' x) + \frac{1}{2}x'\Sigma x \mid \quad l'x + x_0 = 1\} \tag{4.2}$$

that is closely related to (4.1).

Lemma 4.1. *Let Assumption 4.1 be satisfied. The optimal solution for (4.2) is*

$$\begin{cases} x_0 = 1 - t(Q_{21} - \mu_0)/Q_{11}, \\ x_i = t[(\mu_i - \mu_0) + b_i(\theta_{51} - \mu_0\theta_{41})]/\sigma_i, \quad i = 1, \cdots, n-1, \\ x_n = t(\mu_0\theta_{41} - \theta_{51}), \end{cases} \tag{4.3}$$

where $\theta_{41}, \theta_{51}, Q_{11}$ *and* Q_{21} *are defined as from (2.5) to (2.9) for* $k = 1$ *and the multiplier for the budget constraint is* $u = t\mu_0$.

Proof: By the KKT conditions, we have that

$$\begin{cases} x_0 + l'x = 1, \\ t\mu - \Sigma x = ul, \;\; u = t\mu_0. \end{cases} \tag{4.4}$$

Hence

$$\begin{cases} x_0 = 1 - tl'\Sigma^{-1}(\mu - \mu_0 l), \\ x = t\Sigma^{-1}(\mu - \mu_0 l). \end{cases} \tag{4.5}$$

(4.5) with (2.20) and (2.21) implies (4.3). $\square$

Now we will continue to consider the optimal problem (4.1). For $k = -1, \cdots, n$, define

$$t_k = \begin{cases} 0, & k = -1, \\ \frac{Q_{11}}{Q_{21} - \mu_0}, & k = 0, \\ \frac{Q_{1k}(1 + b_k\theta_{4k})}{Q_{2k}(1 + b_k\theta_{4k}) - (\mu_k + b_k\theta_{5k})}, & k = 1, \cdots, n-1, \\ \infty, & k = n. \end{cases} \tag{4.6}$$

For $k = 0, 1, \cdots, n$, let

$$x_k = x_k(t) = ((x_k)_0, (x_k)_1, \cdots, (x_k)_n)',$$

$$u_k = u_k(t),$$

$$v_k = v_k(t) = ((v_k)_0, (v_k)_1, \cdots, (v_k)_n)'.$$

For $k = 0$, define

$$(x_0)_i = \begin{cases} 1 - t(Q_{21} - \mu_0)/Q_{11}, & i = 0, \\ t[(\mu_i - \mu_0) + b_i(\theta_{51} - \mu_0\theta_{41})]/\sigma_i, & i = 1, \cdots, n-1, \\ t(\mu_0\theta_{41} - \theta_{51}), & i = n, \end{cases} \tag{4.7}$$

$$u_0 = t\mu_0, \tag{4.8}$$

$$v_0 = 0. \tag{4.9}$$

For $k = 1, \cdots, n-1$, define

$$\begin{cases} (x_k)_i = 0, & i = 0, \cdots, k-1, \\ (x_k)_i = (x_k(t))_i & \\ = [Q_{1k}(1 + b_i\theta_{4k}) + t(\mu_i + b_i\theta_{5k} - Q_{2k}(1 + b_i\theta_{4k}))]/\sigma_i, & i = k, \cdots, n-1, \\ (x_k)_n = -\theta_{4k}Q_{1k} + t[-\theta_{5k} + Q_{2k}\theta_{4k}], & \end{cases} \tag{4.10}$$

$$\begin{cases} (x_n)_i = 0, \quad i = 0, \cdots, n-1, \\ (x_n)_n = 1. \end{cases} \tag{4.11}$$

For $k = 1, \cdots, n-1$, define

$$u_k(t) = -Q_{1k} + tQ_{2k}, \tag{4.12}$$

$$u_n(t) = -\sigma_n + t\mu_n, \tag{4.13}$$

$$\begin{cases} (v_k)_0 = (v_k(t))_0 = -Q_{1k} + t(Q_{2k} - \mu_0), \\ (v_k)_i = (v_k(t))_i \\ = -Q_{1k}(1 + b_i\theta_{4k}) + t[Q_{2k}(1 + b_i\theta_{4k}) - b_i\theta_{5k} - \mu_i], \quad i = 1, \cdots, k-1, \\ (v_k)_i = 0, \qquad i = k, \cdots, n, \end{cases} \tag{4.14}$$

$$\begin{cases} (v_n)_0 = -\sigma_n + t(\mu_n - \mu_0) \\ (v_n)_i = v_n(t))_i \\ = (-\sigma_n + b_i) + t(\mu_n - \mu_i) \quad i = 1, \cdots, n-1, \\ (v_n)_n = 0. \end{cases} \tag{4.15}$$

where $\theta_{4k}, \theta_{5k}, Q_{1k}$, and Q_{2k} are given from (2.5) to (2.9).

Lemma 4.2. *For $k = 1, \cdots, n-1$,*

$$Q_{2k} - \mu_0 > 0. \tag{4.16}$$

Proof: From Assumption 4.1 (4) and Assumption 4.1 (5), we have

$$\theta_{5k} - \mu_0\theta_{4k} \le 0, \tag{4.17}$$

$$\theta_{2k} - \mu_0\theta_{1k} > 0. \tag{4.18}$$

(4.17) and (4.18) imply

$$\theta_{2k} + \theta_{4k}\theta_{5k}(\sigma_n - \theta_{3k}) - \mu_0(\theta_{1k} + \theta_{4k}^2(\sigma_n - \theta_{3k})) > 0.$$

Thus (4.16) is satisfied. □

Lemma 4.3. *For* $k = 0, \cdots, n$,

$$(v_k(t))_0 \geq 0, \quad t \geq t_0. \tag{4.19}$$

Proof: The KKT conditions for (4.1) are

$$x_0 \geq 0, \quad x \geq 0, \quad l'x + x_0 = 1, \tag{4.20}$$

$$t\mu - \Sigma x = ul - v, \quad v \geq 0, \tag{4.21}$$

$$t\mu_0 = u - v_0, \quad v_0 \geq 0, \tag{4.22}$$

$$v_0 x_0 = 0, \quad v'x = 0, \tag{4.23}$$

where v_0 is the multiplier for the constraint $x_0 \geq 0$ and v is the n-vector of multipliers corresponding to the lower bounds $x_i \geq 0$ for $i = 1, \cdots, n$.

From (4.22), we obtain

$$(v_k)_0 = u_k - t\mu_0,$$

for all $t \geq t_0$.

Substitution of u_k from (4.12) gives

$$(v_k)_0 = -Q_{1k} + t(Q_{2k} - \mu_0). \tag{4.24}$$

From (4.8) and Lemma 4.2, we know that $(v_k)_0$ is increasing in t for any interval $[t_{k-1}, t_k]$. Consequently, (4.19) will be established by verifying that $(v_k(t_{k-1}))_0 \geq 0$. Observe first that from (4.8) and (4.6)

$$(v_k(t_{k-1}))_0 = -Q_{1k} + \frac{Q_{1(k-1)}(1 + b_{k-1}\theta_{4(k-1)})}{Q_{2(k-1)}(1 + b_{k-1}\theta_{4(k-1)}) - (\mu_{k-1} + b_{k-1}\theta_{5(k-1)})}(Q_{2k} - \mu_0).$$

Using Lemma 3.1 gives

$$(v_k(t_{k-1}))_0 = -Q_{1k} + \frac{Q_{1k}(1 + b_{k-1}\theta_{4k})}{Q_{2k}(1 + b_{k-1}\theta_{4k}) - (\mu_{k-1} + b_{k-1}\theta_{5k})}(Q_{2k} - \mu_0).$$

Further re-arranging leads to

$$(v_k(t_{k-1}))_0 = \frac{Q_{1k}(\mu_{k-1} - \mu_0 + b_{k-1}(\theta_{5k} - \mu_0\theta_{4k}))}{Q_{2k}(1 + b_{k-1}\theta_{4k}) - (\mu_{k-1} + b_{k-1}\theta_{5k})}.$$

From Assumption 4.1 (3) — (6), we obtain

$$\mu_{k-1} - \mu_0 + b_{k-1}(\theta_{5k} - \mu_0\theta_{4k}) \geq \mu_1 - \mu_0 + b_1(\theta_{5k} - \mu_0\theta_{4k}) \geq 0. \tag{4.25}$$

Now (3.20) implies

$$Q_{2k}(1 + b_{k-1}\theta_{4k}) - (\mu_{k-1} + b_{k-1}\theta_{5k})$$

$$\geq Q_{2k}(1 + b_k\theta_{4k}) - (\mu_k + b_k\theta_{5k}) \geq 0. \tag{4.26}$$

In addition, (4.25) and (4.26) imply

$$(v_k(t_{k-1}))_0 \geq 0,$$

and this completes the proof. □

4.2 A Closed Form Solution for Assets with Lower Bounds and a Risk Free Asset

This principal result for (4.1) with $t \geq 0$ is the following theorem.

Theorem 4.1. *Let Assumption 4.1 be satisfied and let $t_{-1}, \cdots, t_n$ be defined by (4.6). Then for $k = 0, \cdots, n$,* *(a)* $t_{k-1} < t_k$, *for* $k = 0, \cdots, n$,

(b) $x(t) = x_k(t)$, *for all* $t \in [t_{k-1}, t_k]$, *is optimal for (4.1) with* $x_0(t)$ *being given by (4.7) and* $x_k(t)$ *being given by (4.10) and (4.11), for* $k = 1, \cdots, n$,

(c) the multiplier for the budget constraint is given by $u(t) = u_k(t)$ *for all* $t \in [t_{k-1}, t_k]$, *where* $u_0(t)$ *is given by (4.8) and* $u_k(t)$ *is given by (4.12) and (4.13),*

(d) the multipliers for the lower bounds are given by $v(t) = v_k(t)$ *for all* $t \in [t_{k-1}, t_k]$, *where* $v_0(t)$ *is given by (4.9) and* $v_k(t)$ *is given by (4.14) and (4.15).*

Proof:

From lemma 4.2, we obtain that

$$t_0 = \frac{Q_{11}}{Q_{21} - \mu_0} > t_{-1} = 0. \tag{4.27}$$

From Assumption 4.1 (6), we obtain

$$\mu_0(1 + b_1\theta_{41}) < \mu_1 + b_1\theta_{51}.$$

Therefore,

$$Q_{11}Q_{21}(1 + b_1\theta_{41}) - Q_{11}(\mu_1 + b_1\theta_{51}) < Q_{11}Q_{21}(1 + b_1\theta_{41}) - \mu_0 Q_{11}(1 + b_1\theta_{41}),$$

i.e.,

$$t_0 = \frac{Q_{11}}{Q_{21} - \mu_0} < \frac{Q_{11}(1 + b_1\theta_{41})}{Q_{21}(1 + b_1\theta_{41}) - (\mu_1 + b_1\theta_{51})} = t_1. \tag{4.28}$$

This with (4.27) and Lemma 3.5 completes the proof of part (a).

The proof of part (b) proceeds according to the two cases $0 \le t \le t_0$ and $t_0 \le t \le t_n$. The KKT conditions for (4.1) are

$$\begin{cases} x_0 \ge 0, \quad x \ge 0, \quad x_0 + l'x = 1, \\ t\mu - \Sigma x = ul - v, \quad v \ge 0, \\ t\mu_0 = u - v_0, \quad v_0 \ge 0, \\ v_0 x_0 = 0, \quad v'x = 0. \end{cases} \tag{4.29}$$

Comparing (4.29) with (3.34) and by Theorem 3.1 and Lemma 4.3, we find that the proof of part (b) only needs to show that $x_0(t)$ is optimal for $0 \le t \le t_0$, i.e., $x_0(t)$ meets the KKT conditions.

Let x_0, u_0 and v_0 be as in the statement of Theorem 4.1 and let $t \in [0, t_0]$. According to Lemma 4.1, x_0 is the solution and u_0 is the multiplier for the budget constraint of (4.2). (4.7) implies that x_0 is the optimal solution of (4.1) for $t = 0$.

From Lemma 4.2, we have

$$(Q_{21} - \mu_0)/Q_{11} \ge 0. \tag{4.30}$$

Thus from (4.7), $(x_0(t))_0$ is a decreasing function of t. Since $(x_0(t_0))_0 = 0$, it follows that $(x_0(t))_0 \ge 0$ for $0 \le t \le t_0$. Assumptions 4.1 (3) - (6) imply that

$$\mu_i - \mu_0 + b_i(\theta_{51} - \mu_0\theta_{41}) \ge \mu_1 - \mu_0 + b_1(\theta_{51} - \mu_0\theta_{41}) > 0. \tag{4.31}$$

$(x_0(t))_i$ is a increasing function of t. Since $(x_0(0))_i = 0$, it follows that $(x_0(t))_i \ge 0$ for $0 \le t \le t_0$, for $i = 1, \cdots, n-1$.
Hence x_0 satisfies the first partition of (4.29).

Comparing (4.4) with (4.29), it is obvious that x_0 satisfies the second and the third partitions of (4.29). Therefore, x_0 satisfies the KKT conditions (4.29). We proved that x_0 is optimal for $0 \leq t \leq t_0$. (4.7), Assumption 4.1 (2), (4) and (5) imply that $(x_0(t))_n$ is a increasing function of t. Since $(x_0(0))_n = 0$, it follows that $(x_0(t))_n$ is indeed optimal for $0 \leq t \leq t_0$.
The proof of part (b) is now complete.

For part (c), we know, for $i = 0, \cdots, n$,

$$(v_0(t))_i = 0, \quad t \in [0, t_0], \tag{4.32}$$

From (4.8), for $k = 1, \cdots, n$,

$$(v_k)_0 = -Q_{1k} + t(Q_{2k} - \mu_0), \tag{4.33}$$

which agrees with the statement of Theorem 4.1.

From Lemma 4.3, we have

$$(v_k(t))_0 \geq 0, \quad for \quad all \quad t \quad with \quad t \geq t_0. \tag{4.34}$$

This with (4.32) and Theorem 3.1 completes the proof of part (d). Thus, all of the KKT conditions for (4.1) are satisfied and the proof is complete. □

This principle result for (4.1) is as follows. For $t = t_{-1} = 0$, only the risk free asset 0 is held, *i.e.*, $x_0(0) = 1$ and $x_i(0) = 0$ for $i = 1, \cdots, n$. As t is increased, the risk free asset 0 is reduced and all risky assets are increased from zero. At $t = t_0$, the risk free asset 0 is reduced to its lower bound 0 and remains there for all $t \geq t_0$. Furthermore, at $t = t_0$, all of the risky assets strictly exceed their lower bounds. As t is increased beyond t_0, the process continues precisely as described by Theorem 3.1. Thus, as t is increased from zero in (4.1), the risk free asset is reduced to its lower bound first, then the first risky asset, then second risky asset and so on. For $t_{-1} \leq t \leq t_0$, the first piece of the efficient frontier for (4.1) in (σ_p, μ_p) space is a straight line, namely the Capital Market Line(CML) space. The remainder of the efficient frontier is piece-wise hyperbolic. The CML meets the efficient frontier for the risky assets at some point at that part for the frontier corresponding to its first parametric interval, where all risky assets strictly exceed their lower bounds. This is illustrated in Figure 4.1.

4.3 Example

Example 4.1. *Find all points on the efficient frontier for (4.1) for the problem with* $n = 3$, $\mu = (1, 2, 3)'$, $\mu_0 = 0.5$ *and*

$$\Sigma = \begin{bmatrix} 1 & 0 & 0.2 \\ 0 & 4 & 0.1 \\ 0.2 & 0.1 & 3 \end{bmatrix}.$$

The results of applying Theorem 4.1 are summarized in Table 4.1. Observe that in the first interval, $x_0(t) > 0$ *for* $t_{-1} \leq t < t_0$, $(x_0(t))_0$ *is decreasing in* t *and is reduced to 0 for* $t = t_0$. *In interval 2,* $(x_1(t))_0 = 0$, $(x_1(t))_1$ *is decreasing in* t *and is reduced to 0 when* $t = t_1$. *Furthermore,the multiplier for the non-negativity constraint on asset 0, namely* $(v_1(t))_0$, *is increasing in* t. *In interval 3,* $(x_1(t))_0 = 0$ *and* $(x_1(t))_1 = 0$, $(x_1(t))_2$ *is decreasing in* t *and is reduced to 0 when* $t = t_1$. *The multipliers for the non-negativity constraints on asset 0 and asset 1, namely* $(v_2(t))_0$ *and* $(v_2(t))_1$, *is increasing in* t. *In interval 4, everything is*

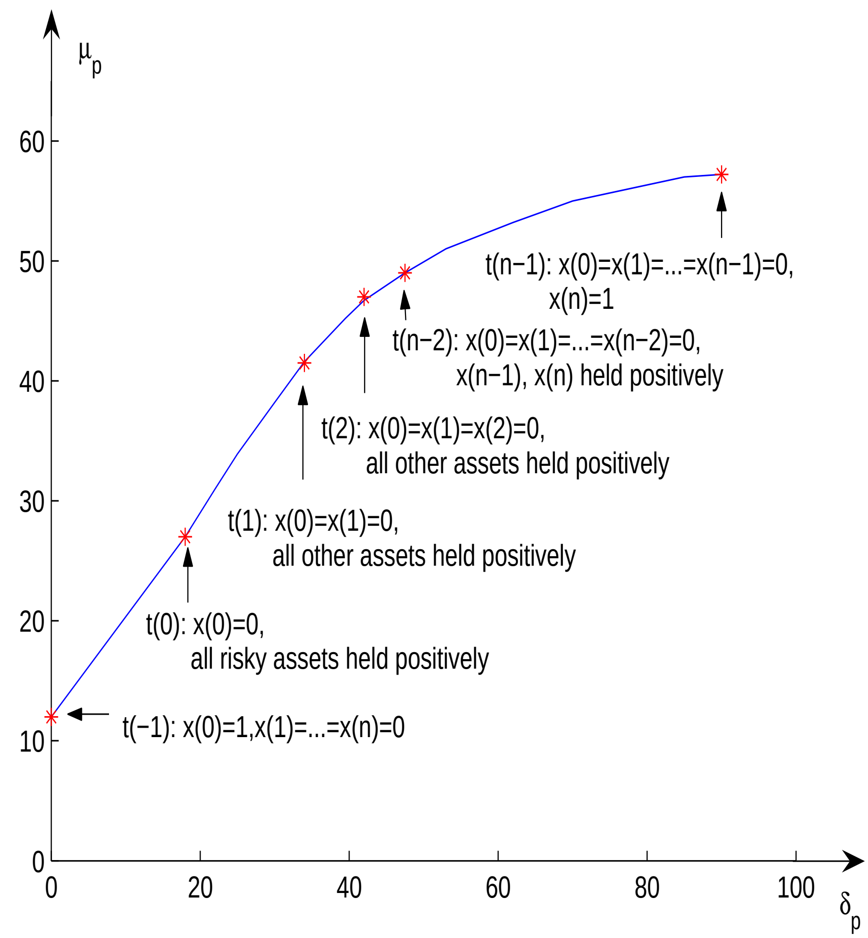

Figure 4.1: Efficient Frontier for Risk Assets/ a Risk Free Asset with No Short Sales

placed in asset 3 and the multipliers for the non-negativity constraints on asset 0, asset 1 and asset 2, namely $(v_3(t))_0$ $(v_3(t))_1$ *and* $(v_3(t))_2$*, respectively, are increasing functions of* t.

Comparing Table 4.1 with Table 4.2, we find that the solution of example 4.1 agrees with results obtained by QP algorithm when $t = 0, t = 0.5, t = 0.6693, t = 1, t = 1.0284, t = 2, t = 2.9$.

Table 4.1: Optimal Solution for Example 4.1

Interval 1:	$t_{-}1 = 0 \leq t \leq 0.6693 = t_0$	
μ_0 =0.5	u_0 = 0.5t	
index	optimal portfolio $x_0(t)$	dual variables $v_0(t)$
0	1.0000 - 1.4941t	0.0000 + 0.0000t
1	0.0000 + 0.3402t	0.0000 + 0.0000t
2	0.0000 + 0.3550t	0.0000 + 0.0000t
3	0.0000 + 0.7988t	0.0000 + 0.0000t
Interval 2:	$t_0 = 0.6693 \leq t \leq 1.0284 = t_1$	
Q_{11} = 0.6882	Q_{21} = 1.5282	u_1 = -0.6882 + 1.5282t
index	optimal portfolio $x_1(t)$	dual variables $v_1(t)$
0	0.0000 + 0.0000t	-0.6882 + 1.0000t
1	0.6521 - 0.6341t	0.0000 + 0.0000t
2	0.1675 + 0.1047t	0.0000 + 0.0000t
3	0.1803 + 0.5294t	0.0000 + 0.0000t
Interval 3:	$t_1 = 1.0284 \leq t \leq 2.9000 = t_2$	
Q_{12} = 1.7632	Q_{22} = 2.5735	u_2 = -1.7632+2.5735t
index	optimal portfolio $x_2(t)$	dual variables $v_2(t)$
0	0.0000 + 0.0000t	-1.7632 + 0.0000t
1	0.0000 + 0.0000t	-1.6485 + 0.6029t
2	0.4265 - 0.1471t	0.0000 + 0.0000t
3	0.5735 + 0.1471t	0.0000 + 0.0000t
Interval 4:	$t_2 = 2.9000 \leq t \leq \infty = t_3$	
σ_3 = 3	μ_3 = 3	u_3 = -3 + 3t
index	optimal portfolio $x_3(t)$	dual variables $v_3(t)$
0	0.0000 + 0.0000t	-3.0000 + 2.5000t
1	0.0000 + 0.0000t	-2.8000 + 2.0000t
2	0.0000 + 0.0000t	-2.9000 + 1.0000t
3	1.0000 + 0.0000t	0.0000 + 0.0000t

Table 4.2: Results of Example 4.1 Using QP Algorithm

t	Efficient Portfolios			
	0	1	2	3
0.0000	1.0000	0.0000	0.0000	0.0000
0.5000	0.2530	0.1701	0.1775	0.3994
0.6693	0.0000	0.2272	0.2376	0.5346
1.0000	0.0000	0.0180	0.2722	0.7097
1.0284	0.0000	0.0000	0.2752	0.7248
2.0000	0.0000	0.0000	0.1324	0.8676
2.9000	0.0000	0.0000	0.0000	1.0000

Chapter 5

Reduction to the Special Case of Uncorrelated Assets

In this chapter, we will verify Theorem 3.1 and Theorem 4.1 by a special case. When $b_i = 0$, for $i = 1, \cdots, n-1$, matrix $\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & b_1 \\ 0 & \sigma_2 & \cdots & b_2 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{n-1} \\ b_1 & b_2 & \cdots & \sigma_n \end{bmatrix}$ becomes a diagonal matrix

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}.$$

When $b_i = 0$, for $i = 1, \cdots, n-1$, equations (2.2) — (2.9) become

$$\theta_{1k} = (\frac{1}{\sigma_k} + ... + \frac{1}{\sigma_{n-1}}), \tag{5.1}$$

$$\theta_{2k} = (\frac{\mu_k}{\sigma_k} + ... + \frac{\mu_{n-1}}{\sigma_{n-1}}), \tag{5.2}$$

$$\theta_{3k} = 0, \tag{5.3}$$

$$\theta_{4k} = -\frac{1}{\sigma_n}, \tag{5.4}$$

$$\theta_{5k} = -\frac{\mu_n}{\sigma_n}, \tag{5.5}$$

$$\theta_{6k} = (\frac{\mu_k^2}{\sigma_k} + ... + \frac{\mu_{n-1}^2}{\sigma_{n-1}}), \tag{5.6}$$

$$Q_{1k} = \frac{1}{\theta_{1k} + \frac{1}{\sigma_n}}, \tag{5.7}$$

$$Q_{2k} = Q_{1k}(\theta_{2k} + \frac{\mu_n}{\sigma_n}). \tag{5.8}$$

5.1 Lower Bounded Assets

The problem to be analyzed is the following n-dimensional problem with lower bounds

$$\min\{-t\mu' x + \frac{1}{2} x' \Sigma x \mid \quad l' x = 1, \quad x \geq 0\}, \tag{5.9}$$

where Σ is a diagonal matrix.

In the case discussed here, Assumption 3.1 can be weakened to the following assumption.

Assumption 5.1.

1) $\sigma_i > 0, for\, i = 1, \cdots n,$

2) $\mu_{i+1} - \mu_i > 0, for\, i = 1, \cdots n-1.$

For $k = 0, 1, \cdots, n$, define

$$t_k = \begin{cases} 0, & k = 0, \\ \frac{Q_{1k}}{Q_{2k}-\mu_k}, & k = 1, \cdots, n-1, \\ \infty, & k = n. \end{cases} \tag{5.10}$$

For $k = 1, \cdots, n$, let

$$x_k = x_k(t) = ((x_k)_1, (x_k)_2, \cdots, (x_k)_n),$$

$$u_k = u_k(t),$$

$$v_k = v_k(t) = ((v_k)_1, (v_k)_2, \cdots, (v_k)_n).$$

For $k = 1, \cdots, n-1$, define

$$\begin{cases} (x_k)_i = 0, & i = 1, \cdots, k-1, \\ (x_k)_i = (x_k(t))_i & \\ = (Q_{1k} + t(\mu_i - Q_{2k})/\sigma_i, & i = k, \cdots, n, \end{cases} \tag{5.11}$$

and define

$$\begin{cases} (x_n)_i = 0, & i = 1, \cdots, n-1, \\ (x_n)_n = 1. \end{cases} \tag{5.12}$$

Furthermore, for $k = 1, \cdots, n-1$, define

$$u_k(t) = -Q_{1k} + tQ_{2k}, \tag{5.13}$$

$$u_n(t) = -\sigma_n + t\mu_n, \tag{5.14}$$

$$\begin{cases} (v_k)_i = (v_k(t))_i = -Q_{1k} + t(Q_{2k} - \mu_i) & i = 1, \cdots, k-1, \\ (v_k)_i = 0, & i = k, \cdots, n. \end{cases} \tag{5.15}$$

Corollary 5.1. *Let $\sum$ be diagonal, let Assumption 5.1 be satisfied and $t_0, \cdots, t_n$ be defined by (5.10). Then for $k = 1, \cdots, n$,*

(a) $t_{k-1} < t_k$;

(b) $x(t) = x_k(t)$, for all $t \in [t_{k-1}, t_k]$, is optimal for (5.9) with $x_k(t)$ being given by (5.11) and (5.12);

(c) the multiplier for the budget constraint is given by $u(t) = u_k(t)$ for all $t \in [t_{k-1}, t_k]$, where $u_k(t)$ is given by (5.13) and (5.14);

(d) the multiplier for the lower bounds are given by $v(t) = v_k(t)$ for all $t \in [t_{k-1}, t_k]$, where $v_k(t)$ is given by (5.15).

This is precisely Theorem 3.1 of Best and Hlouskova [1].

5.2 Example

Example 5.1. *Find all points on the efficient frontier for (5.9) for the problem with $n = 3$, $\mu = (1, 2, 3)'$ and*

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

The results of applying Corollary 5.1 are summarized in Table 5.1. Observe that in the first interval, $x_1(t) > 0$ for $t_0 \leq t < t_1$, $x_{11}(t)$ is decreasing in t and is reduced to 0 for $t = t_1$. In interval 2, $x_{21} = 0$, $x_{22}(t)$ is decreasing in t and is reduced to 0 when $t = t_2$. Furthermore, the multiplier for the non-negativity constraint on asset 1, namely v_{21}, is increasing in t. In interval 3, everything is placed in asset 3 and the multipliers for the non-negativity constraints on asset 1 and asset 2, namely v_{31} and v_{32}, respectively, are increasing functions of t.

5.3 Lower Bounded Assets with a Risk Free Asset

In this section, we will consider the following (n+1)-dimensional problem with lower bounds

$$\min\{-t(\mu_0 x_0 + \mu' x) + \frac{1}{2} x' \Sigma x \mid \quad l' x + x_0 = 1, \quad x \geq 0, \quad x_0 \geq 0\} \tag{5.16}$$

where Σ is a diagonal matrix.

In the case discussed here, Assumption 4.1 can be weakened to the following assumption.

Assumption 5.2.

1) $\sigma_i > 0, for\, i = 1, \cdots n,$

2) $\mu_{i+1} - \mu_i > 0, for\, i = 0, \cdots n-1.$

Table 5.1: Optimal Solution for Example 5.1

Interval 1:	$t_0 = 0 \leq t \leq 1.0909 = t_1$	
Q_{11} = 0.6316	Q_{21} = 1.5789	u_1 = -0.6316 + 1.5789t
index	optimal portfolio $x_1(t)$	dual variables $v_1(t)$
1	0.6316 - 0.5789t	0.0000 + 0.0000t
2	0.1579 + 0.1053t	0.0000 + 0.0000t
3	0.2105 + 0.4737t	0.0000 + 0.0000t
Interval 2:	$t_1 = 1.0909 \leq t \leq 3.0000 = t_2$	
Q_{12} = 1.7143	Q_{22} = 2.5714	u_2 = -1.7143+2.5714t
index	optimal portfolio $x_2(t)$	dual variables $v_2(t)$
1	0.0000 + 0.0000t	-1.7143 + 0.5714t
2	0.4286 - 0.1429t	0.0000 + 0.0000t
3	0.5714 + 0.1429t	0.0000 + 0.0000t
Interval 3:	$t_2 = 3.0000 \leq t \leq \infty = t_3$	
σ_3 = 3	μ_3 = 3	u_3 = -3 + 3t
index	optimal portfolio $x_3(t)$	dual variables $v_3(t)$
1	0.0000 + 0.0000t	-3.0000 + 2.0000t
2	0.0000 + 0.0000t	-3.0000 + 1.0000t
3	1.0000 + 0.0000t	0.0000 + 0.0000t

For $k = -1, \cdots, n$, define

$$t_k = \begin{cases} 0, & k = -1, \\ \frac{Q_{11}}{Q_{21} - \mu_0}, & k = 0, \\ \frac{Q_{1k}}{Q_{2k} - \mu_k}, & k = 1, \cdots, n-1, \\ \infty, & k = n. \end{cases} \tag{5.17}$$

For $k = 0, 1, \cdots, n$, let

$$x_k = x_k(t) = ((x_k)_0, (x_k)_1, \cdots, (x_k)_n),$$

$$u_k = u_k(t),$$

$$v_k = v_k(t) = ((v_k)_0, (v_k)_1, \cdots, (v_k)_n).$$

For $k = 0$, define

$$(x_0)_i = \begin{cases} 1 - t(Q_{21} - \mu_0)/Q_{11}, & i = 0, \\ \frac{t(\mu_i - \mu_0)}{\sigma_i}, & i = 1, \cdots, n, \end{cases} \tag{5.18}$$

$$u_0 = t\mu_0, \tag{5.19}$$

$$v_0 = 0, \tag{5.20}$$

For $k = 1, \cdots, n-1$, define

$$\begin{cases} (x_k)_i = 0, & i = 0, \cdots, k-1, \\ (x_k)_i = (x_k(t))_i & \\ = Q_{1k} + t(\mu_i - Q_{2k})/\sigma_i, & i = k, \cdots, n, \end{cases} \tag{5.21}$$

$$\begin{cases} (x_n)_i = 0, \quad i = 0, \cdots, n-1, \\ (x_n)_n = 1. \end{cases} \tag{5.22}$$

For $k = 1, \cdots, n-1$, define

$$u_k(t) = -Q_{1k} + tQ_{2k}, \tag{5.23}$$

$$u_n(t) = -\sigma_n + t\mu_n, \tag{5.24}$$

$$\begin{cases} (v_k)_i = (v_k(t))_i = -Q_{1k} + t(Q_{2k} - \mu_i) & i = 0, \cdots, k-1, \\ (v_k)_i = 0, & i = k, \cdots, n. \end{cases} \tag{5.25}$$

Corollary 5.2. *Let $\sum$ be diagonal, Assumption 5.2 be satisfied and let $t_-1, \cdots, t_n$ be defined by (5.17). Then for $k = 0, \cdots, n$,*

(a) $t_{k-1} < t_k, for k = 0, \cdots, n$;

(b) $x(t) = x_k(t)$, for all $t \in [t_{k-1}, t_k]$, is optimal for (5.16) with $x_0(t)$ being given by (5.18) and $x_k(t)$ being given by (5.21) and (5.22), for $k = 1, \cdots, n$;

(c) the multiplier for the budget constraint is given by $u(t) = u_k(t)$ for all $t \in [t_{k-1}, t_k]$, where $u_0(t)$ is given by (5.19) and $u_k(t)$ is given by (5.23) and (5.24);

(d)the multiplier for the lower bounds are given by $v(t) = v_k(t)$ for all $t \in [t_{k-1}, t_k]$, where $v_0(t)$ is given by (5.20) and $v_k(t)$ is given by (5.25).

This is precisely Theorem 4.1 of Best and Hlouskova [1].

5.4 Example

Example 5.2. *Find all points on the efficient frontier for (5.16) for the problem with* $n = 3$, $\mu = (1, 2, 3)'$, $\mu_0 = 0.5$ *and*

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

The results of applying Corollary 5.2 are summarized in Table 5.2. Observe that in the first interval, $x_0(t) > 0$ *for* $t_{-1} \leq t < t_0$, $(x_0(t))_0$ *is decreasing in* t *and is reduced to 0 for* $t = t_0$. *In interval 2,* $(x_1(t))_0$, $(x_1(t))_1$ *is decreasing in* t *and is reduced to 0 when* $t = t_1$. *Furthermore,the multiplier for the non-negativity constraint on asset 0, namely* $(v_1)(t)_0$, *is increasing in* t. *In interval 3,* $(x_2(t))_0 = 0$ *and* $(x_2(t))_1 = 0$, $(x_2(t))_2$ *is decreasing in* t *and is reduced to 0 when* $t = t_2$. *The multipliers for the non-negativity constraints on asset 0 and asset 1, namely* $(v_2(t))_0$ *and* $(v_2(t))_1$, *is increasing in* t. *In interval 4, everything is placed in asset 3 and the multipliers for the non-negativity constraints on asset 0, asset 1 and asset 2, namely* $(v_3(t))_0$ $(v_3(t))_1$ *and* $(v_3(t))_2$, *respectively, are increasing functions of* t.

Table 5.2: Optimal Solution for Example 5.2

Interval 1:	$t\_1 = 0 \leq t \leq 0.5854 = t_0$	
μ_0 =0.5	u_0 = 0.5t	
index	optimal portfolio $x_0(t)$	dual variables $v_0(t)$
0	1.0000 - 1.7083t	0.0000 + 0.0000t
1	0.0000 + 0.5000t	0.0000 + 0.0000t
2	0.0000 + 0.3750t	0.0000 + 0.0000t
3	0.0000 + 0.8333t	0.0000 + 0.0000t
Interval 2:	$t_0 = 0.5854 \leq t \leq 1.0909 = t_1$	
Q_{11} = 0.6316	Q_{21} = 1.5789	u_1 = -0.6316 + 1.5789t
index	optimal portfolio $x_1(t)$	dual variables $v_1(t)$
0	0.0000 + 0.0000t	-0.6316 + 1.0000t
1	0.6316 - 0.5789t	0.0000 + 0.0000t
2	0.1579 + 0.1053t	0.0000 + 0.0000t
3	0.2105 + 0.4737t	0.0000 + 0.0000t
Interval 3:	$t_1 = 1.0909 \leq t \leq 3.0000 = t_2$	
Q_{12} = 1.7143	Q_{22} = 2.5714	u_2 = -1.7143+2.5714t
index	optimal portfolio $x_2(t)$	dual variables $v_2(t)$
0	0.0000 + 0.0000t	-1.7143 +0.0000t
1	0.0000 + 0.0000t	-1.7143 + 0.5714t
2	0.4286 - 0.1429t	0.0000 + 0.0000t
3	0.5714 + 0.1429t	0.0000 + 0.0000t
Interval 4:	$t_2 = 3.0000 \leq t \leq \infty = t_3$	
σ_3 = 3	μ_3 = 3	u_3 = -3 + 3t
index	optimal portfolio $x_3(t)$	dual variables $v_3(t)$
0	0.0000 + 0.0000t	-3.0000 + 2.5000t
1	0.0000 + 0.0000t	-3.0000 + 2.0000t
2	0.0000 + 0.0000t	-3.0000 + 1.0000t
3	1.0000 + 0.0000t	0.0000 + 0.0000t

Chapter 6

Conclusions

In this essay, we first gave a brief overview of the mean-variance portfolio selection optimization. Further, we discussed three portfolio selection problems of risky, partially correlated assets. In Chapter 2, we discussed a portfolio selection problem of risky, partially correlated assets with an equality constraint and under some technical assumptions developed a closed form solution for the model. In Chapter 3, we discussed a portfolio selection problem of risky, partially correlated assets subject to lower bounds on all asset holdings. And we obtained a closed form solution for the optimal problem with no short sales. In Chapter 4, we also considered the case when the problem in Chapter 3 was augmented by a risk free asset. Under some technical assumptions we obtained a closed form solution for all portfolio corresponding to the efficient frontier about the the third model. In Chapter 5, we found that these results from the essay extend those from [1].

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