A Review of Computational Lower Bounds of Garsia Entropy

by

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Author’s Declaration

I hereby declare that I am the sole author of this paper. This is a true copy of the research paper, including any required final revisions, as accepted by my examiners.

I understand that my research paper may be made electronically available to the public.

Abstract

For $\beta \in (1, 2)$ the Bernoulli convolution is defined as the weak star limit of $\nu^n = 1/2^n \cdot \sum_{a_1 \ldots a_n \in \{0,1\}^n} \delta_{\sum_{i=1}^n a_i \cdot \beta^{-i}}$. The Garsia entropy is defined as $\lim_{n \to \infty} 1/n \cdot H_n(\beta, p) = -1 \cdot \sum_{a \in \{0,1\}^n} m_p[a] \cdot log(M_n[a_1 \ldots a_n])$. The Garsia Entropy can be used to determine the Hausdorff dimension of a Bernoulli convolution as $\dim_H(\nu_\beta) = \min(1, H(\beta)/\log(\beta))$ [5].

This paper will review and compare algorithms implemented in [4] and [1] to efficiently provide lower and upper bounds on the Garsia entropy and thus the Hausdorff dimension of the Bernoulli convolutions for specific classes of algebraic numbers β.

Acknowledgements

I would like to acknowledge the advice and expertise of Dr. Kevin Hare in assisting me in completing this research project. I would like to thank Chérisse Mike, Dr. Jeff Orchard, Dr. Henry Wolkowicz, Dr. Hans De Sterck, and Dr. Ryan Browne of the Computational Mathematics program for financial and Academic support. Thank you to Dr. Jason Bell for serving as an external reader of this paper.

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Chapter 1

Introduction

The Bernoulli convolution, defined formally in Definition 1.1, is a measure of the redundancy of representations $\pi_\beta(a) = \sum_i a_i \cdot \beta^{-i}$ for binary words in $\{0,1\}^{\mathbb{N}}$.

Definition 1.1. For $\beta \in (1,2)$ the **Bernoulli convolution** is the weak star limit of

$$\nu_p^n = \sum_{a_1 \dots a_n \in \{0,1\}^n} m_p[a] \cdot \delta_{\sum_{i=1}^n a_i \cdot \beta^{-i}}$$

The Bernoulli convolution can be rewritten as $\nu_{\beta,p} = m_p \circ \pi_\beta^{-1}$, with the Bernoulli measure m_p defined in definition 1.2.

Definition 1.2. Take $p \in (0,1)$ then the **Bernoulli measure** m_p is the measure that gives weight p to 0 and $1-p$ to 1 for a word in $\{0,1\}^{\mathbb{N}}$.

When $p = .5$ the Bernoulli measure m_p gives equal weight to all words. This refers to the unbiased Bernoulli Convolution ν_β [1]. Through the rest of this paper, unless otherwise mentioned, we will assume $p = .5$. There is an active literature on Bernoulli convolutions, much of which focuses on determining how its Hausdorff dimension, defined in Definition 1.3, varies for different classes of β.

Definition 1.3. The **Hausdorff dimension** of a measure ν_β is defined as

$$\dim_H(\nu_\beta) = \lim_{r \to 0^+} \log(\nu_\beta(x-r, x+r))/\log(r) = \alpha$$

a.e. $x \in \mathbb{R}$.

Many authors have investigated for which class of β is $\dim_H(\nu_\beta) < 1$ or for which class of β is $\dim_H(\nu_\beta) = 1$. Their work has shown that there is a very small set of β with $\dim_H(\nu_\beta) < 1$ and it is suspected that the only β for which this is true are PV numbers. Recall that β is **Pisot–Vijayaraghavan (PV)** if $\|\beta\| > 1$ and $\|\beta_j\| < 1, j = 2 \dots d$ for

$\beta_j, j = 1 \ldots d$ roots of the minimal polynomial of β. It was shown by Garsia in [3] if β is PV then $\dim_H(\nu_\beta) < 1$. Solomyak's results in [6] lend credance to the idea that PV numbers are the only class for which this is true as he showed $\dim_H(\nu_\beta) = 1$ for β a.e. $\in (1, 2)$.

Hochman in [5] showed that for algebraic $\beta \in (1, 2)$ a function called the Garsia entropy could be used to determine $\dim_H(\nu_\beta)$ via the relation in Equation 1.1. Recall that a number β is an **algebraic integer** if its a root of $x^n + c_{n-1} \cdot x^{n-1} + \cdots + c_0$ with $c_i \in \mathbb{Z}$.

$$\dim_H(\nu_{\beta,p}) = \min(1, H(\beta)/\log(\beta)). \tag{1.1}$$

The Garsia entropy $H(\beta)$ is defined in Definition 1.4 along with the finite Garsia entropy $H_n(\beta)$. It is another measure of binary words having the same representation in β.

Definition 1.4. The **Garsia entropy** is defined as $\lim_{n\to\infty} H_n(\beta)$ with

$$H_n(\beta, p) = -1 \cdot \sum_{a\in\{0,1\}^n} m_p[a] \cdot \log(M_n[a_1...a_n]).$$

The quantity $M_n[a_1...a_n]$ denotes the sum of the Bernoulli measures of the words b having the representation $\pi_\beta(a)$

$$M_n[a_1...a_n] = \sum_{\substack{b\in\{0,1\}^n \\ \sum_i b_i\cdot\beta^{-i}=\sum_i a_i\cdot\beta^{-i}}} m_p[b]. \tag{1.2}$$

For more background on Bernoulli convolutions see the recent review paper by Varju [7]. Based on the results of [5] in Equation 1.1 Akiyama et al. in [1] and Hare et al. in [4] created algorithms to obtain lower bounds on the Garsia entropy that are tractable for many hyperbolic and algebraic β, and algebraic β respectively. Recall that β is **hyperbolic** if $\| \beta \| > 1$ and $\| \beta_j \| \neq 1, j = 2 \ldots d$ for $\beta_j, j = 1 \ldots d$ roots of the minimal polynomial of β. This paper will review the bounds and performance of algorithms from the aforementioned papers for specific algebraic polynomials.

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Chapter 2

Algorithms to Bound Garsia Entropy

2.1 Random Matrices and Bounds on Bernoulli Convolutions in [1]

Let β_j for $j = 1 \ldots d$ refer to the roots of a degree d algebraic polynomial.The authors divide these roots, called Galois conjugates, into 3 potentially empty but distinct groups. Let $|\beta_j| \in (1, 2)$ for $j = 1 \ldots r$ refer to the Galois conjugates in order of decreasing value with modulus greater than 1. Let $j = r+1 \ldots r+\ell$ denote the Galois conjugates of modulus 1 and $j = r+\ell+1 \ldots d$ refers to the Galois conjugates with modulus less than 1 in absolute value. The algorithms in [1] work for algebraic and hyperbolic β ($\ell = 0$) by using all Galois conjugates with modulus greater than one to construct matrices whose Lyaponov exponents and products provide lower-bounds on the Garsia entropy.

Define the set $V_{\beta,n} = \{x = \sum \epsilon_i \cdot \beta^{n-i}\}$ for $\epsilon_i \in \{-1, 0, 1\}$ and $|(\sum \epsilon_i \cdot \beta_j^{n-i}| \leq 1/(\| \beta_j \| - 1)$ for $j = 1 \ldots r$.

Define the set $V_\beta = \cup_n V_{\beta,n}$. When β is algebraic and hyperbolic the set V_β will be finite and can be utilized to provide lower-bounds on the Garsia entropy and thus the Hausdorff dimension of the p-Bernoulli convolution. The set V_β is constructed as below.

1. Let $V_{\beta,0} = \{1, 0, -1\}$, $A_0 = \{1, 0, -1\}$.
2. Let $V_{\beta,n} = V_{\beta,n-1} \cup A_n$ for $A_n = \{\beta \cdot x - \epsilon_n, \epsilon_n \in \{-1, 0, 1\}, x \in A_{n-1}, \beta \cdot x - \epsilon_n \in V_\beta\}$.
3. Stop when $V_{\beta,n} = V_{\beta,n-1}$.

After constructing the finite set V_β lower-bounds on $H(\beta)$ can be obtained through operations on two matrices M_0 and M_1. The matrices M_0 and M_1 are constructed as below.

1. Create a directed graph G by giving a weighted edge $e \in \{-1, 0, 1\}$ for all $x, y \in V_\beta$ when $y = \beta \cdot x + \epsilon$.

2. Prune G by removing vertices with no path from x to 0 for nodes x in G, creating a pruned graph $G^{'}$.

3. Define M_0 a $|G'|x|G'|$ matrix whose i,j entries are $1-p$ if $x_j = \beta x_i - 1$, p if $x_j = \beta x_i$, else 0 for x_i, x_j in $G^{'}$.

4. Define M_1 a $|G'|x|G'|$ matrix whose i,j entries are $1-p$ if $x_j = \beta x_i$, p if $x_j = \beta x_i + 1$, else 0 for x_i, x_j in $G^{'}$.

After constructing the sets V_β and the pruned directed graph/set $G^{'}$ the authors construct three lower-bounds for $H(\beta)$, two of which are tractable to compute for many hyperbolic β.

2.1.1 Proof of convergence for hyperbolic β

The authors prove that their algorithm converges for β hyperbolic by creating a set that contains the differences between elements in $V_{\beta,n}$. They obtain lower-bounds on the size of elements in this set that are equivalent to the minimum possible spacing between elements of $V_{\beta,n}$. lower-bounds on minimal spacing provide an upper-bound on the maximum possible number of elements of $V_{\beta,n}$. When $\ell = 0$ they show that this upper-bound is constant for all n and then as $V_{\beta,n} \subset V_{\beta,n+1}$ the set V_β will be finite. We know $V_{\beta,n} \subset V_{\beta,n+1}$ by definition as any x in $V_{\beta,n}$ will meet the same defining inequality in $V_{\beta,n+1}$ with $\epsilon_{n+1} = 0$. Note also that $V_{\beta,n} \subset [-1/(\beta-1), 1/(\beta-1)]$. In Lemma 2.4 of [1] the authors show that $|V_\beta| \leq C(\beta) \cdot (n+1)^\ell + 1$. When β is hyperbolic $\ell = 0$, $V_{\beta,n}$ is bounded for all n by a constant and thus $|V_\beta|$ is finite as $V_{\beta,n} \subset V_{\beta,n+1}$.

To prove Lemma 2.4 the authors create a set $V^{'}_{\beta,n} \subset [-2/(\beta-1), 2/(\beta-1)]$ defined as $V^{'}_{\beta,n} = \{x = \sum_{i=0}^{n} \epsilon_i \cdot \beta^{n-i} : \epsilon_i \in \{-2,-1,0,1,2\}$ and $|\sum_{i=0}^{n} \epsilon_i \cdot \beta_j^{n-i}| \leq 2/(|\beta_j| - 1)\}$

Using Lemma 2.2 of [1], that $\prod_{j=1...d} P(\beta_j)$ is an integer for β algebraic and P a polynomial with integer coefficients, the product $\prod_{j=1}^{d} |x_j| \geq 1$ for $x_j = \sum_{i=0}^{n} \epsilon_i \cdot \beta_j^{n-i}$. They then use the properties of the beta conjugates to provide bounds on the magnitudes $|x_j|$.

For $j \in \{r+\ell+1...d\}$ the magnitude $|x_j| \leq 2/(1-\beta_j)$. For $j \in \{1...r\}$ the magnitude $|x_j| \leq 2/(\beta_j - 1)$. For $j \in \{r+1...r+\ell\}$ the magnitude $|x_j| \leq \sum_{i=0}^{n} 2 \cdot 1^{n-i} = 2 \cdot (n+1)$.

Thus for any $x \in V^{'}_{\beta,n}$, excluding 0, $|x| \geq 1/(\prod_{j \in 2...d} |X_j|) \geq C_0(n)$. $C_0(n)$ is defined using the inequalities for each Galois conjugate as

$$C_0(n) = 2^{1-d} \cdot \left(\prod_{j \in \{2...r\}} (|\beta_j| - 1) \right) \cdot (1/(n+1)^\ell) \cdot \left(\prod_{j \in \{r+\ell+1...d\}} \left(1 - |\beta_j|\right) \right). \tag{2.1}$$

Because $C_0(n)$ provides a lower-bound on the size of elements in $V^{'}_{\beta,n}$ and the difference between any elements in $V_{\beta,n}$ will be in $V^{'}_{\beta,n}$ any two elements in $V_{\beta,n}$ must be separated by at least $C_0(n)$.

Therefore $V_{\beta,n}$ will contain at most $1+\mu([-1/(\beta-1), 1/(\beta-1)])/C_0(n) = C(\beta)\cdot(n+1)^{\ell}+1$ elements. Then for β hyperbolic, $\ell = 0$ and hence as $V_{\beta,n} \subset V_{\beta,n+1}$ the set $V_\beta = \cup_n V_{\beta,n}$ will be finite and the algorithm in the prior section will converge when $V_{\beta,n} = V_{\beta,n-1}$.

2.1.2 Lower-bound A

The matrices M_0 and M_1, defined in section 2.1.0, provide a lower-bound on the Garsia entropy in Equations 2.2 and 2.3 via the product $M_{a_1}\cdot\cdots\cdot M_{a_n}$.

$$L_n(\beta,p) = -\sup_{i\in A}\sum_{a\in\{0,1\}^n} m_p[a_1\ldots a_n]\cdot\log(\sum_{j\in A}(M_{a_1}\cdot\cdots\cdot M_{a_n})_{ij}). \tag{2.2}$$

The set A indexes the rows and columns of M_0 and M_1. Proposition 3.3 and Theorem 1.2 of [1] state the below inequality and thus that the quantity L_n converges to $H(\beta,p)$ from below as n approaches ∞ and is a lower-bound on the Garsia entropy.

$$\begin{aligned} 1/n\cdot H_n(\beta,p) - 1/n\cdot\log(C(\beta)\cdot(n+l)^{\ell}+1) &\leq 1/n\cdot L_n(\beta,p) \\ &\leq H(\beta,p) \\ &\leq 1/n\cdot H_n(\beta,p). \end{aligned} \tag{2.3}$$

2.1.2.1 Proof of lower-bound A

The matrices M_0 and M_1 are created such that they encode the relation

$$(M_{a_1}...M_{a_n})_{i,j} = \sum_{\substack{b_1......b_n\in\{0,1\}^n \\ \beta^n\cdot x_i+\sum_{l=1}^n(a_l-b_l)\cdot\beta^{n-l}=x_j}} m_p[b_1\ldots b_n] \tag{2.4}$$

for a word a and x_i and x_j from G'. In Lemma 3.2 from [1] they prove that $L_n(\beta,p)$ is super-additive, that is $L_{n+m}(\beta,p) \geq L_n(\beta,p) + L_m(\beta,p)$.

They then rely on the finite Garsia entropy H_n being subadditive. Recall that a sequence a_n is **subadditive** if $a_{n+m} \leq a_n + a_m$ for all $n,m\in\mathbb{N}$. Since $H_{n+m} \leq H_n + H_m$ and H_n is an upper-bound on L_n then $H(\beta) \in (1\cdot L_n, 1/n\cdot H_n)$ for all n $\in N$. They then create the set of π_β maps for words of length n, $X_n = \{\sum_{i=1}^n a_i\cdot\beta^{-i} : a_i\in\{0,1\}\}$ and rewrite L_n as below, with $M_{x,n}$ referring to the product $M_{a_1}\ldots M_{a_n}$ for any a such that $x = \sum_{i=1}^n a_i\cdot\beta^{-i}$.

$$L_n(\beta,p) = -\sum_{x\in X_n}(M_{x,n})_{1,1}\cdot\log((M_{x,n})_{1,1}) - \sup_{i\in A}\sum_{x\in X_n}(M_{x,n})_{1,1}\cdot\log(\sum_{j\in A}(M_{x,n})_{i,j}/(M_{x,n})_{1,1}). \tag{2.5}$$

The first term in the equation above is H_n and they bound L_n by moving the sum in the latter term inside the log function, as the log function is concave. Therefore $L_n \geq H_n - \sup_{i\in A} \log(\sum_{x\in X_n, j\in A}(M_{x,n})_{i,j})$.

Then using the definition of M_0 and M_1 as an encoding for the sum of measures of words b whose Bernouli projections relate to the word a in Equation 2.4 the authors rewrite $\sum_{x\in X_n, j\in A}(M_{x,n})_{i,j}$ as $\sum_{x\in X_n, j\in A} \sum_{\substack{b_1\dots b_n\in\{0,1\}^n \\ \beta^n\cdot x_i+\beta^n\cdot x-\sum_{l=1}^n b_l\cdot\beta^{n-l}=x_j}} m_p[b_1\dots b_n]$ which is equal to $\sum_{b_1\dots b_n\in\{0,1\}^n} m_p[b_1\dots b_n]\cdot|X_n(i,\beta)|$. The set $X_n(i,\beta) = \{x\in X_n : \beta^n\cdot x_i + (\beta^n\cdot x - \sum_{l=1}^n \beta^{n-l}\cdot b_l) \in V_\beta\}$.

They bound the quantity $X_n(i,\beta)$ using Lemma 2.4 which bounds the size of $V_{\beta,n}$ as $|V(\beta,n) \leq C(\beta)\cdot(n+1)^\ell + 1$. Therefore $\sum_{x\in X_n, j\in A}(M_{x,n})_{i,j} \leq C(\beta)\cdot(n+1)^\ell + 1$ and they prove Theorems 1.2 and Proposition 3.3 as $L_n \geq H_n - \log(C(\beta)\cdot(n+1)^\ell + 1)$.

2.1.3 Lower-bound B

When β is hyperbolic the authors provide a more computationally efficient lower-bound on the Garsia entropy by moving the supremum i in A in the definition of L_n inside the summation. They define the below quantity with $\|\ \|$ the row-sum norm.

$$L_n(\beta,p)' = -\sum_{a\in\{0,1\}^n} m_p[a_1\dots a_n]\cdot\log(\|(M_{a_1}\dots M_{a_n})\|). \tag{2.6}$$

Lower-bound B converges to $H(\beta)$ for hyperbolic β and $L_n(\beta,p)' \leq L_n$.

2.1.3.1 Proof of Lower-bound B

Since the supremum over i in A is moved inside the summation in L_n', then lower-bound B $L_n' \leq L_n$.

They apply Lemma 9.9 from [8] to obtain the inequality $L_n(\beta,p)' \geq H_n(\beta,p) - \log(\sum_{x\in X_n}\|M_{x,n}\|)$.

Since $\sum_{x\in X_n}\|M_{x,n}\| = \sum_{x\in X_n}\max_{i\in A}\sum_{j\in A}(M_{x,n})_{i,j} \leq \sum_{x\in X_n}\sum_{i\in A}\sum_{j\in A}(M_{x,n})_{i,j}$.Rearranging the sums you can then obtain $\sum_{i\in A}\sum_{x\in X_n}\sum_{j\in A}(M_{x,n})_{i,j} \leq \sum_{i\in A}(C(\beta)\cdot(n+1)^\ell+1) \leq |A|\cdot\left(C(\beta)\cdot(n+1)^\ell+1\right)$.

Then for β hyperbolic, $\ell = 0$ and hence $1/n\cdot L_n' \leq 1/n\cdot H_n - 1/n\cdot|A|\cdot(C(\beta)+1)$ and thus $1/n\cdot L_n'$ converges to $H(\beta)$ from below.

2.1.4 Lower-bound C

For hyperbolic β the authors obtain an additional computationally tractable lower-bound on the Garsia entropy of the Bernoulli convolution as

$$-\,ln(\lambda) \leq H(\beta,p) \tag{2.7}$$

for λ the largest eigenvalue of the matrix $(1-p)\cdot M_0 + p\cdot M_1$.

2.1.4.1 Proof of lower-bound C

This result is proved by providing a lower-bound on the quantity $L_n^{'}$ by replacing the row sum norm with the 1 norm $\| \ \|_1 = \sum_{i,j} |M_{i,j}|$.

$1/n\cdot L_n^{'} \geq -1/n\cdot\log(\sum_{a_1\ldots a_n\in\{0,1\}^n} m_p[a_1\ldots a_n]\cdot \| M_{a_1}\ldots M_{a_n} \|_1)$. Because M_0 and M_1 are positive matrices $\| M_0 \|_1 + \| M_1 \|_1 = \| M_0 + M_1 \|_1$ $-1/n\cdot\log(\sum_{a_1\ldots a_n\in\{0,1\}^n} m_p[a_1\ldots a_n]\cdot \| M_{a_1}\ldots M_{a_n} \|_1) = -1/n\cdot \| ((1-p)\cdot M_0 + p\cdot M_1)^n \|_1$. As n approaches infinity $1/n\cdot \| ((1-p)\cdot M_0 + p\cdot M_1)^n \|_1$ approaches $\log(\lambda)$ and thus $H(\beta, p) \geq -\log(\lambda)$.

2.2 Bounding Garsia Entropy in [4]

The algorithm in [4] works by defining the maps T_0 and T_1 in Equations 2.8 and 2.9 for algebraic $\beta_1 \in (1,2)$ and $|\beta_2| \in (1,2)$. Note that in contrast with the constraints to use [1], the algorithm in [1] does not require β to be hyperbolic, only algebraic.

$$T_0(x,y) = (x/\beta_1, y/\beta_2). \tag{2.8}$$

$$T_1(x,y) = (x/\beta_1 + 1, y/\beta_2 + 1)\,. \tag{2.9}$$

Then define the region I_{β_1,β_2} or I_β as follows

- If $\beta_2 > 1$ then $I_{\beta_1,\beta_2} = [0, 1/(\beta_1 - 1)] \times [0, 1/(\beta_2 - 1)]$.
- If $\beta_2 < -1$ then $I_{\beta_1,\beta_2} = [0, 1/(\beta_1 - 1)] \times [\beta_2/(\beta_2^2 - 1), \beta_2^2/(\beta_2^2 - 1)]$.

The region I_β has analogous definitions when $\beta_2 \in C$ or when using more than two Galois conjugates and formulas can be found in [4].

Then $L_n/(n\cdot\log(\beta)) \leq H(\beta)/\log(\beta)$ for L_n as defined in Equations 2.10 and 2.11.

$$L_n/(n\cdot\log(\beta)) = (n\cdot\log(2) - \log(m_n))/(n\cdot\log(\beta)). \tag{2.10}$$

$$m_n = \max_{(x,y)\in I_{\beta_1,\beta_2}} \#a_1a_2\ldots a_n : (x,y) \in T_{a_1}\circ T_{a_2}\circ\ldots\circ T_{a_n}(I_{\beta_1,\beta_2}). \tag{2.11}$$

The term m_n for a given (β_1, β_2) was obtained by calculating all 2^n rectangles $T_{a_1}\ldots T_{a_n}(I_\beta)$ and then the intersections among rectangles, intersections of intersections, etc. until none are left. Each intersection was associated with its ancestors from the original set of 2^n

rectangles. This process is visualized in Figures 2.1 through 2.5 for $n = 4$ and $\beta_1 = 1.2987$, $\beta_2 = -1.541$. Figure 2.5 plots all maps and intersections on the same graph, with original maps in black, intersections between two maps in yellow, between three maps in blue, and between four maps in red.

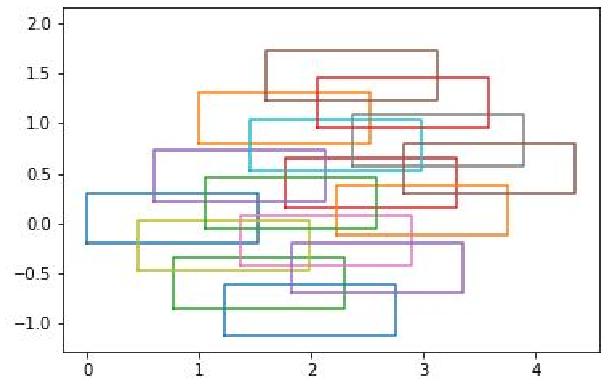

Figure 2.1 All mappings of $T_{a_1}...T_{a_n}(I_\beta)$

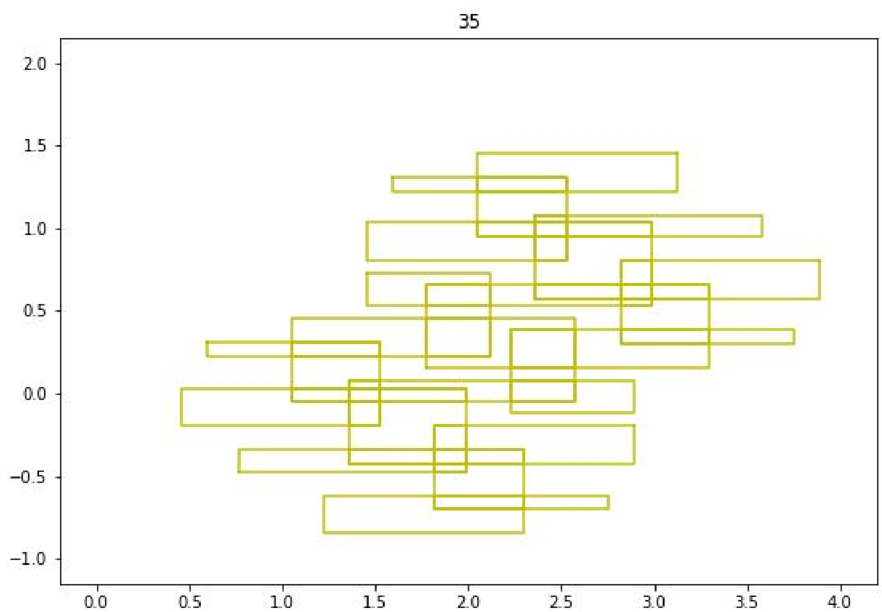

Figure 2.2 Intersections of two maps $T_{a_1}...T_{a_n}(I_\beta)$

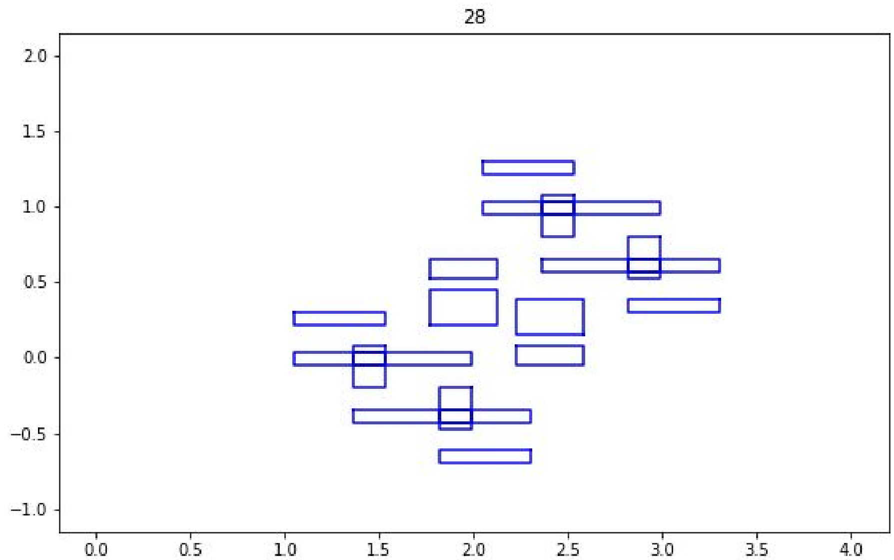

Figure 2.3 Intersections of three maps $T_{a_1}...T_{a_n}(I_\beta)$

2.2.1 Proof of [4]

The quantity L_n is proven to be a lower-bound for $H(\beta)$ as m_n is a max taken over I_β. Recall that the region I_β is constructed so it will contain the $\pi_\beta(a)$ representations which define the Garsia entropy, and Garsia entropy can be defined using a term similar to m_n.

Define the H_n per Equations 2.12 and 2.13.

$$H_n(\beta) = -\sum_{a\in\{0,1\}^n} 1/2^n \cdot \log(\mathcal{N}_n(a,\beta)/2^n). \tag{2.12}$$

$$\mathcal{N}_n(a,\beta) = \#\{b \in \{0,1\}^n : \sum a_i \cdot \beta^{-i} = \sum b_i \cdot \beta^{-i}\}. \tag{2.13}$$

Consider $K = \{\sum_{i=1}^{\infty} a_i \cdot (\beta_1^{-i}, \beta_2^{-i})\}$ for $a_i \in \{0,1\}^{\mathbb{N}}$. By construction $K \subset I_\beta$ and $K = T_0(K) \cup T_1(K)$. Therefore $m_n \geq \mathcal{N}_n(a,\beta)$. They show that $L_n(\beta_1,\beta_2)$ is supadditive and therefore $H_n(\beta) \geq -\sum_{a\in\{0,1\}^n} 1/2^n \cdot \log(m_n/2^n) = L_n$. Recall that a sequence a_n is **supadditive** if $a_{n+m} \geq a_n + a_m$ for all $n,m \in \mathbb{N}$.

Therefore $H(\beta) = \lim_{n\to\infty} H_n/n \geq \lim_{n\to\infty} L_n/n \geq L_n/n$.

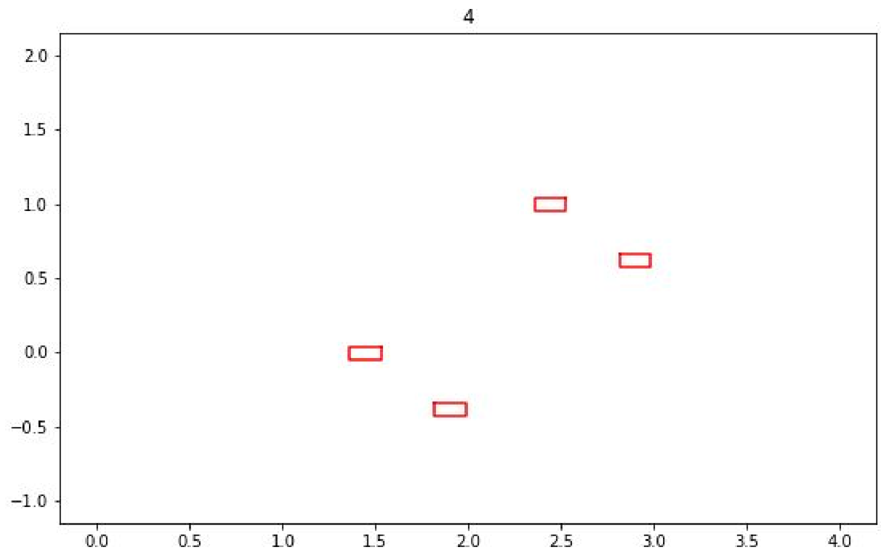

Figure 2.4 Intersections of four maps $T_{a_1}...T_{a_n}(I_\beta)$

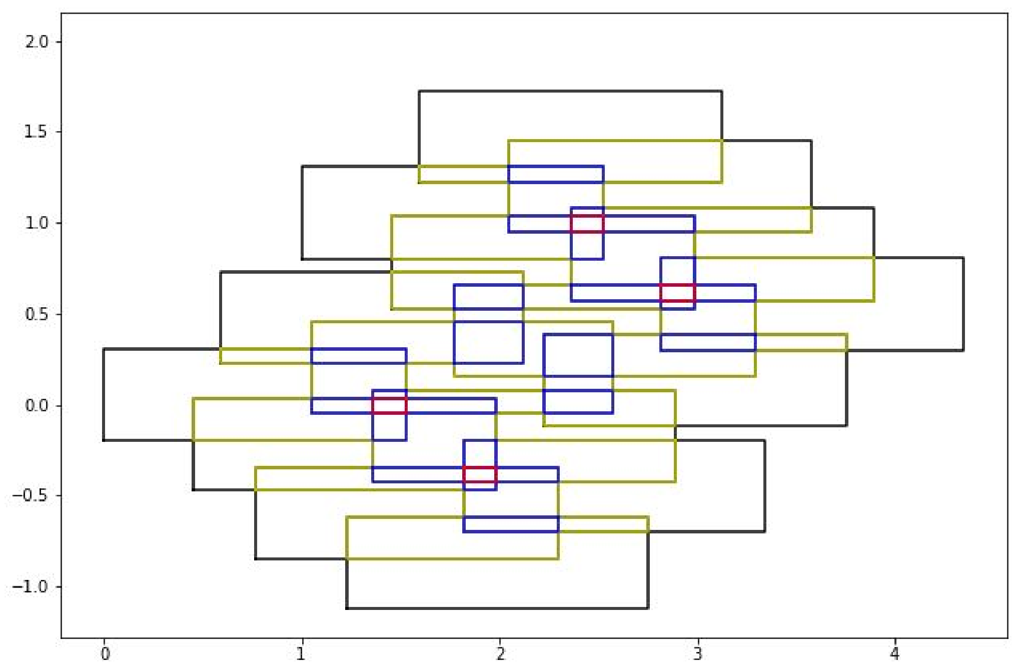

Figure 2.5 All maps and intersections of $T_{a_1}...T_{a_n}(I_\beta)$

2.3 Convergence of lower-bounds in [1] and [4] to the Garsia entropy

The lower-bounds described in the prior section differ in whether or not they will converge exactly to the Garsia entropy. Lower-bound B in [1] will converge to $H(\beta)$ while lower-bound C in [1] and the lower-bound obtained in [4] will not. However as the main intent of bounding Garsia entropy is to utilize Hochman's result in Equation 1.1 to determine the Hausdorff dimension it is mainly of interest to calculate whether $H(\beta)/\log(\beta) > 1$ via the lower-bounds. The authors of [4] and [1] include more detailed discussions of the convergence of their lower-bounds to concepts related to but distinct from $\dim_H$.

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Chapter 3

Numerical Results and Comparisons

3.1 Performance of Akiyama et al. [1]

The performance of the methods of Akyiyama et al. [1] varies widely for algebraic and hyperbolic β. Factors such as having Galois conjugates with modulus close to 1 or increasing the degree of the minimal polynomial (and thus the total number of Galois conjugates) can cause blow-up in the size of the finite set V_β and thus the time it takes to determine the set. This can be seen by $C_0(n)$ defined in Equation 2.1. Galois conjugates close in modulus to 1 or just increasing the number of Galois conjugates will decrease this value and the upper-bound on the size of V_β is proportional to $C_0(n)^{-1}$.

As an illustration consider the polynomials $\beta^5+\beta^4-\beta^2-\beta-1$ and $\beta^5+\beta^4-\beta^2-\beta-1$. Using Equation 2.1 the upper bounds on the size of V_β respectively are approximately 11,709,697 and 83,412. The actual sizes of $|G^{'}|$ are shown in [1] as 6485 and 13 respectively. Large differences in the upper bounds for each of these polynomials correspond to differences in the actual finite set size and thus how quickly the algorithm converges in each case.

Examining more computational results in [1] reinforces the wide variability in algorithm performance. The size of the pruned graphs in Figure 3.1 vary from 5 to 1253.

As $|V_\beta|$ increases the time it takes to calculate it does as well. Table 3.1 lists the set size and clock computation time in seconds running [1] on the specified polynomials.

Polynomial	$\lvert V_\beta\rvert$	time(s)
$\beta^3-\beta^2-\beta-1$	7	57
$\beta^4-\beta^3-\beta^2+\beta-1$	69	30
$\beta^4+\beta^3-\beta^2-\beta-1$	69	24

Table 3.1 Calculation Times for given polynomials

Polynomial	β	type	lower bound $\left(\frac{-\log\lambda}{\log\beta}\right)$	size of $\mathcal{G}'$
x^2-x-1	1.6180	Pisot	0.99240	5
x^3-x^2-x-1	1.8393	Pisot	0.96422	7
x^3-x^2-1	1.4656	Pisot	0.99912	49
x^3-x-1	1.3247	Pisot	0.99999	179
$x^4-x^3-x^2-x-1$	1.9276	Pisot	0.97333	9
$x^4-x^3-x^2+x-1$	1.5129	not Pisot	1.38670	21
x^4-x^3-1	1.3803	Pisot	0.99999	1253
$x^4-x^3+x^2-x-1$	1.2906	not Pisot	2.50349	9
x^4-x^2-1	1.2720	not Pisot	1.98480	25
x^4-x-1	1.2207	not Pisot	1.61576	1253
$x^4+x^3-x^2-x-1$	1.1787	not Pisot	3.49147	21

Figure 3.1 table 1 from [1]

Calculating the finite Garsia entropy per [1] increases exponentially in n as it involves summing over all 2^n words $a \in \{0,1\}^n$. Figure 3.2 displays clock time to calculate H_n using methods of [1] for the polynomial $\beta^4 - \beta^3 - \beta^2 + \beta - 1$.

Excluding the cost of calculating V_β itself, calculating $L_n^{'}$ will increase exponentially in n as well, for the same reasons as calculating the finite Garsia entropy. Figure 3.3 displays the clock calculation time to determine lower-bound B using methods of [1] for the polynomial $\beta^4 - \beta^3 - \beta^2 + \beta - 1$.

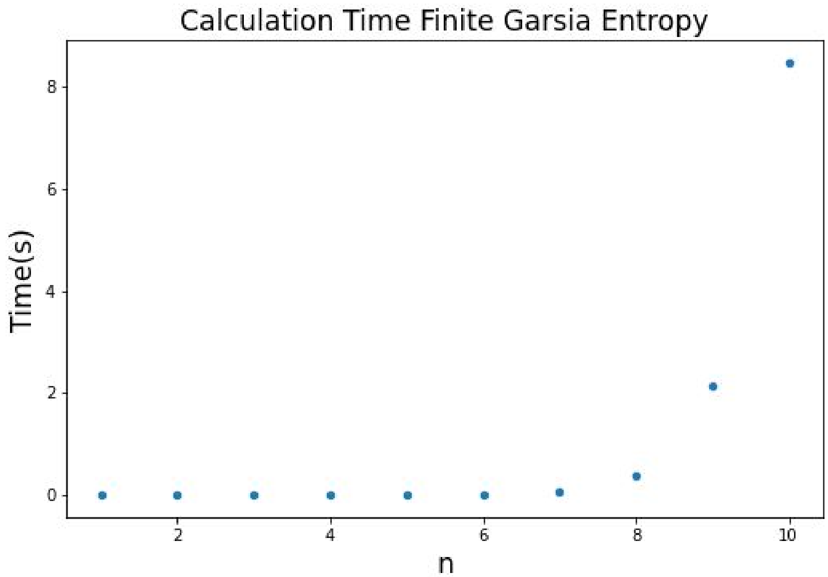

Figure 3.2 Calculation Times for [1] of H_n, $\beta_1 = 1.5129$, $\beta_2 = -1.1787$

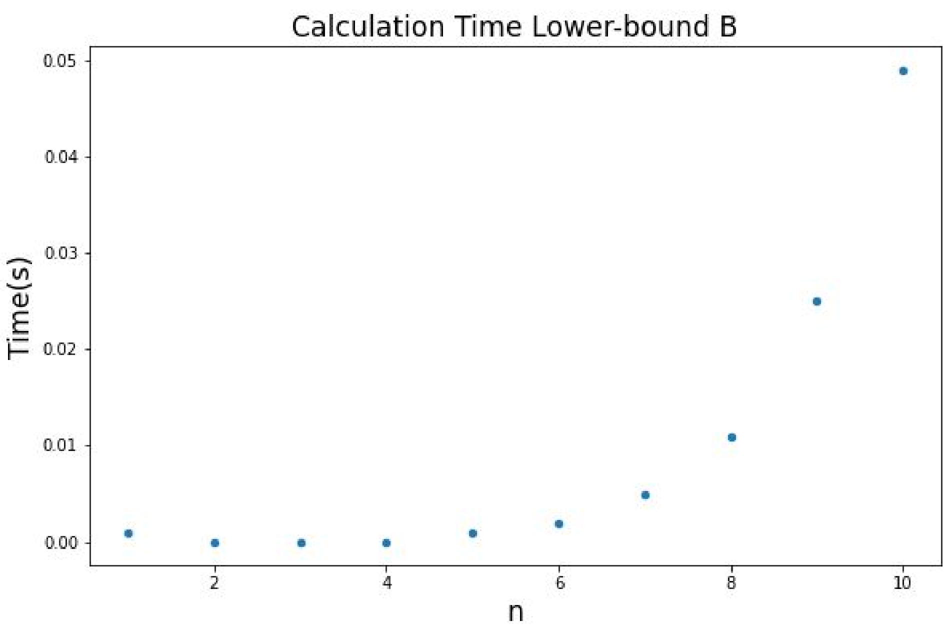

Figure 3.3 Calculation Time of Lower-bound B in [1], $\beta_1 = 1.5129$, $\beta_2 = -1.1787$

3.2 Performance of Hare et al. [4]

The computational costs of computing the lower-bound in [4] also increases exponentially in n as it involves calculating the max number of intersections between the 2^n maps of the region I_β. In contrast with [1] however there is no fixed cost comparable to calculating the finite set V_β and thus computations can still be tractable if a lower-bound greater than 1 is achieved at a low n. Figure 3.4 shows an exponential increase in calculation time for the polynomial $\beta^6 - \beta^4 + \beta^3 - 2 \cdot \beta^2 + \beta - 1$.

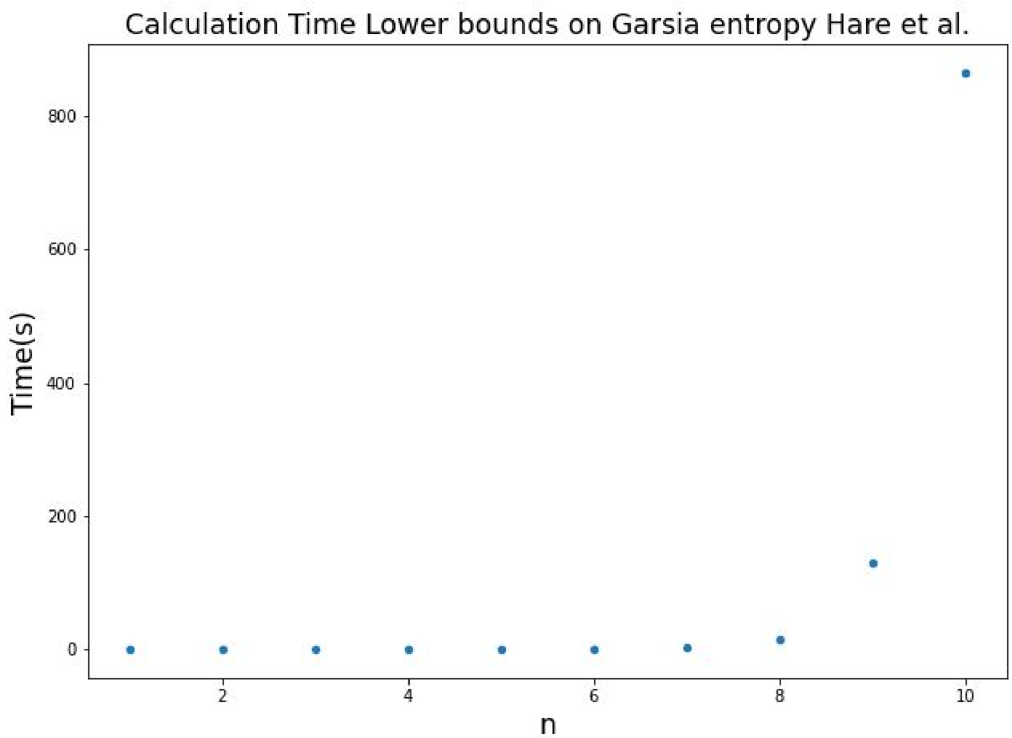

Figure 3.4 Calculation Time for [4], $\beta_1 = 1.2045$, $\beta_2 = -1.6454$

Figures 3.5, 3.6 and 3.7 show performance data using Hare et al. to obtain bounds for the polynomial $\beta^5 + \beta^4 - \beta^3 - \beta^2 - 1$. This has two roots of modulus greater than 1 in absolute value, $\beta_1 = 1.14$, $\beta_2 = -1.68$. Figure 3.5 shows that calculation times become expensive at $n = 10$, taking over an hour to calculate the lower-bound, and about 25x longer than the calculation time at the same n for the polynomial in Figure 3.4. Figure 3.6 though shows that calculating bounds at high n is unnecessary above $n = 2$ as we have already obtained a lower-bound greater than 1 and thus proven that $\dim_H = 1$. Figure 3.7 shows the calculation times for $n = 1$ to 5 and illustrates that the method in Hare et al. has rapidly determined the dimension of the Bernoulli Convolution defined by $\beta_1 = 1.14$, $\beta_2 = -1.68$. In contrast after an hour of calculation time the algorithm in Akiyama et al. did not converge for the same polynomial $\beta^5 + \beta^4 - \beta^3 - \beta^2 - 1$, and $|G'| = 139$ from [1]. For this polynomial the performance of Hare et al. was orders of magnitude faster in determining $\dim_H$ than was Akiyama et al.

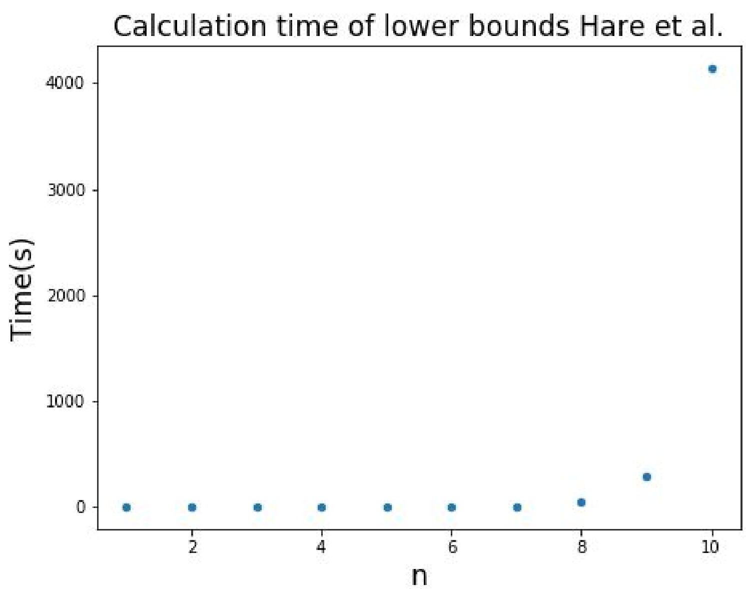

Figure 3.5 Calculation Times for [4], $\beta_1 = 1.14$, $\beta_2 = -1.68$

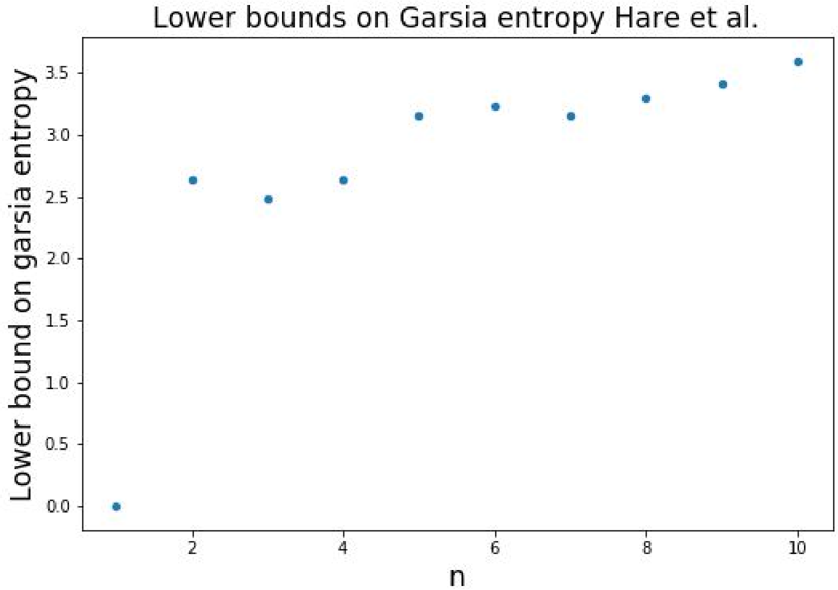

Figure 3.6 Bounds from [4], $\beta_1 = 1.14$, $\beta_2 = -1.68$

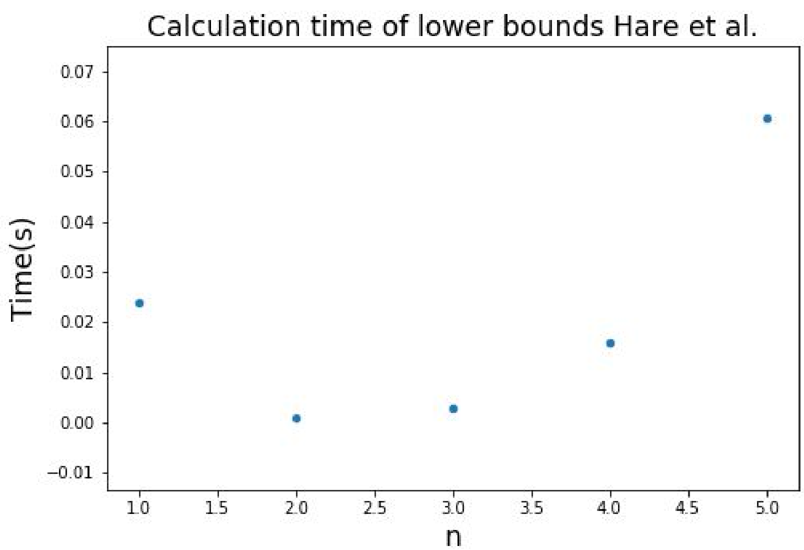

Figure 3.7 Calculation Times from [4], $\beta_1 = 1.14$, $\beta_2 = -1.68$

3.3 Further Numerical Results

3.3.1 $\beta^3 - \beta^2 - \beta - 1, \beta = 1.8393$

The polynomial $\beta^3 - \beta^2 - \beta - 1$ is PV with root $\beta = 1.8393$. As β is PV $\dim_H < 1$ based on the results of Garsia in [3]. The algorithm in Akiyama et al. can then be used to provide bounds on $\dim_H$ between lower-bound B, lower-bound C and 1. The finite set $|G^{'}| = 7$ and is shown in Figure 3.8. Figure 3.9 shows the convergence of H_n and lower-bound B to a value below one, and that lower-bound C quickly provides a tight bound on the Garsia entropy for $\beta = 1.8393$.

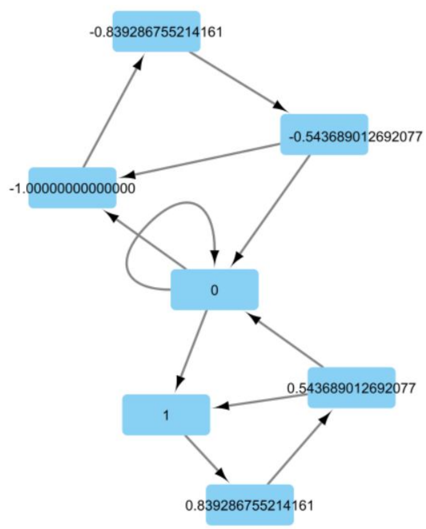

Figure 3.8 Structure of V_β for $\beta = 1.8393$

3.3.2 $\beta^4 - \beta^3 - \beta^2 + \beta - 1, \beta_1 = 1.5129, \beta_2 = -1.1787$

The polynomial $\beta^4 - \beta^3 - \beta^2 + \beta - 1$ is not PV. It's roots of modulus greater than one are β_1=1.5129, β_2=-1.1787. The finite set $|V_\beta| = 69$ and the pruned graph $|G^{'}| = 21$. Figures 3.10 and 3.11. Figure 3.12 shows that lower-bound C reveals $\dim_H = 1$ with quick calculation time of 30 seconds given in table 3.1. In contrast Figures 3.13 and 3.14 show that [4] takes orders of magnitude more computation time to obtain a lower-bound on Garsia entropy that is greater than 1. In [4] a lower-bound > 1 is obtained first at $n = 12$, Figure 3.13 shows the lower-bounds obtained up to $n = 7$ and Figure 3.14 that beyond n=7 the computation times increase exponentially from 500 s at n=7. Thus for $\beta^4 - \beta^3 - \beta^2 + \beta - 1$ calculations using [1] show that $\dim_H = 1$ in orders of magnitude less time than do calculations using [4].

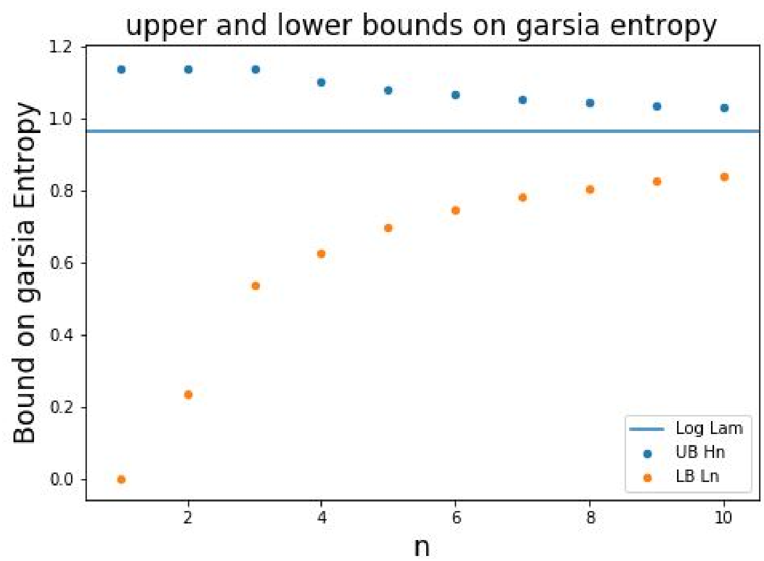

Figure 3.9 Bounds from [1], $\beta = 1.8393$

3.3.3 $\beta^4 + \beta^3 - \beta^2 - \beta - 1$, $\beta_1 = 1.1787$, $\beta_2 = -1.5128$

The polynomial $\beta^4 + \beta^3 - \beta^2 - \beta - 1$ is not PV. It's roots of modulus greater than 1 are β_1=1.1787, β_2=-1.5128. The finite set $|V_\beta| = 69$ and $|G'| = 21$ as found by [1]. The time to determine V_β was 24 seconds as shown in table 3.1. Figure 3.15 shows that $\dim_H = 1$ based on lower-bound C and lower-bound B quickly increasing beyond 1. [4] is also quickly able to determine that $\dim_H = 1$ as shown in Figures 3.16 and 3.17. At $n = 3$ the lower-bound in Hare et al. is greater than 1 and this is calculated in less than 1 second per Figure 3.17.

3.3.4 $\beta^7 - \beta^5 - \beta^3 - \beta - 1$, $\beta_1 = 1.2986$, $\beta_2 = -1.54133$

The polynomial $\beta^7 - \beta^5 - \beta^3 - \beta - 1$ is not PV. Its roots with modulus greater than 1 in absolute value are $\beta_1 = 1.2986$, $\beta_2 = -1.54133$. Figures 3.18 and 3.19 show that [4] is able to quickly show that $\dim_H = 1$. The lower-bound at n=6 is greater than 1 and the computation time is around 100 seconds. In contrast as this is a seventh order polynomial the computation time using [1] is likely to be quite large. The bound on $|V_\beta|$ is 2,719,611,421, calculated using Equation 2.1, and thus $|V_\beta|$ is likely to be quite large and take an inordinate amount of time to compute.

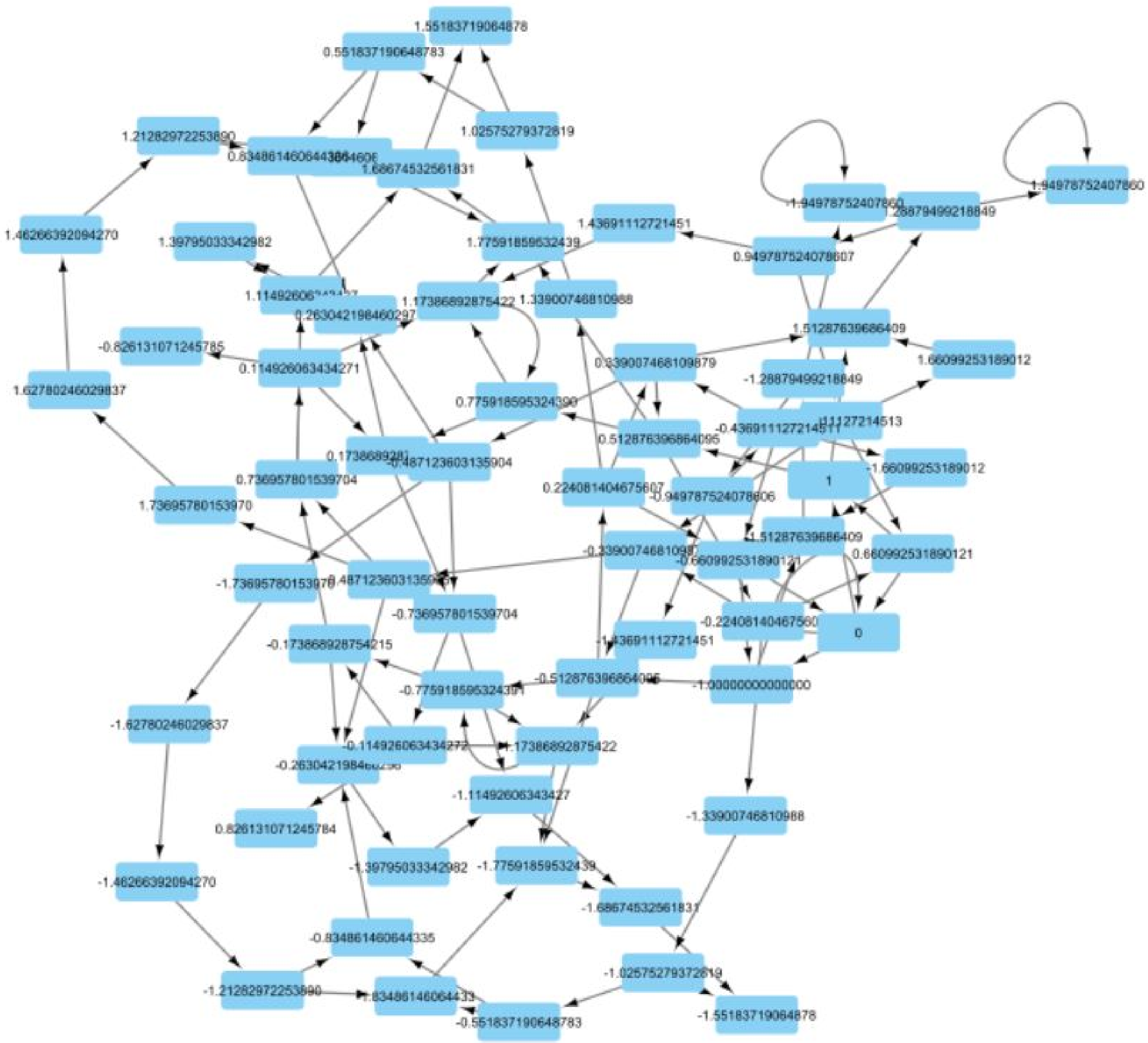

Figure 3.10 Structure of V_β for β_1=1.5129, $\beta_2 = -1.1787$

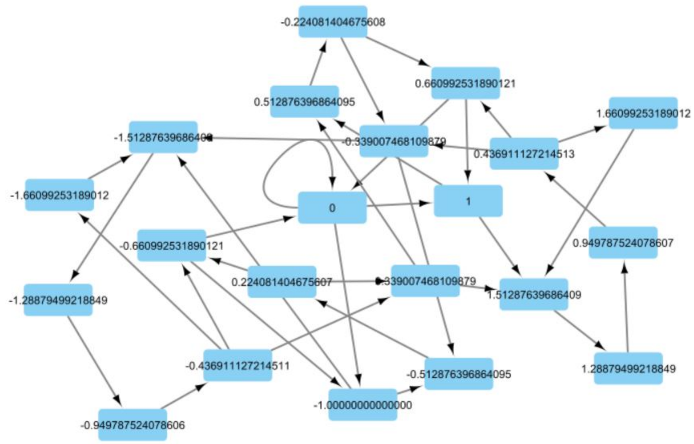

Figure 3.11 Structure of $G^{'}$ for $\beta_1 = 1.5129$, $\beta_2 = -1.1787$

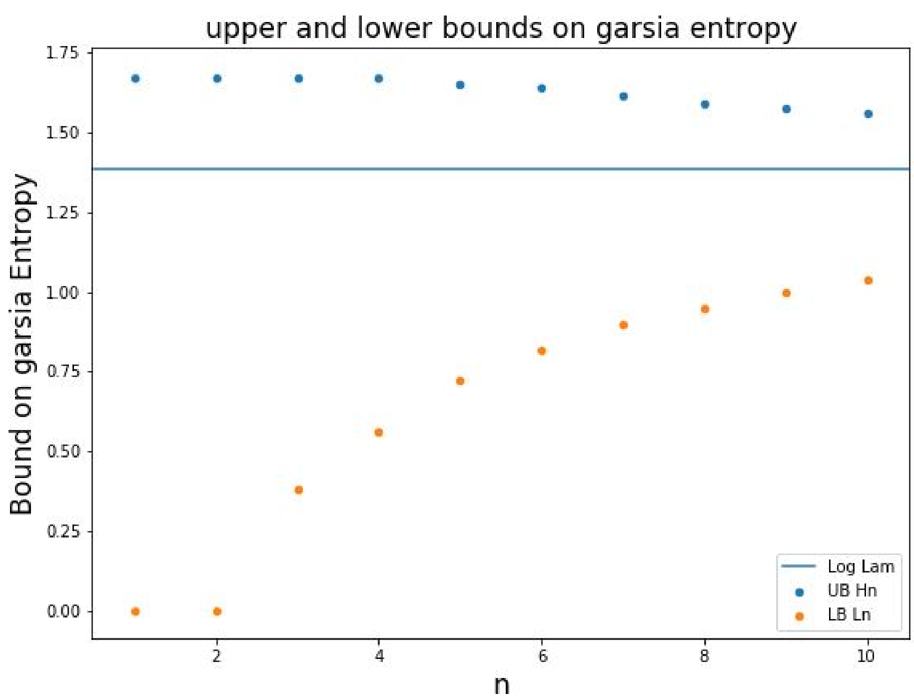

Figure 3.12 Bounds from [1], β_1 = 1.5129, β_2=-1.1787

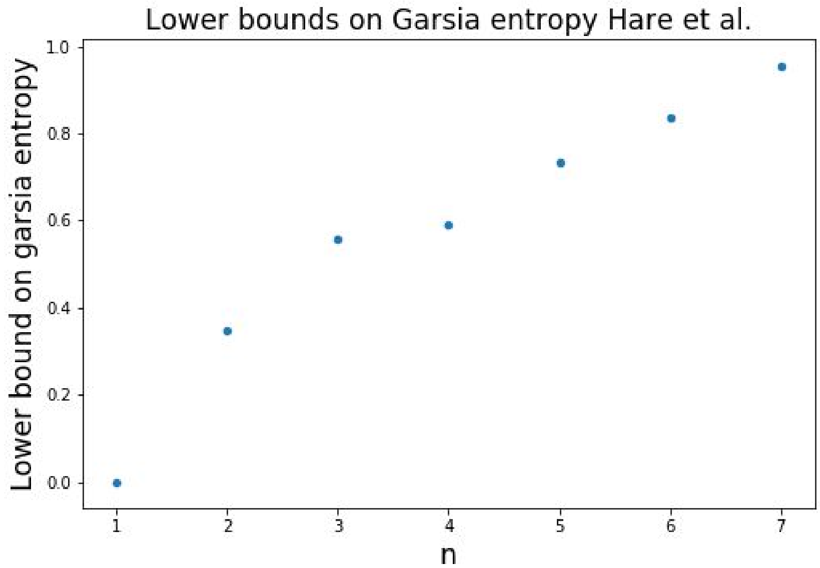

Figure 3.13 Bounds from [4], β_1=1.5129, $\beta_2 = -1.1787$

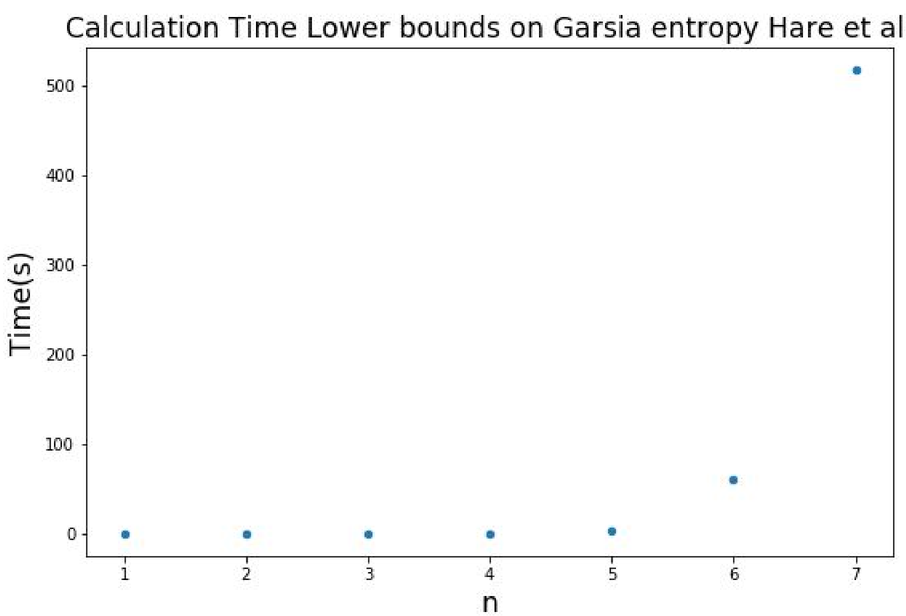

Figure 3.14 Calculation time for [4], $\beta_1 = 1.5129$, $\beta_2 = -1.1787$

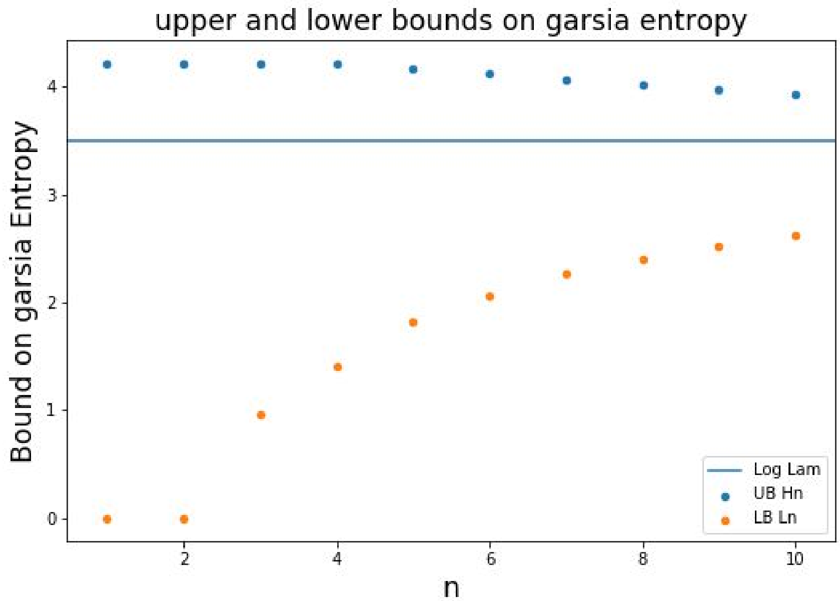

Figure 3.15 Bounds from [1], $\beta_1 = 1.1787$, $\beta_2 = -1.5128$

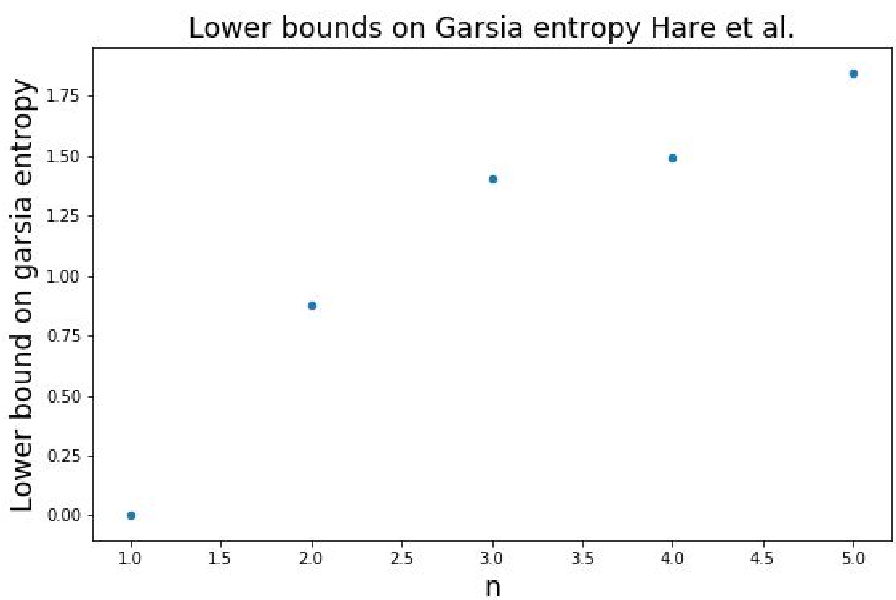

Figure 3.16 Bounds from [4], $\beta_1 = 1.1787$, $\beta_2 = -1.5128$

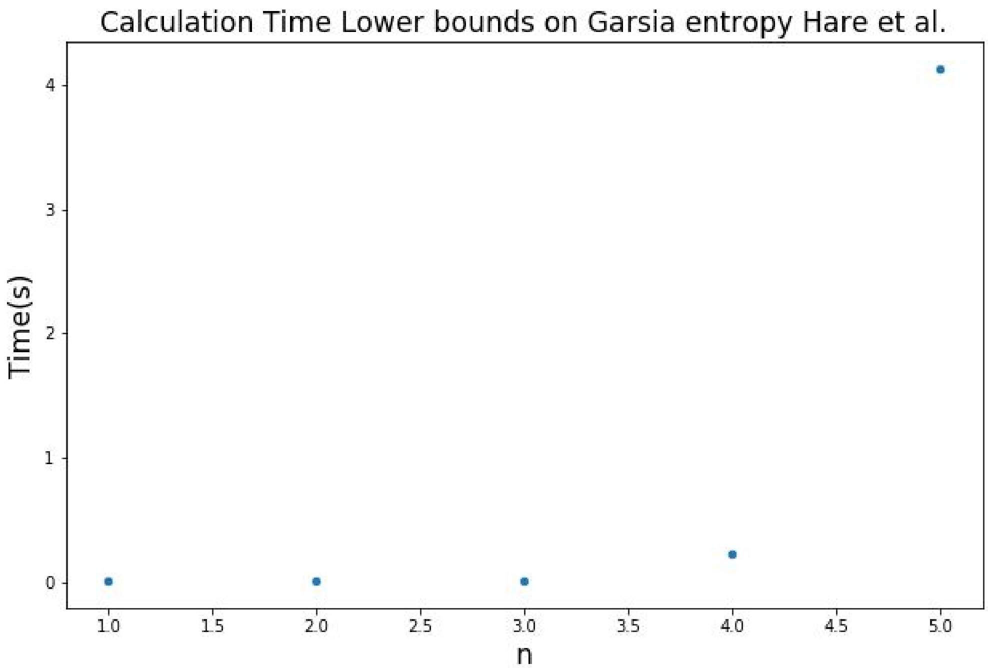

Figure 3.17 Calculation time for [4], $\beta_1 = 1.1787$, $\beta_2 = -1.5128$

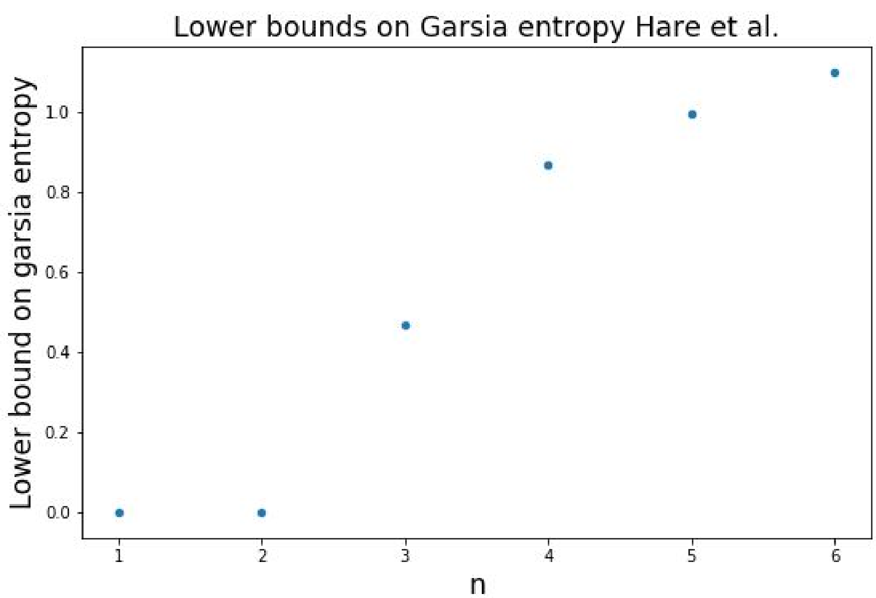

Figure 3.18 Bounds from [4], $\beta_1 = 1.2986$, $\beta_2 = -1.54133$

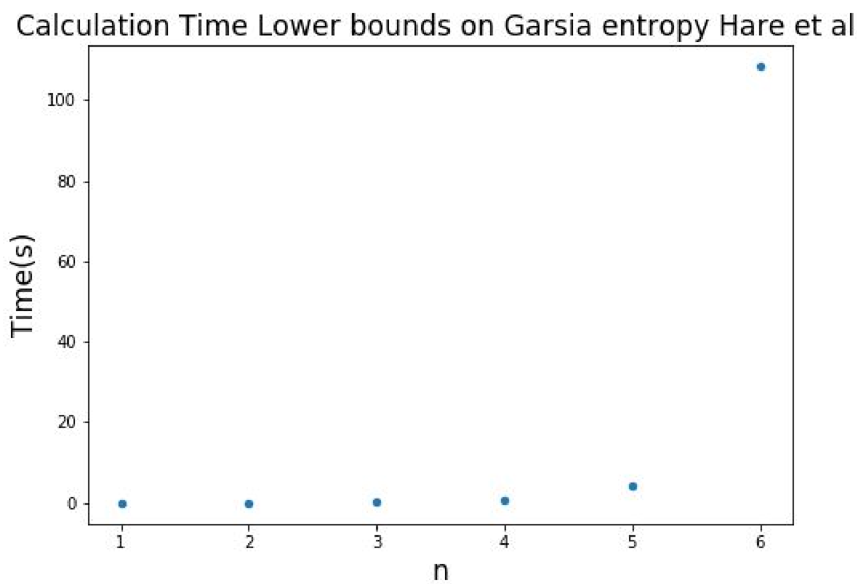

Figure 3.19 Calculation time for [4], $\beta_1 = 1.2986$, $\beta_2 = -1.54133$

3.4 Summary and Conclusions

Calculating how $\dim_H(\nu_\beta)$ varies for different categories of algebraic β is an active area of mathematics. Computational and theoretical evidence discussed in this paper supports PV numbers being the only class of β with $\dim_H < 1$. The algorithms in [1] and [4] build off the theoretical results of [5] and are able to determine $\dim_H$ for algebraic and hyperbolic β and algebraic β respectively. Numerical results in the prior sections show that for many low-degree algebraic and hyperbolic polynomials both methods can quickly determine $\dim_H$. Table 3.2 summarizes the time to determine $\dim_H = 1$ of [4] and [1] for the non PV polynomials listed in the previous section.

The bounds in [1] quickly become intractable to compute however when β has roots in modulus close to 1 or when the degree of the polynomial increases. The method of [4] is not as effected by these concerns which is shown by its several orders of magnitude superior performance compared to [1] to show $\dim_H = 1$ for the polynomials $\beta^7 - \beta^5 - \beta^3 - \beta - 1$ and $\beta^5 + \beta^4 - \beta^3 - \beta^2 - 1$. An additional benefit of [4] is that it is able to prove results for non-hyperbolic β whereas [1] is restricted to hyperbolic β. There are polynomials, however, such as $\beta^4 - \beta^3 - \beta^2 + \beta - 1$ that [1] is able to determine $\dim_H$ with performance several orders of magnitude smaller than that of [4]. The full papers of [4] and [1] in addition contain numerous examples requiring inordinate amounts of computation time to determine $\dim_H$ for both methods. Further research needs to be done to develop tractable algorithms to calculate $\dim_H$, such as roots of higher order polynomials, for β that are not tractable for either [1] or [4].

Polynomial	time(s) [1]	time(s) [4]
$\beta^4 - \beta^3 - \beta^2 + \beta - 1$	30	>3600
$\beta^4 + \beta^3 - \beta^2 - \beta - 1$	24	.1
$\beta^7 - \beta^5 - \beta^3 - \beta - 1$	>3600	120
$\beta^5 + \beta^4 - \beta^3 - \beta^2 - 1$	>3600	.03

Table 3.2 Calculation Times to determine $\dim_H = 1$ for [1] and [4]

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