Computing Lower Bounds of the Measure of Convex Dimension Traps for the Baker's Map

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The Baker's map is a dynamical system defined on the unit square. For any open set F, the survivor sets $\mathcal{J}(F)$ are given by the points in X whose orbits do not intersect F. Dimension traps are sets F such that $\dim_H \mathcal{J}(F) = 0$, where $\dim_H$ is the Hausdorff Dimension of a set. In this project, we will compute lower bounds of the measures of dimension traps. We obtain a bound of approximately 0.12 for the measure of convex dimension traps with mirror symmetry. Currently, there is no conjectured value for the exact bound.

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Chapter 1

Introduction

Dynamics is the study of iterating functions that map sets to themselves. Usually there is a space X and a function $f : X \to X$. The goal is to notice patterns that occur as f is iterated. There are some notation that is useful when is discussing problems in dynamical systems.

Definition 1.0.1. For $n \geq 1$, we define $f^n(x) = \underbrace{f \circ f \circ f \cdots f}_{\text{n times}}(x)$, with $f^0(x) = x$.

Definition 1.0.2. A dynamical $f : X \to X$ is **invertible** if there exists $f^{-1} : X \to X$ such that $f \circ f^{-1} = id_X$. For $n < 0$, we define $f^n(x) = (f^{-1})^{-n}(x)$. **In this report we will be restricting our attention to invertible maps**.

Definition 1.0.3. The **orbit** for a point $x \in X$ is given by

$$\mathcal{O}(x) = \{f^n(x) \mid n \in \mathbb{Z}\}$$

Definition 1.0.4. If we have an open set H, the **survivor set** for H is defined as

$$\begin{aligned} \mathcal{J}(H) &= \{x : f^n(x) \notin H \text{ for all } n \in \mathbb{Z}\} \\ &= \{x : \mathcal{O}(x) \cap H = \emptyset\}. \end{aligned}$$

Definition 1.0.5. A **periodic point** of period $T > 0$ is a point $w \in X$ such that $f^T(w) = w$, where T is the smallest such number that satisfies $f^T(w) = w$.

Definition 1.0.6. An **eventually periodic point** is a point $w \in X$ such that there exists $s > 0$ such that $f^s(w)$ is periodic. Note that in an invertible dynamical system a point is eventually periodic if and only if it is periodic.

Definition 1.0.7. A **fixed point** is a point $w \in X$ such that $f(w) = w$. A fixed point is **attracting** if there is an open set $U \subset X$ with $w \in U$ such that for all $x \in U$, $\lim_{n\to\infty} f^n(x) = w$. A fixed point is **repelling** if $U \subset X$ with $w \in U$ such that for all $x \in U$, $\exists k \in \mathbb{N}, f^k(x) \notin U$. Note that for a repelling point, it is not necessarily true that $f^k(x) \notin U$ for sufficiently large k. It is only necessary that it is true for some $k \in \mathbb{N}$.

1.0.1 One Dimensional Billiards on a Circle

An example of a Dynamical system is set $X = [0, 1)$ with the function $f(x) = x + \theta \mod 1$. One way to represent this system is to think of points on a circle in the Cartesian plane. Applying one iteration of f can be seen as moving θ distance along a circle with circumference equal to 1.

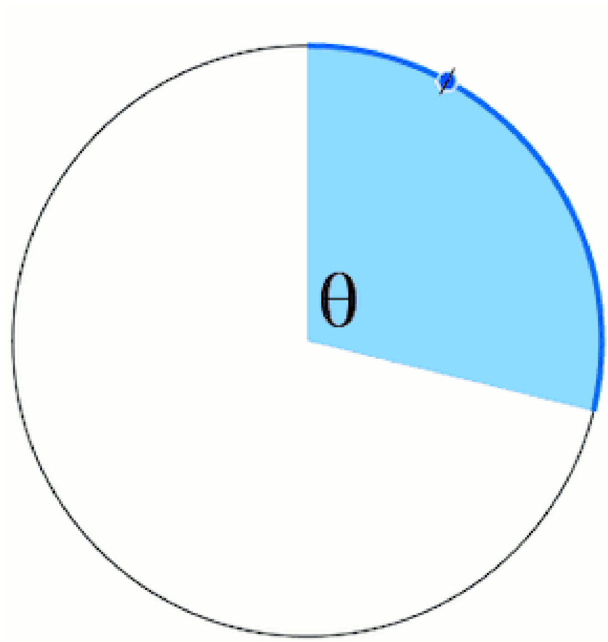


Figure 1.1: Adding by θ mod 1 can be seen as moving along a circle. By Siefkenj - Own work, CC BY 4.0, https://commons.wikimedia.org/wiki/File:Sturmian-sequence-from-irrational-rotation.gif

Some interesting properties of this system are can be found based on whether θ is rational or irrational.

Theorem 1.0.1. If θ is rational, then the orbit $\mathcal{O}(x)$ is finite for any $x \in X$.

Definition 1.0.8. The **closure** of a set S is given by

$$\bar{S} = S \cup \{\lim_{n \to} x_n : x_n \in S \text{ for all } n \in \mathbb{N}\}.$$

We say that S is **dense** in X if $\bar{S} = X$.

Theorem 1.0.2. If θ is irrational, then the orbit $\mathcal{O}(x)$ is dense in $[0, 1]$.

Theorem 1.0.3. (Three-gap Theorem)
Let $\theta \in \mathbb{R}$ be any real number. Let $S = \{i\theta \mod 1 : 0 \leq i \leq n\}$. Let D be the set of distances between adjacent points in S. Then $|D| \leq 3$.

1.1 Hausdorff dimension

There are several notions that need to be defined before we can define Hausdorff dimension.

Definition 1.1.1. The **diameter** of a set $U \in \mathbb{R}^n$ is given by $\mathrm{Diam}(U) = \sup_{x,y\in U} |x-y|$.

Definition 1.1.2. A **δ-cover** of F is a countable collection of open sets $\{U_i\}_{i=1}^{\infty}$ such that

$$F \subset \bigcup_{i=1}^{\infty} U_i$$

with $0 \leq \mathrm{Diam}(U_i) \leq \delta$. In principle such a cover need not be countable. For our purposes it suffices to restrict to countable covers.

Definition 1.1.3. For $s \geq 0$, we define

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} \mathrm{Diam}(U)^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}. \tag{1.1}$$

Definition 1.1.4. We define the **s-dimensional Hausdorff measure** of F as

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F).$$

Lemma 1.1.1. For any δ-cover $\{U_i\}$ and any $t, s \in \mathbb{R}$ with $t > s$ with $\delta < 1$ we have that

$$\sum_{i=1}^{\infty} \mathrm{Diam}(U_i)^t \leq \sum_{i=1}^{\infty} \mathrm{Diam}(U_i)^{t-s}\mathrm{Diam}(U_i)^s \leq \delta^{t-s}\mathrm{Diam}(U_i)^s.$$

It then follows that if $\mathcal{H}^t(F) > 0$, then $\mathcal{H}^s(F) = \infty$ for all $s < t$. It also follows that $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$ for all $s < t$.

Definition 1.1.5. There is a critical value where $\mathcal{H}^s(F)$ jumps from ∞ to 0, which is called the **Hausdorff dimension** of F, denoted $\dim_H F$.

$$\dim_H F := \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$$

Note that $\mathcal{H}^{\dim_H F}(F)$ can be 0, finite, or infinite.

Definition 1.1.6. We define F as a **dimension trap** if $F \subset \mathbb{R}^n$ is open and $\dim_H \mathcal{J}(F) = 0$, where $\mathcal{J}(F)$ is the survivor set of F from the dynamical system of the Baker's map.

1.2 Baker's map

The system we will study is defined as follows. Let $X = [0, 1]^2$ be the unit square and define the Baker's map $B : X \to X$ be given by

$$B(x, y) = \begin{cases} (2x, \frac{y}{2}) & 0 \leq x < \frac{1}{2} \\ (2x - 1, \frac{y+1}{2}) & \frac{1}{2} < x \leq 1 \end{cases}$$

(The map is not defined when $x = \frac{1}{2}$). The name of the Baker's map comes how dough is made. A common method of making dough is to cut the dough in half and stack the dough on top of each other, much like the above operation.

For $(x, y) \in X$, let $x = 0.x_1x_2x_3\ldots, y = 0.y_1y_2y_3\ldots$ be the binary expansions of the coordinates. We will associate with each point (x, y) a bi-infinite sequence $\ldots y_3y_2y_1 \cdot x_1x_2x_3\ldots$. We notice then that

$$\begin{aligned} B(\ldots y_3y_2y_1 \cdot x_1x_2x_3\ldots) &= B(x, y) \\ &= B(0.x_1x_2x_3\ldots,\ \ 0.y_1y_2y_3\ldots) \\ &= (0.x_2x_3\ldots,\ \ 0.x_1y_1y_2y_3\ldots) \\ &= \ldots y_3y_2y_1x_1 \cdot x_2x_3\ldots. \end{aligned}$$

$$\begin{aligned} B^{-1}(\ldots y_3y_2y_1 \cdot x_1x_2x_3\ldots) &= B^{-1}(x, y) \\ &= B(0.x_1x_2x_3\ldots,\ \ 0.y_1y_2y_3\ldots) \\ &= (0.y_1x_1x_2x_3\ldots,\ \ 0.y_2y_3\ldots) \\ &= \ldots y_3y_2y_1 \cdot x_1x_2x_3\ldots. \end{aligned}$$

We can then interpret the Baker's map as shifting the decimal point of bi-infinite representation of points in the unit square. Note that a number such as $x = \frac{1}{2}$ can be represented both as 0.1000... or 0.01111.... We see that, if $x = \frac{1}{2}, 2x - 1 = 0$. Hence, we may assume $\frac{1}{2} = 0.1000\ldots$ instead of 0.01111...). Points with different binary expansions can be ignored as either their forward or backwards orbit goes to $(0, 0)$ or $(1, 1)$. We also ignore points with orbits containing ...00001111... and ...11110000.... as those lie the boundary of the unit square.

Chapter 2

Observations

2.1 Finding a Lower Bound for the area of dimension traps for the Baker's map

The goal of this project will be to estimate a lower bound on the measure of specific subset of dimension traps for the Baker's map. We will need several definitions first.

Definition 2.1.1. Let V be a set of strings. We first define the powers of V^i by

$$\begin{aligned} V^0 &= \{\epsilon\} \\ V^1 &= V \\ V^{i+1} &= \{wv : w \in V^i \text{ and } v \in V\} \text{ for each } i > 0. \end{aligned}$$

The **Kleene star** V^* is given by

$$V^* = \bigcup_{i \geq 0} V^i$$

Definition 2.1.2. For any c be a binary word. That is $c \in \{0,1\}^*$. Let $c^n = \underbrace{cc\ldots c}_{\text{n copies}}$ be the concatenation of c with itself n times.

Definition 2.1.3. For $S \subset X$ and $x \in X$, let $d(x, S) = \inf_{s \in S} |x - s|$.

Definition 2.1.4. For $A, B \subset X$, let $D(A, B) = \max_{a \in A}(d(a, B))$.

Definition 2.1.5. For $A, B \subset X$, let $\mathcal{D}(A, B) = \max(D(A, B), D(B, A))$.

Definition 2.1.6. Let a, b be binary strings. Assume $\nexists j, k \in \mathbb{N}$ such that $a^j = b^k$. For $n \in \mathbb{N}$, the n-th Cantor sets associated a, b are given by

$$C(a, b, n) = \{ab^n, b^{n+1}\}^*.$$

Note that the orbit of $C(a, b, n)$ under the Baker's map, is $C(a, b, n)$. That is, $\mathcal{O}(C(a, b, n)) = C(a, b, n)$.

Definition 2.1.7. Let a, b be given as above. Let $C(a, b)$ be the closed set (i.e compact) set that is limit of the sequence of sets $C(a, b, n)$, which is defined as the set that satisfies

$$\lim_{n \to \infty} \mathcal{D}(C(a, b, n), C(a, b)) = 0.$$

It can be shown that $C(a, b) = \overline{\mathcal{O}(....bbb \cdot abbb...)}$, where $\overline{\mathcal{O}(....bbb \cdot abbb...)}$ is the closure of $\mathcal{O}(....bbb \cdot abbb...)$. Note that $C(a, b)$ itself is not a Cantor set, as it contains only countably many points.

Definition 2.1.8. A set $U \subset \mathbb{R}^2$ is **convex** if for all $x, y \in U$ and $\lambda \in [0, 1], \lambda x + (1 - \lambda) y \in U$.

Definition 2.1.9. The **convex hull** of points $\{(x_1, y_1), \ldots \{x_k, y_k)\}$ is the convex set C satisfying such that for any other convex set C' containing $\{(x_1, y_1), \ldots \{x_k, y_k)\}$, we have $C \subset C'$.

It can be shown that if a set S is convex, it contains the convex hull of any finite collection of points in S.

Our goal is to estimate the lower bound on the measure of dimension traps. In particular, we will follow the work by Clark, Hare and Sidorov [1] Let $\mathcal{T}$ be the set of all convex, open dimension traps which include the points $(1/2, 0), (1/2, 1)$. We will also require traps in $\mathcal{T}$ have mirror symmetry, meaning that if $T \in \mathcal{T}$, then $(x, y) \in T \iff (1-x, 1-y) \in T$. Our goal is estimate $\delta = \inf_{T \in \mathcal{T}} \mathcal{L}(T)$, where $\mathcal{L}$ is the Lebesgue measure.

Definition 2.1.10. For $\epsilon > 0$, let $B_\epsilon \subset \mathbb{R}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \epsilon\}$.

Definition 2.1.11. For $A, B \subset \mathbb{R}^2$, let $A + B = \{(x_1 + x_2, y_1 + y_2) : (x_1, y_1) \in A, (x_2, y_2) \in B\}$.

Let $F \subset \mathbb{R}^2$. It can be shown that if $\overline{F} \cap C(a, b) = \emptyset$, then $\dim_H(\mathcal{J}(F)) > 0$. In particular, there exists $n \in \mathbb{N}$ such that $C(a, b, n) \subset \mathcal{J}(F)$, and $\dim_H C(a, b, n)) > 0$. To see this, we note that $\mathcal{D}(C(a, b), C(a, b, n)) \to 0$, and hence $\forall \epsilon > 0, \exists n$ such that $\mathcal{D}((C(a, b), C(a, b, n)) < \epsilon$. We can choose ϵ sufficiently small so that $(C(a, b) + B_\epsilon) \cap \overline{F} = \emptyset$. Then $C(a, b, n) \cap F = \emptyset$, and $C(a, b, n) \subset \mathcal{J}(F)$. This gives us that $\dim_H(\mathcal{J}(F)) > 0$.
The following Lemma can also be shown using the above fact.

Lemma 2.1.1. Let a, b be defined as in Definition 2.1.5. For $T \in \mathcal{T}$ and $\epsilon > 0$,

$$C(a, b) \cap (T + B_\epsilon) \neq \emptyset.$$

We can show how this Lemma is used in a few examples.

Example 2.1.1. We first note that $C(0, 1), C(1, 0)$ are given by

$$C(0,1) = \bigcup_{k \geq 1} \{(1, 1 - 2^{-k}), (1 - 2^{-k}, 1)\}$$
$$C(1,0) = \bigcup_{k \geq 1} \{(0, 2^{-k}), (2^{-k}, 0)\}$$

When $\mathcal{T}$ was defined, we enforced that for all $T \in \mathcal{T}, (1/2, 1)$ and $(1/2, 0) \in T$. We can see that Lemma 2.1.1 is satisfied for the choices of $a = 0, b = 1$ and $a = 1, b = 0$, since $(1/2, 1) \in C(0, 1)$ and $(1/2, 0) \in C(1, 0)$.

Example 2.1.2. Using Lemma 2.1.1, we will bound the size of dimension traps $T \in \mathcal{T}$. Consider the pair of strings $(10, 01)$. Strings in $C(10, 01)$ must be of one of the following forms.

$$\ldots 1010 \cdot \underbrace{1010\ldots 0101}_{\text{first } k \text{ digits, where } k \text{ is odd}} 100101\ldots = \left(\frac{2}{3} + \frac{1}{2^{k+1}} - \frac{1}{2^{k+2}}, \frac{1}{3}\right)$$

$$\ldots 0101 \cdot \underbrace{0101\ldots 0101}_{k \text{ digits, where } k \text{ is even}} 100101\ldots = \left(\frac{1}{3} + \frac{1}{2^{k+1}} - \frac{1}{2^{k+2}}, \frac{2}{3}\right)$$

$$\ldots 010110 \underbrace{0101\ldots 0101}_{k \text{ digits, where } k \text{ is odd}} \cdot 1010\ldots = \left(\frac{2}{3}, \frac{1}{3} + \frac{1}{2^{k+1}} - \frac{1}{2^{k+2}}\right)$$

$$\ldots 010110 \underbrace{0101\ldots 0101}_{k \text{ digits, where } k \text{ is even}} \cdot 0101\ldots = \left(\frac{1}{3}, \frac{2}{3} + \frac{1}{2^{k+1}} - \frac{1}{2^{k+2}}\right)$$

$$\ldots 01011 \cdot 00101\ldots = \left(\frac{2}{6} + \frac{1}{2}, \frac{1}{6}\right) = \left(\frac{5}{6}, \frac{1}{6}\right)$$

Let $\delta' = 0.13$ be a fixed hyperparameter. Consider $T \in \mathcal{T}$ such that $\mathcal{L}(T) < \delta'$. As we also enforced that $(1/2, 0), (1/2, 1) \in T$, and that T is convex and satisfies relational symmetry $(x, y) \in T \iff (1-x, 1-y) \in T$, then we know that for any point $(x, y) \in T$, the convex hull with vertices $[(1/2, 0), (1/2, 1), (x, y), (1-x, 1-y)]$ must also be contained in T.

By Lemma 2.1.1, given $\epsilon > 0$, there must be at least one point in $C(a, b) \cap (T + B_\epsilon)$. We can check each of the points in $(x, y) \in C(01, 10)$ to see if the convex hull generated $[(1/2, 0), (1/2, 1), (x, y), (1-x, 1-y)]$ has area less than 0.13.
We can see that the only such points that satisfy the above condition are

$$\ldots 010101.0110010101\ldots = \left(\frac{1}{3} + \frac{1}{2^3} - \frac{1}{2^4}, \frac{2}{3}\right) = \left(\frac{19}{48}, \frac{2}{3}\right)$$
$$\ldots 010101.1001010101\ldots = \left(\frac{1}{3} + \frac{1}{2^1} - \frac{1}{2^2}, \frac{2}{3}\right) = \left(\frac{7}{12}, \frac{2}{3}\right).$$

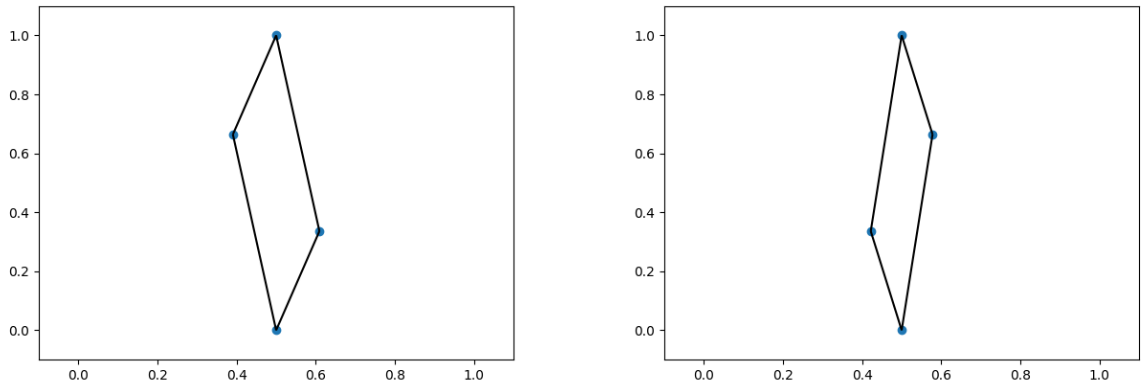


Figure 2.1: Polygons generated from $C(01, 10)$

By Lemma 2.1.1, for any $\epsilon > 0$, $C(01, 10) \cap (T + B_\epsilon) \neq \emptyset$. We then get that T must include either the convex hull of $[(1/2, 0), (1/2, 1), (\frac{19}{48}, \frac{2}{3}), (1 - \frac{19}{48}, \frac{2}{3})]$ or $[(1/2, 0), (1/2, 1), (\frac{7}{12}, \frac{2}{3}), (1 - \frac{7}{12}, \frac{2}{3})]$.

The above convex hulls have areas 0.109375 and 0.0781250 respectively. From this we can that if T is a dimension trap, $(1/2, 0), (1/2, 1) \in T$, and T is closed under symmetry $(x, y) \in T \iff (1 - x, 1 - y)$, then $\mathcal{L}(T) \geq 0.0781250$. Note that we picked $\delta' = 0.13$ based on previous work by Clark, Hare and Sidorov [1]

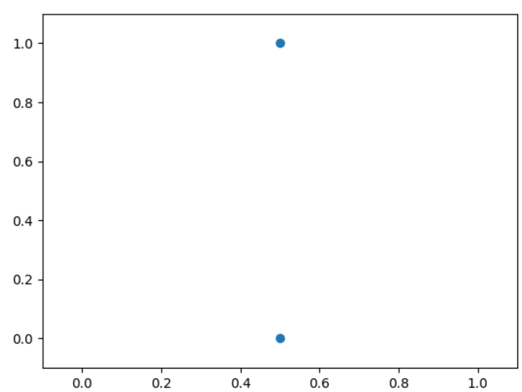
We repeat this process on other Cantor sets to improve the bound.

2.2 Algorithm

Our goal is to find a lower bound on size of dimension traps. In particular, following work by Clark, Hare and Sidorov, we will bound the size of convex, open dimension traps T that have rotational symmetry $(x, y) \in T \iff (1-x, 1-y) \in T$.

We store our polygons as lists of points, representing the vertices of the polygon. Our initial list of polygons is the array A_1 containing the polygon $[(1/2, 0), (1/2, 1)]$, and we set our hyperparameter $\delta' = 0.13$.

Figure 2.2: Initial Points



Our algorithm is composed of 4 nested loops.

1. On the first loop, we iterate from $i = 2$ and increase i by 1 at every iteration. Given $i \geq 2$, let $\{(a_j, b_j) : 1 \leq j \leq l$ denote the set pairs of binary strings of length i, where l is the number of binary strings of length i.

   (a) On the second loop, we iterate through $j = 1$ of all pairs (a_j, b_j) and generate points in $C(a_j, b_j)$. Note that as $C(a_j, b_j)$ is an infinite set, we only generate points in that set up to a certain length, which is determined by our choice of $\epsilon > 0$. In this project we generated points up to length 14. Next, we create a new array B_j of polygons T that intersect $C(a_j, b_j)$ and that also have $\mathcal{L}(T) < \delta'$. We set $B_0 = A_{i-1}$.

i. Given k, let $\{T_k : 1 \leq k \leq m\}$ denote the set polygons that intersected the previous Cantor set, where m is the number of polygons that intersected the previous Cantor set. We create an array C_k for the polygons created from adding points from $C(a_j, b_j)$ to T_k

   A. On the fourth loop, we iterate through the points in $C(a_j, b_j)$. For each point $c_k = (c_k x, c_k y)$, we create a new polygon $T'_{kc} = \text{hull}(T_k \cup \{c\} \cup \{(1 - c_k x, 1 - c_k y\})$. We store all such T'_{kc} that satisfy $\mathcal{L}(T'_{kc}) < 0.13$ in an array C_k.

ii. At the end of the third loop we append all polygons in C_k to B_j

(b) At the end of the second loop, we append all polygons in B_j to A_i

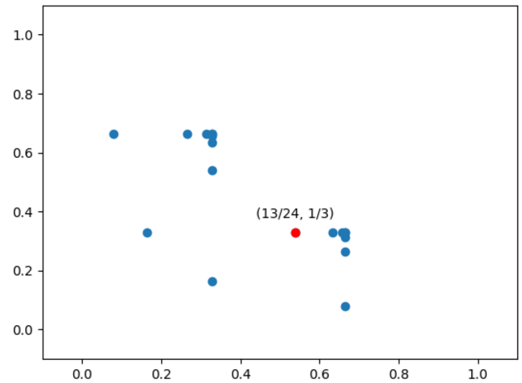
2. At the end of the first loop, we calculate the minimum of the area of all polygons in A_i. This represents a lower bound on the set of dimension traps $\mathcal{T}$ discussed above.

It may be that our δ' value is set too small, and so at the i-th iteration, we cannot a find a single point from Cantor sets associated with any of the j pairs to generate a new polygon that has measure smaller than δ'. In this we case we terminating early, and we know that δ' has become a lower bound of smallest possible dimension trap.

We can look at the first iteration of the algorithm as an example. At iteration 2, we generate pairs $(00, 01), (00, 10), (00, 11), (01, 00), (01, 10), (01, 11), (10, 00), (10, 01), (10, 11),$ $(11, 00), (11, 01), (11, 10)$.

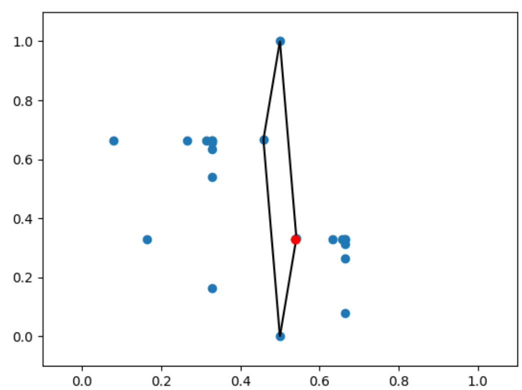
We consider iterate through the initial set of $A_1 = B_0 = \{[(1/2, 0), (1/2, 1)]\}$.
For the first pair $(00, 01)$, we generate points in the Cantor Set $C(00, 01)$.

Figure 2.3: Points in Set $C(00, 01)$



There is only one polygon we can create from these points to the $\{[(1/2, 0), (1/2, 1)]\}$ that has area less than 0.13. This polygon comes from taking the string $\ldots 0101010.1000101\ldots$, which corresponds to the point $(1/3, 2/3 - 1/2^3) = (1/3, 13/24)$. The polygon created from this point has vertices $(1/2, 0), (1/2, 1), (1/3, 13/24), (2/3, 11/24)$, coming from the constraint that we are looking for dimension traps that are closed under symmetry $(x, y) \iff (1 - x, 1 - y)$. This polygon has area equal to 0.0390625.

Figure 2.4: Surviving Polygon created from Set $C(00, 01)$



We append this polygon B_1. The algorithm would continue and repeat the process for $C(00,10)$ and $C(00,11)$, until it has gone through all the sets associated with each pair.

2.3 Dimension of $C(a, b, n)$

As noted in [1], the Hausdorff Dimension and Box counting Dimension of $C(a, b, n)$ are equal. We can give some insight on how to show $\dim_H C(a, b, n) > 0$ by showing $\dim_{\text{box}} C(a, b, n) > 0$.

Definition 2.3.1. The box counting dimension of a set S is given by

$$\dim_{\text{box}}(S) := \lim_{\epsilon \to 0} \frac{\log(N(\epsilon))}{\log(\frac{1}{\epsilon})}$$

where $N(\epsilon)$ counts the minimal number of boxes with side length ϵ needed to cover the set. We will give an example of calculating the box dimension below.

Example 2.3.1. To calculate $\dim_{\text{box}}(C(0, 1, 0))$, we can first calculate the number of words $(x, y) = y_k y_{k-1} \dots y_1 \cdot x_1 \dots x_1 \dots x_k$ that are belong in $C(0, 1, 0)$ as a function of k. Recall that $C(0, 1, 0) = \{01^0, 1^1\}^* = \{0, 1\}^*$. We can see that there are 2^{2k} options for each character x_k, y_k.

For each word $y_k y_{k-1} \dots y_1 \cdot x_1 \dots x_1 \dots x_k$, we can form a square with coordinates

$$\begin{aligned}
\dots 000 y_k y_{k-1} \dots y_1 \cdot x_1 \dots x_1 \dots x_k 000 \dots &= (x, y) \\
\dots 111 y_k y_{k-1} \dots y_1 \cdot x_1 \dots x_1 \dots x_k 000 \dots &= (x, y + 1/2^k) \\
\dots 000 y_k y_{k-1} \dots y_1 \cdot x_1 \dots x_1 \dots x_k 111 \dots &= (x + 1/2^k, y) \\
\dots 111 y_k y_{k-1} \dots y_1 \cdot x_1 \dots x_1 \dots x_k 111 \dots &= (x + 1/2^k, y + 1/2^k).
\end{aligned}$$

This square has area $\frac{1}{2^k} \cdot \frac{1}{2^k}$. We can calculate that for $k \in \mathbb{N}$, and $\epsilon = 1/2^k$,

$$\frac{\log(N(\epsilon))}{\log(1/\epsilon)} = \frac{\log(2^{2k})}{\log(2^k)} = \frac{2k \log(2)}{k \log(2)} = 2$$

As we take $k \to \infty$, we get that $\dim_{\text{box}}(C(0, 1, 0)) = 2$

Example 2.3.2. To calculate $\dim_{\text{box}}(C(0,1,1))$, we can first see that words in $C(0,1,1) = \{01, 11\}^*$ must be in one of two forms. Either they are of the form

$$[y_k y_{k-1}][y_{k-2} y_{k-3}] \dots [y_2 y_1] \cdot [x_1 x_2][x_3 x_4] \dots [x_{k-1} x_k]$$

or

$$[y_k][y_{k-1} y_{k-2}][y_{k-3} y_{k-4}] \dots [y_1 \cdot x_1][x_2 x_3][x_4 x_5] \dots [x_{k-2} x_{k-1}][x_k]$$

where the pairs of characters inside each of the [] are either 01 or 11. The characters on the ends for the second form can be either 0 or 1.
From these two cases, we get that there are $2 \cdot 2^k$ choices, as in each of the two cases, there are k brackets that each have the choice of being 01 or 11.
We can use a similar cover as in the previous example, to get that

$$\frac{\log(N(\epsilon))}{\log(1/\epsilon)} = \frac{\log(2 \cdot 2^k)}{\log(2^k)} = \frac{(k+1)\log(2)}{k \log(2)} = \frac{k+1}{k}$$

As we take $k \to \infty$, we get that $\dim_{\text{box}}(C(0,1,0)) = 1$.

In general, we will see a trend that $\dim_{\text{box}} C(0,1,n) = \frac{2}{n+1}$. We then know that $\dim_H C(0,1,n) > 0$ for $n \in \mathbb{N}$.

2.4 Optimizations

As we go on to further iterations, the number of pairs of binary words increases exponentially. At the ith iteration, we consider binary words of length i, of which there are $m = 2^i$ possible words. As we want to consider pairs of these words, there are $m(m-1) = 2^i \cdot (2^i - 1)$ such pairs!

There were a few ways we could discard polygons at the current iteration.

2.4.1 Polygons that already intersect a given set

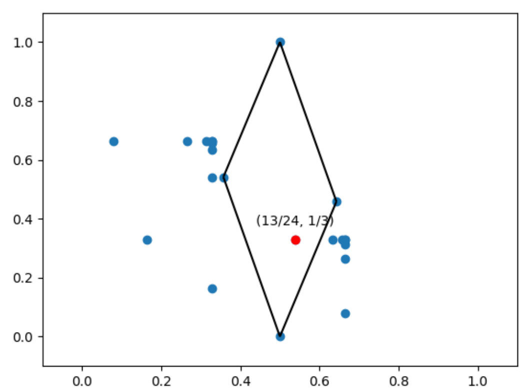
During the algorithm, we used Lemma 2.1.1 to increase the bound on the smallest dimension trap T. For a given pair of strings $a, b \in \{0,1\}^*$, we found points $c \in C(a,b)$ such that $\mathcal{L}(\text{hull}(T \cup \{c\})) > \mathcal{L}(T)$. However, it may be the case that when considering a new pair of strings (a, b), there already exists a point in $C(a, b)$ which lies in the interior of T. If we find such a point, we know that we do not need to consider polygons generated from adding points from set to T.

As an example, suppose we have the polygon given by

$$T = \text{hull}(\{(1/2, 0), (1/2, 1), (5/14, 13/24), (9/14, 11/24)\}).$$

Suppose we are looking to extend the polygon by points in $C(00, 01)$. We see that the point $(13/24, 1/3)$ is already contained with the polygon.

Figure 2.5: Finding a point in a new Cantor set already contained with a polygon



We know that any other polygons generated from extending T will have larger measure. As such, we can break out of the inner loop for adding points from this Cantor set to T, and instead append T itself to the ongoing list of polygons.

2.4.2 Polygons with the same exterior points

It is possible that two polygons from a previous iteration may result in the same polygon in the current iteration after appending a new point from a Cantor set. Consider the polygons given by

$$T_1 = \mathrm{hull}\{(1/2, 0), (1/2, 1), (1/3, 13/24), (2/3, 11/24)\}$$
$$T_2 = \mathrm{hull}\{(1/2, 0), (1/2, 1), (1/3, 10/24), (2/3, 14/24)\}.$$

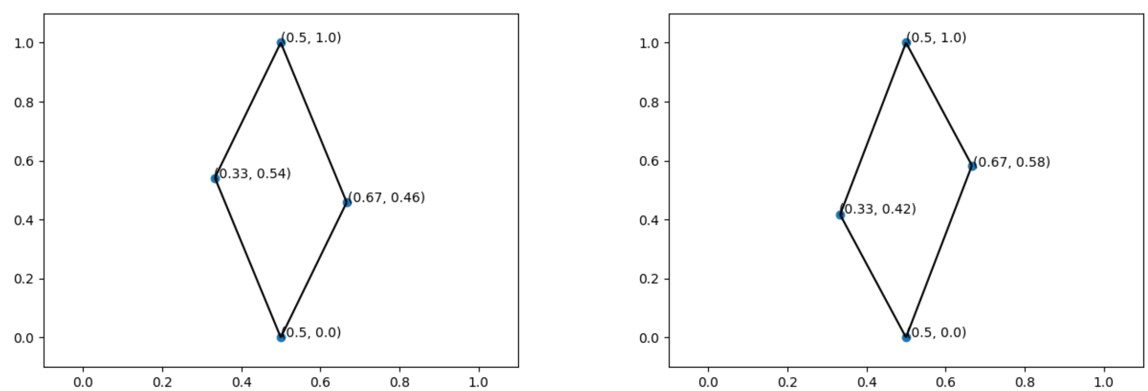
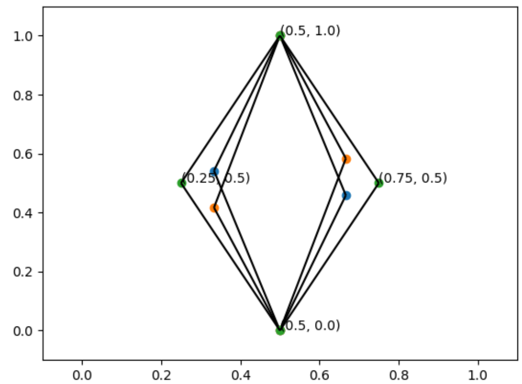


Figure 2.6: Two polygons that are not the same in a given iteration

Suppose a new point $p = (1/4, 1/2)$ belongs to a Cantor Set in the next iteration. For each of the polygons above, adding p along with its mirror under the symmetry $(3/4, 1/2)$ to the polygons and taking the convex hull, we see that we get a new polygon P' with vertices $(1/2, 0), (1/2, 1), (1/4, 1/2), (3/4, 1/2)$.

Figure 2.7: An Polygon that is generated twice from two different polygons



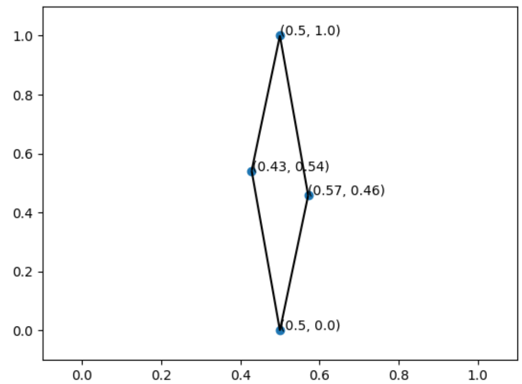
As we identify polygons only by their exterior vertices in the script, we would then get that P' would be generated added to are ongoing list of polygons twice. This decreases the efficiency of our program, so at the end of every iteration, we remove duplicate polygons from our list.

2.4.3 Polygons that contain each other

It is possible that after an iteration is completed, one polygon in the generated list may completely contain another polygon. Consider the polygon given by

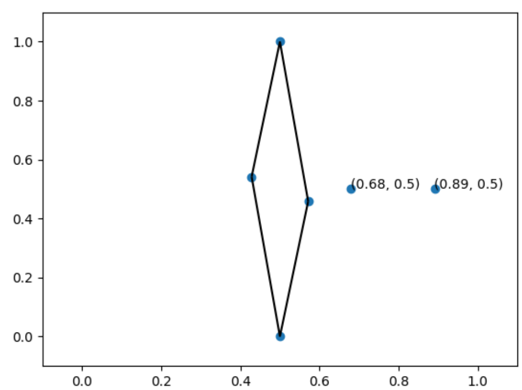
$$T_3 = \text{hull}\{(1/2, 0), (1/2, 1), (1/3, 13/24), (2/3, 11/24)\}.$$

Figure 2.8: T_3, a polygon that will generate polygons T_4, T_5



Suppose the points $(19/28, 1/2), (25/28, 1/2)$ belong to a given Cantor Set.

Figure 2.9: Two points in a given Cantor set

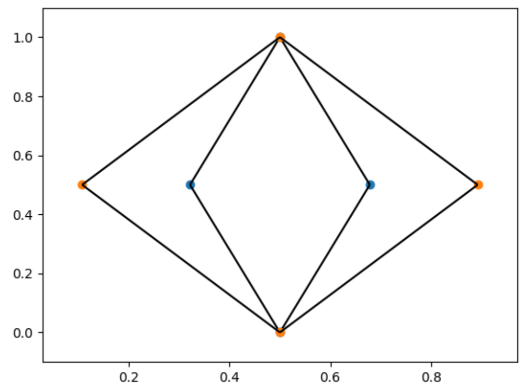


Next, consider the polygons given by

$$T_4 = \text{hull}\{(1/2, 0), (1/2, 1), (1/3, 13/24), (2/3, 11/24)\} \cup (19/28, 1/2), (9/28, 1/2)\}$$
$$T_5 = \text{hull}\{(1/2, 0), (1/2, 1), (1/3, 13/24), (2/3, 11/24)\} \cup (25/28, 1/2), (3/28, 1/2)\}.$$

We see that T_5 completely contains T_4. Furthermore, we know that any polygons generated from appending points from T_5 will contain polygons generated from appending T_4.

Figure 2.10: Polygon that completely contains another Polygon



Such polygons will always have greater or equal measure as $\mathcal{L}(B) \geq \mathcal{L}(A)$ if $B \subset A$. As we are trying to bound $\delta = \inf_{T \in \mathcal{T}} \mathcal{L}(T)$, we see that considering polygons generated from T_5 will not increase the bound. Therefore at the end of every iteration, we discard any polygons that completely contain any other polygons.

2.5 Results

Below is table of several statistics of algorithm. We can see that the time taken increases exponentially as the length of words increases.

Iteration	Best Bound Found	Time Taken (s)	Number of polygons found with area less than 0.13
2	0.090820	0.128545	3
3	0.116455	3.185380	16
4	0.117371	47.517198	42
5	0.117371	170.526676	21
6	0.119385	1434.867396	66

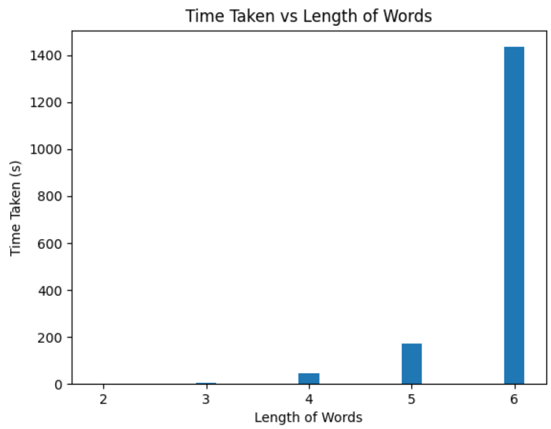


Figure 2.11: Graph of Time Taken at each iteration

The exponential time growth can be seen visually in the graph above. For pairs of words of length 7, we can estimate that it would take approximately 28 hours to compute. We were unable to verify this due to computation power. When considering pairs of length 7 and above, our computer ran out of memory due to the large number of polygons generated.

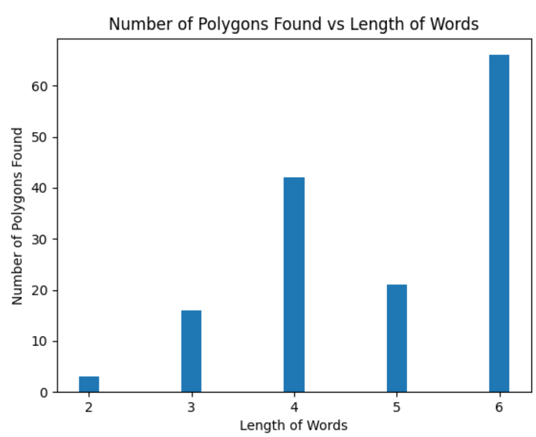


Figure 2.12: Number of polygons found at each iteration with measure less than 0.13

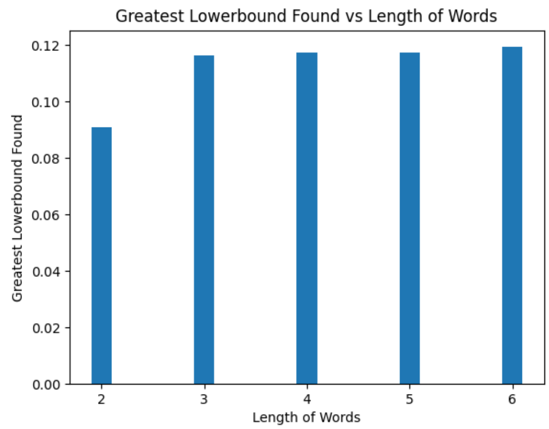


Figure 2.13: Graph of Best Bound Found at each iteration

We also note that the rate of growth of the lower bound decreases sharply as the iteration number increases, and from the graph it seems to follow a logarithmic growth. We can estimate that it would take at least 10 iterations for the bound to grow above 0.12.

Chapter 3

Conclusion

In this project, we have implemented the algorithm in [1] for computing lower bounds and we have estimated a lower bound on the measure of a specific subset of dimension traps for the Baker's map. We generated points in Cantor sets $C(a, b)$ for pairs of $a, b \in \{0, 1\}^*$ of lengths between 2 and 6. We then used the algorithm to generate polygons from those points that the dimension traps would need to contain. We found that time taken per iteration approached an exponential growth, while the growth of the lower bound per iteration approached a logarithmic one. A few questions are left open for further consideration.

1. Is there a method of choosing which pairs of strings at each iteration to consider? We found in this project that many of them do not improve the bound.

2. Would choosing pairs of strings of different lengths increase the bound the differently? In this project we only considered pairs of words of the same length.

3. Is there a way to lower the hyperparameter δ' dynamically at each iteration if the number of polygons we generate becomes to large?

As well, we only considered symmetric dimension traps for the Baker's map. Work has been done considering non symmetric dimension traps in [1], as well as dimension traps with different kinds of symmetry.

References

[1] Lyndsey Clark, Kevin G. Hare, and Nikita Sidorov. The baker's map with a convex hole. *Nonlinearity*, 31:3174–3202, 2018.

[2] Kenneth Falconer. *Fractal geometry: mathematical foundations and applications*. John Wiley & Sons, 2004.

[3] hmakholm left over Monica (https://math.stackexchange.com/users/14366/hmakholm-left-over monica). What is meant by limit of sets? Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/1490451 (version: 2015-10-21).