Robustness of data-driven CVaR optimization using smoothing technique

by

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I hereby declare that I am the sole author of this report. This is a true copy of the report, including any required final revisions, as accepted by my examiners.

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Abstract

Conditional value at risk (CVaR) is an attractive alternative risk measure to value at risk (VaR) because of its coherence property and ability to capture tail risk. In data-driven CVaR optimization where only a small number of scenarios are available, the optimal solutions are prone to estimation errors which make them unreliable. Smoothing technique is a method to approximate the loss exceeding a target function and when used in the CVaR optimization it speeds up the computation dramatically, with a small relative difference. In this paper, we compare the results from the original CVaR optimization with the smoothed version and investigate whether the results from the latter are also prone to estimation errors and become unreliable solutions.

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1 Introduction

Value-at-risk (VaR) is a risk measure of the loss of a portfolio over a specific time horizon with a confidence level β. It is an appealing risk measure because of its ability to aggregate risks of instruments of different asset classes into a single number. It has become an industry standard for risk managers to assess the potential risk of portfolios and for regulators to impose capital requirement on financial institutions. Basel II permits banks to use VaR models as internal market risk models [10]. Despite its popularity, VaR has properties that make it undesirable. Because of its lack of sub-additivity, VaR does not reflect the diversification effect of portfolios, in which the VaR of the diversified portfolio may be greater than the total VaR of each instruments [7]. It is also non-convex which makes VaR optimization very difficult as it contains multiple local minima.

As an extension of VaR, Conditional value-at-risk (CVaR), also known as Expected Shortfall, is a risk measure of the conditional expectation of losses exceeding VaR over a specific time horizon with a confidence level β. VaR is a quantile measure which only tells us to expect a loss not greater than a certain value, while CVaR tells us what is the expected value of the tail loss. Because it can capture the tail risk better, CVaR is proposed to replace VaR for use in internal risk models by the Basel Committee in the consultative document published in 2012 [11]. Unlike VaR, CVaR is a coherent risk measure and has useful properties such as convexity [13]. Thus the CVaR optimization problem is a convex optimization in which the local minimum is in fact global.

Rockafellar and Uryasev [14] formulated portfolio optimization using CVaR as the risk measure. The key of their work was to define a convex auxiliary function as the objective function. The auxiliary function involves an integral which can be approximated by Monte Carlo simulation. The minimization of the function was proved to be equivalent to minimizing CVaR. In addition, their approach does not require solving for VaR first, but it is a by-product of the minimization.

The minimization problem can be translated into a linear programming problem as shown by Rockafellar and Uryasev. But the computation is inefficient for a large number of scenarios even with commercial LP solvers. Alexander, Coleman and Li demonstrated that by using the smoothing technique, such optimization is not only feasible but also significantly faster, at a cost of a small relative difference from the LP problem [16].

The distributions of returns of the underlying assets are often not known in practice. Therefore, historical data is typically a practical choice as input scenarios for the CVaR optimization. However, since the number of relevant historical markets is limited, the optimization is prone to estimation errors and becomes unreliable as shown by Lim, Shanthikumar and Vahn [9].

Lim, Shthikumar and Vahn demonstrated that the estimation errors of mean returns in both mean-variance and mean-CVaR optimization contribute significantly to the variation of the efficient frontiers. Robust optimization can be used to address the uncertainty of the mean returns. Min-max robust optimization generates an optimal portfolio which produces the best worst-case performance. An alternative robust model proposed by Zhu, Coleman and Li [18] is CVaR robust portfolio optimization. Its return performance is measured by CVaR and the optimal portfolio is generated based on the $(1-\beta)$-tail of the mean returns distribution. When β is high, the CVaR robust optimization produces a more robust and more diversified portfolio than the min-max robust optimization, because it takes a set of worst-case scenarios into account, instead of a single one.

These studies offer effective techniques to address the estimation errors of mean returns. Therefore, in this paper, we focus on the study of estimation errors of CVaR in data-driven CVaR optimization, without any estimation errors of the mean returns. The results of the original CVaR formulation by Rockafellar and Uryasev are compared with the smoothed version by Alexander, Coleman and Li, under the same distributions used by Lim, Shanthikumar and Vahn. We show that the smoothing technique helps reducing the variation of mean-CVaR frontiers and global minimum CVaR in all the distributions and it is more effective with higher smoothing resolution.

2 Mathematical Formulation

2.1 Formulation of VaR and CVaR

Let $f(\boldsymbol{x}, \boldsymbol{S})$ be the loss function of a portfolio with decision variable $\boldsymbol{x} \in \mathbb{R}^n$ and random variable $\boldsymbol{S} \in \mathbb{R}^q$. Let $\boldsymbol{x}$ be interpreted as the composition of the portfolio and $\boldsymbol{S}$ as the underlying risk factors. Let $p(\boldsymbol{S})$ denote the probability distribution function for the random variable $\boldsymbol{S}$. For each given $\boldsymbol{x}$, the loss of the portfolio, $f(\boldsymbol{x}, \boldsymbol{S})$, is also a random variable with a distribution induced by $\boldsymbol{S}$. The cumulative distribution function of $f(\boldsymbol{x}, \boldsymbol{S})$ not exceeding a threshold α, for a fixed $\boldsymbol{x}$, is then given by

$$\Phi(\boldsymbol{x}, \alpha) = \int_{f(\boldsymbol{x}, \boldsymbol{S}) \leq \alpha} p(\boldsymbol{S}) d\boldsymbol{S}$$

In general, $\Phi(\boldsymbol{x}, \alpha)$ is not necessarily continuous because of the possibility of jumps. However, in this paper, we assume $\Phi(\boldsymbol{x}, \alpha)$ to be continuous everywhere with respect to α.

The VaR of the loss random variable associated with a portfolio $\boldsymbol{x}$ and a confidence level β, is given by

$$\alpha_\beta(\boldsymbol{x}) = min\,\{\alpha \in \mathbb{R} : \Phi(\boldsymbol{x}, \alpha) \geq \beta\}$$

For $\Phi(\boldsymbol{x}, \alpha)$ is smooth, CVaR is the conditional expectation of all the loss exceeding $\alpha_\beta(\boldsymbol{x})$. It is given by [13]

$$\phi_\beta(\boldsymbol{x}) = \mathbb{E}(f(\boldsymbol{x}, \boldsymbol{S}) | f(\boldsymbol{x}, \boldsymbol{S}) > \alpha_\beta(\boldsymbol{x}))$$

Alternatively, it can be represented by

$$\phi_\beta(\boldsymbol{x}) = \frac{1}{1-\beta} \int\limits_{f(\boldsymbol{x},\boldsymbol{S}) \geq \alpha_\beta(\boldsymbol{x})} f(\boldsymbol{x}, \boldsymbol{S}) p(\boldsymbol{S}) d\boldsymbol{S} \tag{1}$$

as shown by [13] and [14].

Rockafellar and Uryasev proposed an augmented function to characterize both $\phi_\beta(\boldsymbol{x})$ and $\alpha_\beta(\boldsymbol{x})$ as follows [14].

$$F_\beta(\boldsymbol{x}, \alpha) = \alpha + \frac{1}{1-\beta} \int\limits_{\boldsymbol{S} \in \mathbb{R}^m} [f(\boldsymbol{x}, \boldsymbol{S}) - \alpha]^+ p(\boldsymbol{S}) d\boldsymbol{S} \tag{2}$$

where

$$[z]^+ = \begin{cases} z & : z > 0 \\ 0 & : z \leq 0 \end{cases}$$

The augmented function is convex and continuously differentiable. Rockafellar and Uryasev proved that minimizing the augmented function in terms of α is equivalent to solving for CVaR. Hence,

$$\phi_\beta(\boldsymbol{x}) = \min_{\alpha \in \mathbb{R}} \quad F_\beta(\boldsymbol{x}, \alpha) \tag{3}$$

Unlike equation (1), the optimization formulation does not require VaR to be predetermined, but calculated as part of the solution. The integral in equation (2) can be numerically approximated by Monte Carlo simulation as

$$\bar{F}_{\beta}(\boldsymbol{x}, \alpha) = \alpha + \frac{1}{q(1-\beta)} \sum_{i=1}^{q} [f(\boldsymbol{x}, \boldsymbol{S^i}) - \alpha]^{+} \tag{4}$$

Artzner, Delbaen, Eber and Heath [1] call a risk measure $p(.)$ defined on $G \in \mathbb{R}^{\kappa}$ coherent, if it satisfies the following four axioms.

Axiom 1 *Translation invariance. For all* $\boldsymbol{x} \in G, a \in \mathbb{R}$, $p(\boldsymbol{x} + a) = p(\boldsymbol{x}) + a$.

Axiom 2 *Subadditivity. For all* $\boldsymbol{x_1}, \boldsymbol{x_2} \in G$, $p(\boldsymbol{x_1} + \boldsymbol{x_2}) \leq p(\boldsymbol{x_1}) + p(\boldsymbol{x_2})$.

Axiom 3 *Positive homogeneity. For all* $\boldsymbol{x} \in G, \lambda \geq 0$, $p(\lambda \boldsymbol{x}) = \lambda p(\boldsymbol{x})$.

Axiom 4 *Monotonicity. For all* $\boldsymbol{x_1}, \boldsymbol{x_2} \in G$, *if* $\boldsymbol{x_1} \leq \boldsymbol{x_2}$, *then* $p(\boldsymbol{x_1}) \leq p(\boldsymbol{x_2})$.

VaR is not a coherent risk measure because it does not satisfy the subadditivity axiom. CVaR is coherent as proved by Pflug in [13].

Kou, Peng and Heyde [8] call a risk measure $p(.)$ robust if it can adapt to model uncertainty or misspecification, and is not sensitive to small changes in the data. They suggested that a risk measure used in regulatory purposes, such as capital requirement, has to be robust so that it can be enforced consistently among financial institutions. Each institution is allowed to use its own internal model and private data; however, given the exact same portfolio, each should hold at least the same amount of capital requirement. If the risk measure is not robust, the institutions can find a model or manipulate the data such that they only need to hold the least amount of capital. Kou, Peng and Heyde showed that coherent risk measures, including CVaR, are not robust with respect to small changes in the data. CVaR, in particular, is highly model-dependent that its computation relies on the assumptions on the extreme tails of the loss distributions. They proposed a more robust risk measure, Tail Conditional Median (TCM) as a better risk measure for regulatory purpose. The study of TCM is beyond the scope of this paper.

2.2 Formulation of CVaR optimization

The formulation (3) can be extended to formulate the portfolio optimization problem with regard to CVaR. Rockafellar and Uryasev proved that minimizing CVaR is equivalent to minimizing the augmented function $F_{\beta}(\boldsymbol{x}, \alpha)$ with regard to $(\boldsymbol{x}, \alpha)$. That is [14],

$$\min_{\boldsymbol{x}\in\mathbb{R}^m} \quad \alpha_\beta(\boldsymbol{x}) \quad = \min_{(\boldsymbol{x},\alpha)\in\mathbb{R}^m\times\mathbb{R}} \quad F_\beta(\boldsymbol{x},\alpha)$$

It is important to note that the VaR, α^*, which is resolved as part of the solution of the CVaR-optimized portfolio, $\boldsymbol{x}^*$, may not be the optimized VaR. However, the optimized CVaR gives us an upper bound of the optimized VaR for, by definition, $\phi_\beta(\boldsymbol{x}) \geq \alpha_\beta(\boldsymbol{x})$.

In this paper, we let $\boldsymbol{y_i}$ be the uncertain return of instruments under the i^{th} scenario, x_j be the position of the j^{th} instrument in the portfolio and n be the number of instruments in the portfolio. Therefore, the loss function $f(\boldsymbol{x}, \boldsymbol{S^i})$ under the i^{th} scenario can be represented by

$$f(\boldsymbol{x}, \boldsymbol{S^i}) = -\sum_{j=1}^{n} x_j y_{i,j} = -\boldsymbol{x}^T \boldsymbol{y_i}$$

We also consider each x_j as a weight of the portfolio, so that we have a budget constraint of the CVaR optimization problem

$$\sum_{j=1}^{n} x_j = 1$$

Furthermore, it is reasonable to require an optimized portfolio to return an expected amount R. We impose a constraint on the expected return of the portfolio, with g_j the expected return of the j^{th} instrument

$$\sum_{j=1}^{n} x_j g_j \geq R$$

Consider $Y = [\boldsymbol{y_1}, \boldsymbol{y_2}, \ldots, \boldsymbol{y_q}]$ a matrix of excess returns of all the instruments in all scenarios. The complete formulation of the CVaR optimization can be written as

$$\begin{aligned} \min_{(\boldsymbol{x},\alpha)\in\mathbb{R}^m\times\mathbb{R}} \quad & \alpha + \frac{1}{q(1-\beta)} \sum_{i=1}^{q} [-\boldsymbol{x}^T \boldsymbol{y_i} - \alpha]^+ \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j g_j \geq R \\ & \sum_{j=1}^{n} x_j = 1 \end{aligned} \tag{5}$$

By using a range of minimal expected return R, we can generate an efficient frontier of mean-CVaR.

2.3 Linear Programming of CVaR optimization

In order to solve the optimization problem (5) using LP-solvers, it needs to be transformed into a linear programming problem such as the following [13]:

$$\begin{aligned} \min_{(\pi,\alpha)\in\mathbb{R}^m\times\mathbb{R}} \quad & \alpha + \frac{1}{q(1-\beta)}\sum_{i=1}^{q} z_i \\ \text{subject to} \quad & \boldsymbol{z} \geq -\boldsymbol{x}^T Y - \alpha \mathbf{1}^{\boldsymbol{T}} \\ & \sum_{j=1}^{n} x_j g_j \geq R \\ & \sum_{j=1}^{n} x_j = 1 \end{aligned} \tag{6}$$

3 Smoothing Approximation

As shown in the previous section, the plus function, $[.]^+$, in (5) can be transformed into linear programming form by introducing q slack variables and q constraints. The resulting linear programming (6), which consists of $O(q+n)$ variables and $O(q+n)$ constraints, can then be solved by any LP-solvers. In the standard form of LP, $min\left\{\boldsymbol{c}^T\boldsymbol{x}|A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}\right\}$, the matrix A is a dense matrix with size determined by the number of variables and scenarios. As the size of the portfolio or the number of scenarios increase, the computational cost increases drastically and it becomes nearly impossible for a large scale CVaR optimization [16].

Without introducing slack variables and new constraints, another technique for solving minimization problems with a plus function is to apply smoothing approximation. It is a technique to approximate the plus function with a high degree of accuracy. Chen and Mangasarian have shown significant performance gain in solving linear, convex and nonlinear complementarity problems by using the technique [4] [5].

With regard to the CVaR optimization problem, Alexander, Coleman and Li proposed to apply smoothing approximation to address the inefficiency in solving large-scale problems [16]. The piecewise linear approximation in equation (4) is replaced by a continuously differentiable piecewise quadratic approximation $\tilde{F}(\boldsymbol{x}, \alpha)$,

$$\tilde{F}_\beta(\boldsymbol{x}, \alpha) = \alpha + \frac{1}{q(1-\beta)} \sum_{i=1}^{q} \rho_\epsilon(f(\boldsymbol{x}, \boldsymbol{S^i}) - \alpha) \tag{7}$$

where $\rho_\epsilon(z)$ is a smoothing approximation to $max(z, 0)$, with a resolution parameter ϵ. The proposed formulation is

$$\rho_\epsilon(z) = \begin{cases} z & : z \geq \epsilon \\ \frac{z^2}{4\epsilon} + \frac{1}{2}z + \frac{1}{4}\epsilon & : -\epsilon \leq z \leq \epsilon \\ 0 & : otherwise \end{cases} \tag{8}$$

The effect of the smoothing technique is illustrated in Figure 1. The function $f(\alpha) = E([S - \alpha]^+)$ is approximated by

$$\frac{1}{m} \sum_{i=1}^{m} [S_i - \alpha]^+$$

in which S follows a Normal distribution. As the number of samples increases, the smoothing functions show smaller difference from the piecewise linear function.

With the smoothing approximation, the alternative CVaR optimization is a continuous piecewise quadratic convex programming problem

$$\begin{aligned} \min_{(\boldsymbol{x},\alpha) \in \mathbb{R}^m \times \mathbb{R}} \quad & \alpha + \frac{1}{q(1-\beta)} \sum_{i=1}^{q} \rho_\epsilon(-\boldsymbol{x}^T \boldsymbol{y_i} - \alpha) \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j g_j \geq R \\ & \sum_{j=1}^{n} x_j = 1 \end{aligned} \tag{9}$$

The choice of ϵ is problem dependent. In the CVaR optimization problem, Alexander, Coleman and Li suggested a typical range between 0.005 and 0.05. Using an interior point method solver, they demonstrated that the smoothing approximation with $\epsilon = 0.005$ is

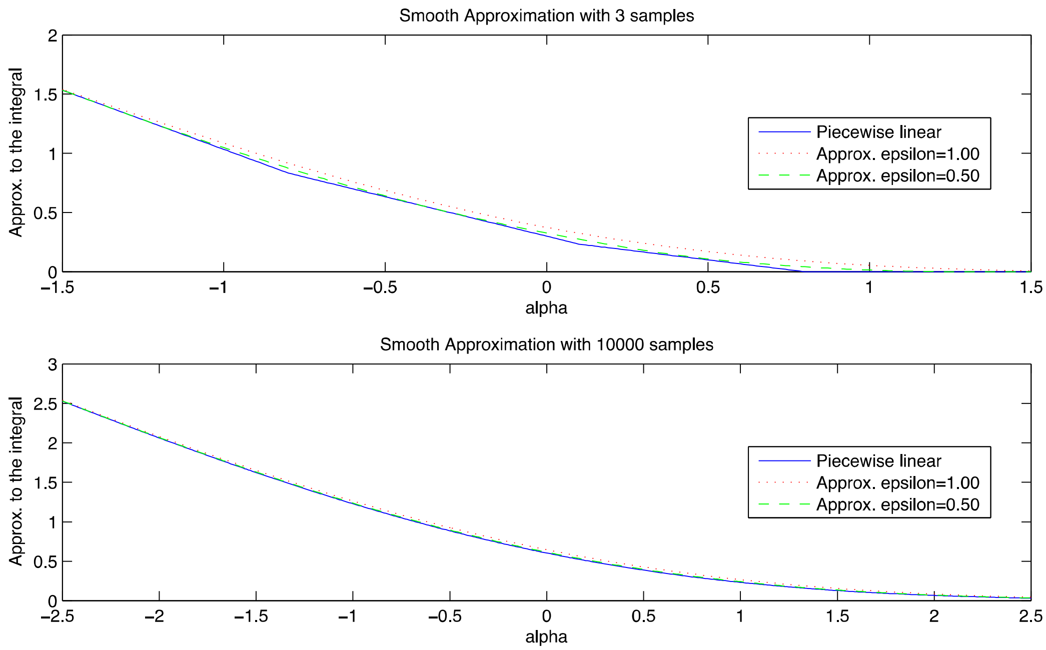

Figure 1: Plus function and smooth approximation

several times faster than the linear programming approach and only has a small relative difference (at most 1.5% in their experiments). For a large number of simulations, a smaller ϵ is recommended to take advantage of the efficiency gain yet with a negligible discrepancy.

4 Data-driven CVaR Optimization

One of the common schemes for scenario generation is by historical simulation. Historical data of the instruments or their underlyings is gathered over a certain period of time, such as the past one year or some stressed periods like the financial crisis in 2008 or the dot-com bubble in 2001. For simple instruments such as common stocks, the historical data is simply their prices. For derivatives, the contract values can either be re-priced or approximated by sensitivities such as delta-gamma-theta approximation [3], although sometimes the sensitivities cannot be easily obtained. In this paper, we will focus on the simple instruments for simplicity, since this is sufficient for our investigation.

Since CVaR is a tail statistic and we are using Monte Carlo simulation to approximate the integral in the CVaR formulation, a large number of observations or scenarios is required for an accurate result. However, in historical simulation, the length of the observation period is typically limited to one to two years, and rarely more than five years. The data-driven CVaR optimization is prone to estimation errors due to insufficient amount of data.

As described in the work of Black and Litterman, in the mean-variance portfolio optimization problem, the optimal portfolio selection is highly sensitive to the expected mean [2]. In the work later by Lim, Shanthikumar and Vahn, they also demonstrated that the sampling errors of the mean have a significant impact on the mean-variance problem as well as on the mean-CVaR problem [9]. In order to prevent the effect of sampling errors in the mean from understanding the effect of the sampling errors on CVaR, we consider the global minimum CVaR (GMC) and true mean-empirical CVaR (TMEC) problems posed by their paper.

4.1 Global Minimum CVaR (GMC)

To avoid the sampling errors from the empirical mean, a natural method is to remove the requirement of minimum return of the portfolio in formulation (5) and (9). The resulting formulations will seek for the global minimum CVaR. In this formulation, if the portfolio consists of a risk-free asset, the solution will become trivial that the optimal portfolio will only consist of the risk-free asset because it has no loss in all scenarios.

4.2 True Mean-Empirical CVaR (TMEC)

Alternatively, to avoid the sampling errors from the mean in our analysis is to assume the true mean is given. Let $\boldsymbol{\mu}$ be the true mean of the returns. The constraint on the expected return of the portfolio in formulation (5) and (9) is replaced by

$$\sum_{j=1}^{n} x_j \mu_j \geq R$$

Since the same mean is used for all experiments, it eliminates the sampling errors of the mean when we compare the results. The variation of the mean-CVaR frontiers can be attributed to the sampling errors of the CVaR.

5 Approach

Lim, Shanthikumar and Vahn, from their experiments [9], observed significant variation in the efficient mean-CVaR frontiers when the number of scenarios that is used to generate each frontier is small, even though the true mean is known in the TMEC problem or the mean is not taken into account in the GMC problem. The main effect of errors on CVaR is the sampling errors from the limited data set, which are amplified by the optimization process. The variation is reduced when the number of scenarios increases. This is a typical situation in data-driven CVaR optimization where only a time span of one or two years of real world market data is used. Due to the variation, they concluded that the CVaR-optimal portfolio is not a reliable solution. It largely underestimates the CVaR and the true risk exposure of the portfolio.

The smoothing approximation is a good estimate of the plus function when a large number of scenarios is used. In this paper, we will investigate whether the same effect of sampling errors will happen in the smoothing approximation. We will compare the results between CVaR optimization under the smoothing approximation and the original optimization. Therefore, the experiments are run under the same distributions of returns by Lim, Shanthikumar and Vahn, and similar evaluation methodology is used.

The distributions in which we simulate the historical data are a multivariate normal distribution, a mixture of multivariate normal and negative exponential distributions, and a mixture of multivariate normal and one-sided power distributions. We use each distribution to generate the excess returns of 5 instruments. The sample histogram of each distribution is shown in Figure 2.

5.1 Multivariate Normal

The excess returns of instruments, Y, in this case follow a multivariate normal distribution. Under the assumption of a normal distribution both the VaR and CVaR can be solved by analytical methods. It does not capture any tail event but will be a base case for the other two distributions.

The data used by Lim, Shanthikumar and Vahn is real historical data of 5 stock indices in North America: Dow Jones Industrial Average (DJI), NASDAQ Composite (IXIC), NYSE Composite (NYA), S&P 100 (OEX) and S&P 500 (GSPC). The time period is from August 3, 1984 to June 1, 2009. The time interval between each scenario is one month [9].

$$Y \sim N(\boldsymbol{\mu}, \Sigma)$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} 26.11 \\ 25.21 \\ 28.90 \\ 28.68 \\ 24.18 \end{pmatrix} * 10^{-4} \qquad \Sigma = \begin{pmatrix} 3.715 & 3.730 & 4.420 & 3.606 & 3.673 \\ 3.730 & 3.908 & 4.943 & 3.732 & 3.916 \\ 4.420 & 4.943 & 8.885 & 4.378 & 5.010 \\ 3.606 & 3.732 & 4.378 & 3.930 & 3.789 \\ 3.673 & 3.916 & 5.010 & 3.799 & 4.027 \end{pmatrix} * 10^{-4}$$

5.2 Multivariate Normal + Negative Exponential Tail

The second case we consider is a mixture of a multivariate normal and a negative exponential distributions. In most of the scenarios, the excess returns will follow a multivariate normal distribution, but with a small probability a perfectly correlated exponential tail loss will occur in all instruments. The tail loss probability follows a Bernoulli distribution.

$$Y \sim (1 - I(p))N(\boldsymbol{\mu}, \Sigma) + I(p)(Z\boldsymbol{e} + \boldsymbol{f})$$

where

$$f_Z(x) = \begin{cases} \lambda e^{\lambda x} & : x \leq 0 \\ 0 & : x > 0 \end{cases}$$

The parameters we use in our experiments are $p = 0.05$, $\lambda = 10$, e is 5x1 vector of ones and $f_i = \mu_i - \sqrt{\Sigma_{ii}}$.

5.3 Multivariate Normal + One-sided Power Tail

The last case we consider is a mixture of a multivariate normal and a one-sided power distributions. The formulation is similar to the second case. The power distribution considered by Lim, Shanthikumar and Vahn is a special case of a Pareto distribution. The Pareto distribution is a heavy-tail distribution with a shape parameter α, a scale parameter x_m and a probability density function [17]

$$f_Z(x) = \begin{cases} \frac{\alpha x_m^{\alpha}}{x^{\alpha+1}} & : x \geq x_m \\ 0 & : x < x_m \end{cases}$$

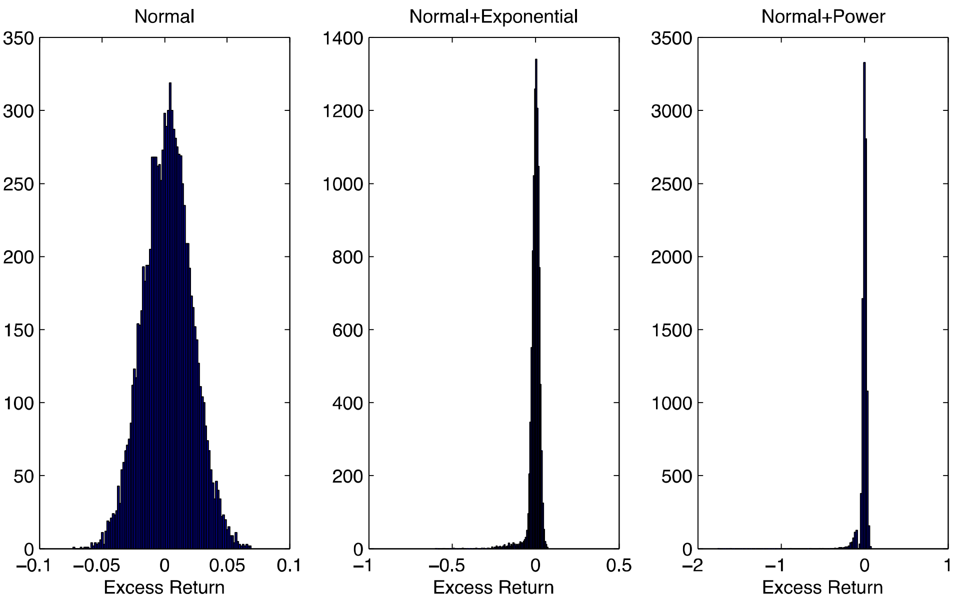

Figure 2: Histogram of 10,000 samples of each distribution

For the special case we set $\alpha = \gamma - 1$, $x_m = -1$. That is,

$$f_Z(x) = \begin{cases} \frac{\gamma-1}{(-x)^\gamma} & : x \le -1 \\ 0 & : x > -1 \end{cases}$$

With a small probability, a perfectly correlated power tail loss will occur to all instruments. The excess returns Y follows

$$Y \sim (1 - I(p))N(\boldsymbol{\mu}, \Sigma) + I(p)Z(\gamma)\boldsymbol{f}$$

The parameters in our experiments are $p = 0.05$, $\gamma = 3.5$ and $f_i = \mu_i - 5\sqrt{\Sigma_{ii}}$.

5.4 Algorithms for Efficient Frontiers

In this section we describe the algorithms for generating efficient frontiers in the TMEC and GMC cases with the original CVaR optimization problem and the smoothing approximation.

5.4.1 Algorithm for TMEC

This is the algorithm to generate an efficient frontier in the TMEC case. Repeat the process for as many frontiers as necessary.

1. Generate $Y = [\boldsymbol{y_1}, .., \boldsymbol{y_q}]$ samples from excess return distribution M

2. For $R = [R_1, .., R_d]$, range of minimum expected return

   (a) Solve for the optimal portfolio $\boldsymbol{x}^*$ for R, using formulation (5) for the original CVaR optimization, and formulation (9) for the smoothing approximation.

   (b) Generate $D = [\boldsymbol{d_1}, .., \boldsymbol{d_m}]$ samples from distribution M, for a large m.

   (c) Solve for the true CVaR using Monte Carlo simulation with D and the formulation (4).

   (d) Compute for the expected mean by $Ex = \sum_{j=1} x_j \mu_j$

   (e) Plot the coordinate $(CVaR, Ex)$ to construct the frontier.

5.4.2 Algorithm for GMC

The algorithm in the GMC case is very similar to the TMEC case, except that it does not require specifying a minimum expected return. The result is not an efficient frontier but a single point. Repeat the process to generate a scatter plot of minimum CVaR and minimum mean of returns.

1. Generate $Y = [\boldsymbol{y_1}, .., \boldsymbol{y_q}]$ samples from excess return distribution M

2. Solve for the optimal portfolio $\boldsymbol{x}^*$, using formulation (5) for the original CVaR optimization, and formulation (9) for the smoothing approximation.

3. Generate $D = [\boldsymbol{d_1}, .., \boldsymbol{d_m}]$ samples from distribution M, for a large m.

4. Solve for the true CVaR using Monte Carlo simulation with D and the formulation (4).

5. Compute for the expected mean by $Ex = \sum_{j=1} x_j \mu_j$

6. Plot the coordinate $(CVaR, Ex)$.

5.4.3 Software Package

CVX is an optimization modeling language implemented in Matlab. It can solve standard optimization problems including linear programs (LP), quadratic programs (QP), second-order cone programs (SOCP) and semidefinite programs (SDP) [6]. Problems can be defined in CVX as close as how they are written. In addition, some of the common functions such as max and min are directly supported. Hence, in our experiments, we will use CVX to solve for the optimal portfolios in formulation (5). The smoothing technique in formulation (9) is a piecewise function which is not supported in CVX. We will use the 'fmincon' function from the Optimization Toolbox in Matlab.

6 Experiments

In the following experiments, in order to keep the simulated data consistent, the scenario generation for each distribution is started by the same random seed and they are generated prior to the execution of the experiments. In computing the true CVaR with the optimal portfolios, the same $10,000$ simulated scenarios are used across all experiments, such that there is not any variation in computing the true CVaR.

In Experiment 1 and 2 we will replicate the TMEC and GMC observations made by Lim, Shanthikumar and Vahn in the original CVaR formulation, and compare them with the results of the smoothing technique. In Experiment 3 and 4, we will investigate the effect of the smoothing parameter ϵ on the results. In Experiment 5 and 6, we will vary the shape of the distributions and examine how it affects the effectiveness of the smoothing technique.

6.1 Experiment 1

In this experiment, we compare the impact on the efficient frontiers in the TMEC setting between the original CVaR optimization problem and the smoothed version, with respect to the number of simulated scenarios q. The smoothing parameter is fixed at $\epsilon = 0.005$ and $\epsilon = 0.05$ to show any difference at different smoothness resolution. The experiment is run under all three distributions.

The results in the Normal distribution are shown in Figure 3. We observe that the mean-CVaR frontiers generated by the smoothing technique vary less as the parameter ϵ increases. The frontiers by $\epsilon = 0.05$ are apparently more concentrated than those by

$\epsilon = 0.005$ and the original formulation. Table 1 that shows the standard deviation in the upper, middle and lower sections of the frontiers confirms the observation in the plots. With $\epsilon = 0.005$, the smoothed version reduces the standard deviation by about 5-6% with 50 scenarios and 8-18% with 400 over the original formulation. With $\epsilon = 0.05$ the standard deviation is significantly reduced by at least 30% with 50 scenarios and 53% with 400.

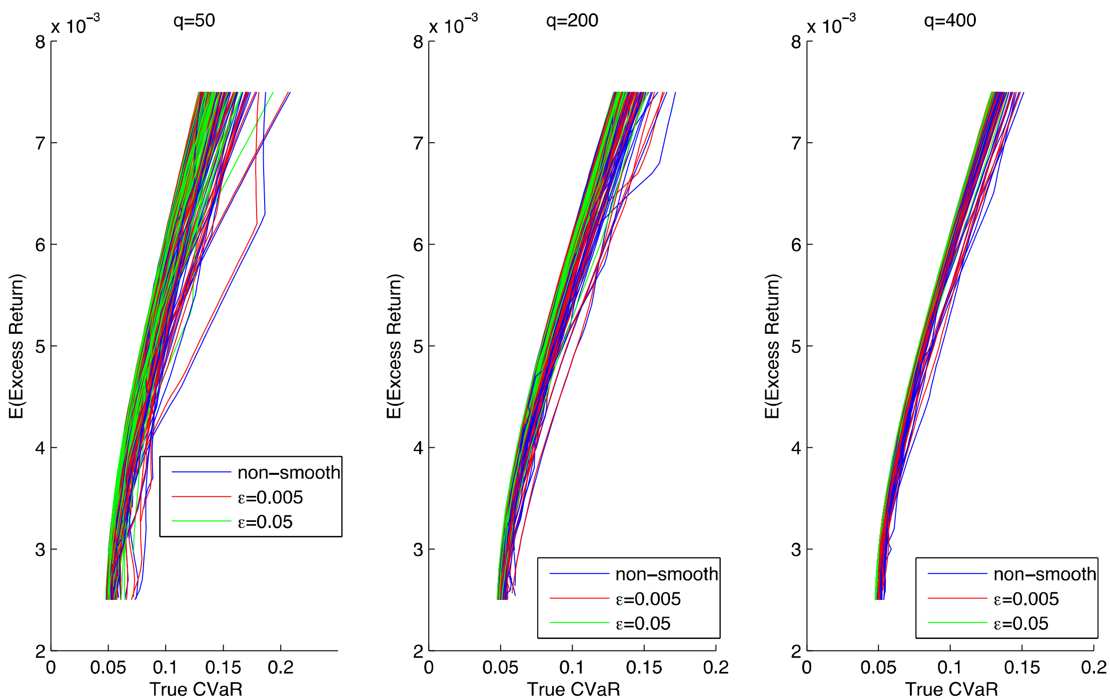

Figure 3: Experiment 1: Efficient frontiers in TMEC in Normal distribution.

In Figure 4 and Table 2 , the results in the mixture of the Normal and Exponential distributions also indicate that the smoothing technique reduces the standard deviation from the original formulation, although it is not as significant as in the Normal distribution. The smoothed version with $\epsilon = 0.05$ reduces 11-14% of variation with 50 scenarios and 21-40% with 400.

The results in the mixture of the Normal and one-sided Power distributions are illustrated in Figure 5. As in the other two distributions, we observe that as the smoothing parameter increases, the variation of the frontiers is less. The detailed comparison is shown in Table 3. The reduction in variation is similar to the case in the Normal distribution.

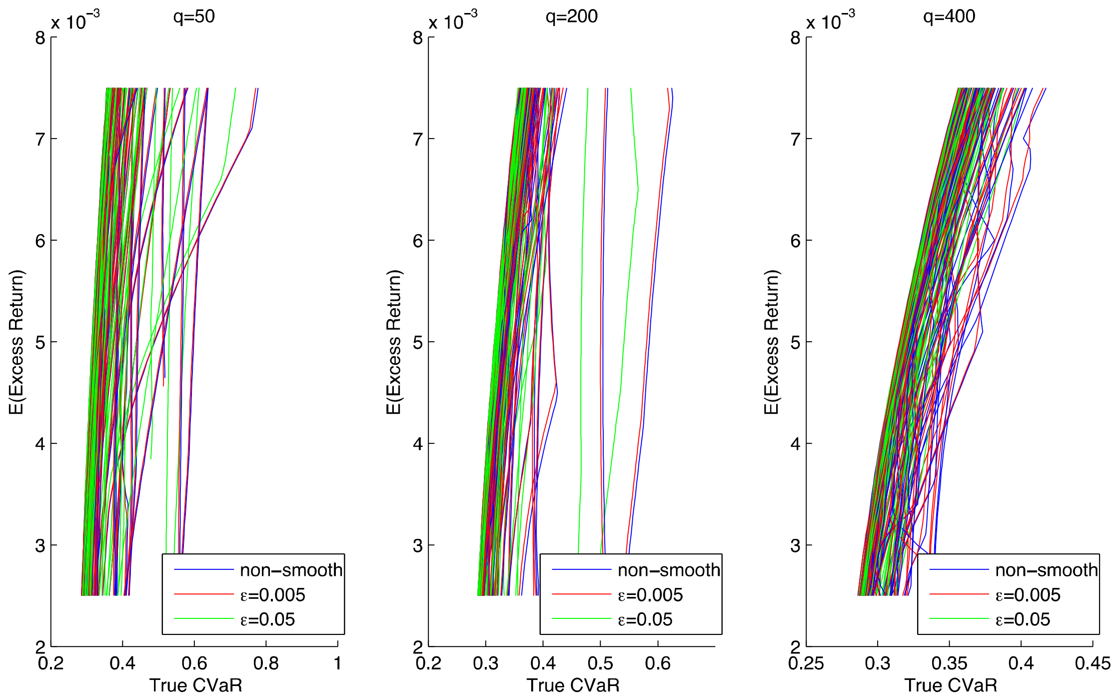

Figure 4: Experiment 1: Efficient frontiers in TMEC in Normal+Exponential distribution.

The smoothed version with $\epsilon = 0.005$ reduces 5-6% standard deviation with 50 scenarios, while with $\epsilon = 0.05$ it reduces 32-43%. With 400 scenarios, the former reduces 1-18% and the latter 44-54%.

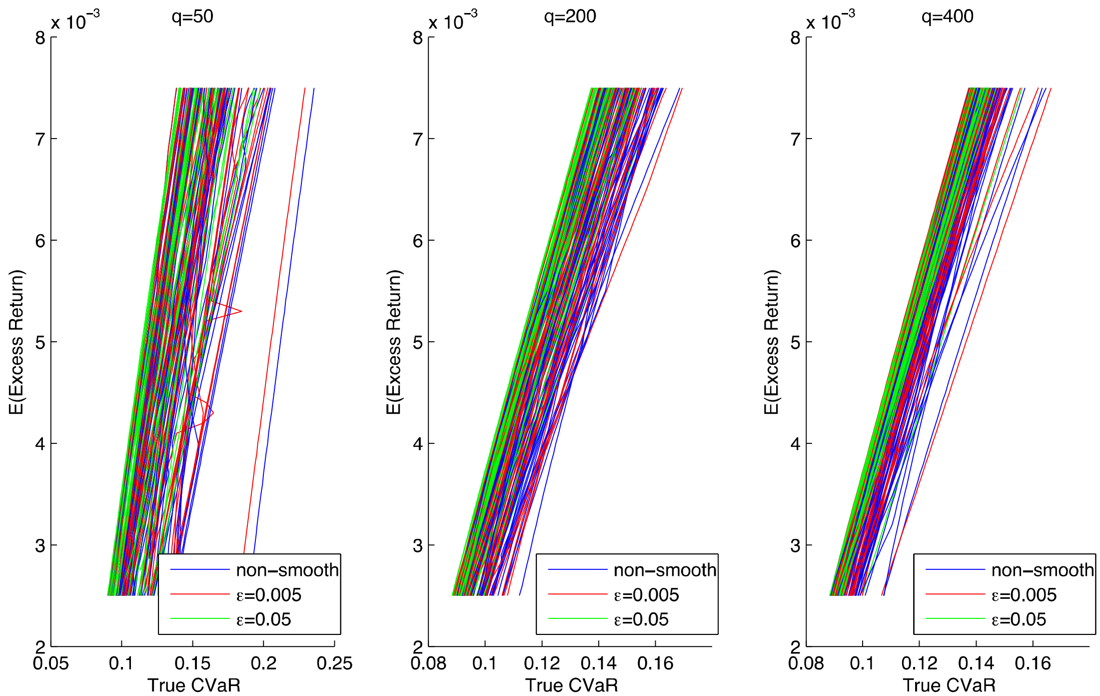

Figure 5: Experiment 1: Efficient frontiers in TMEC in Normal+Power distribution.

6.2 Experiment 2

This experiment is similar to the Experiment 1, but we measure the variation of minimum CVaR in the GMC setting with various number of scenarios q. We only consider the CVaR produced by both formulations because the expected mean is not the subject of our study.

The scatter plot of the minimum CVaR and the corresponding expected mean in Figure 6 shows that as the smoothing effect increases, the minimum CVaR generated by the smoothed formulation is more stable than the original version. Results in Table 4 confirm the same observation that even when more scenarios are available, the smoothed version consistently produces less variation. The reduction in standard deviation is close to the results of TMEC in Experiment 1.

The improvement of the smoothed version over the original is less obvious in the mixture of the Normal and Exponential distributions as shown in Figure 7. In Table 5, we observe that the results by the smoothed version with $\epsilon = 0.005$ contain only slightly lower standard

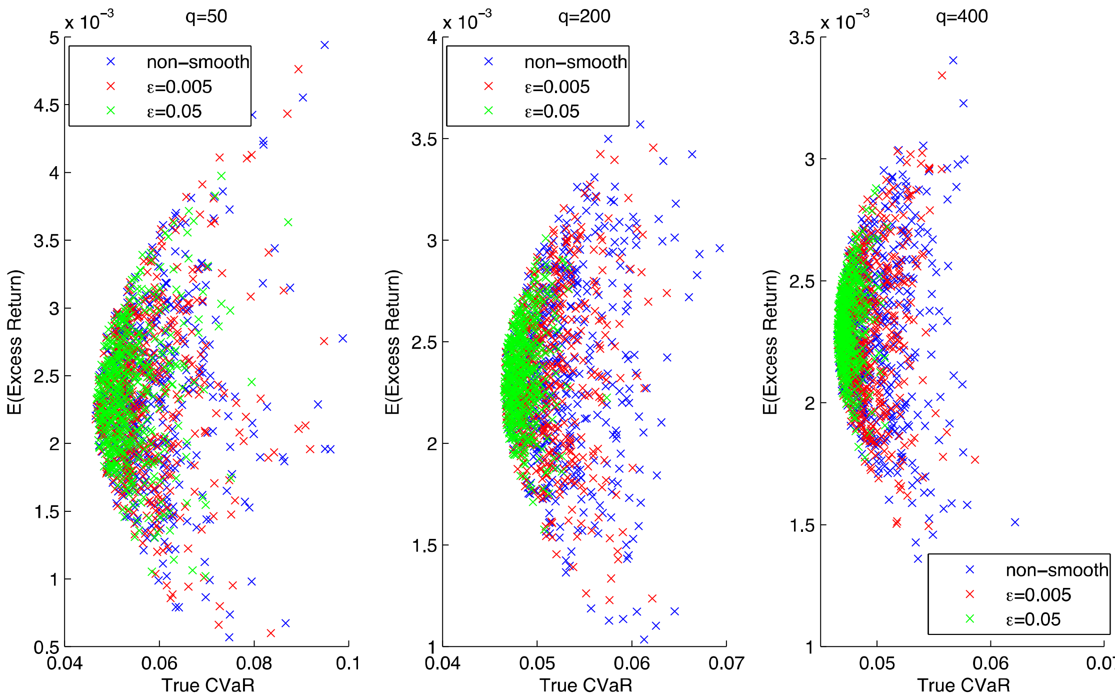

Figure 6: Experiment 2: Scatter plot of true CVaRs and minimum expected means in Normal distribution.

deviation than the original with 50 and 200 scenarios. However, with 400 scenarios, it actually increases the deviation. The smoothed version with $\epsilon = 0.05$ reduces the deviation by 12-25%, which is similar to the observations in Experiment 1.

The deviation reduction in the mixture of Normal and the Power distributions is the worst among the three distributions. From the results in Figure 8 and Table 6, the smoothed version with $\epsilon = 0.005$ reduces only 0.37% and 1.14% with 50 and 200 scenarios, respectively; however, it increases deviation with 400 scenarios. The smoothed version with $\epsilon = 0.05$ only reduces 2-10%.

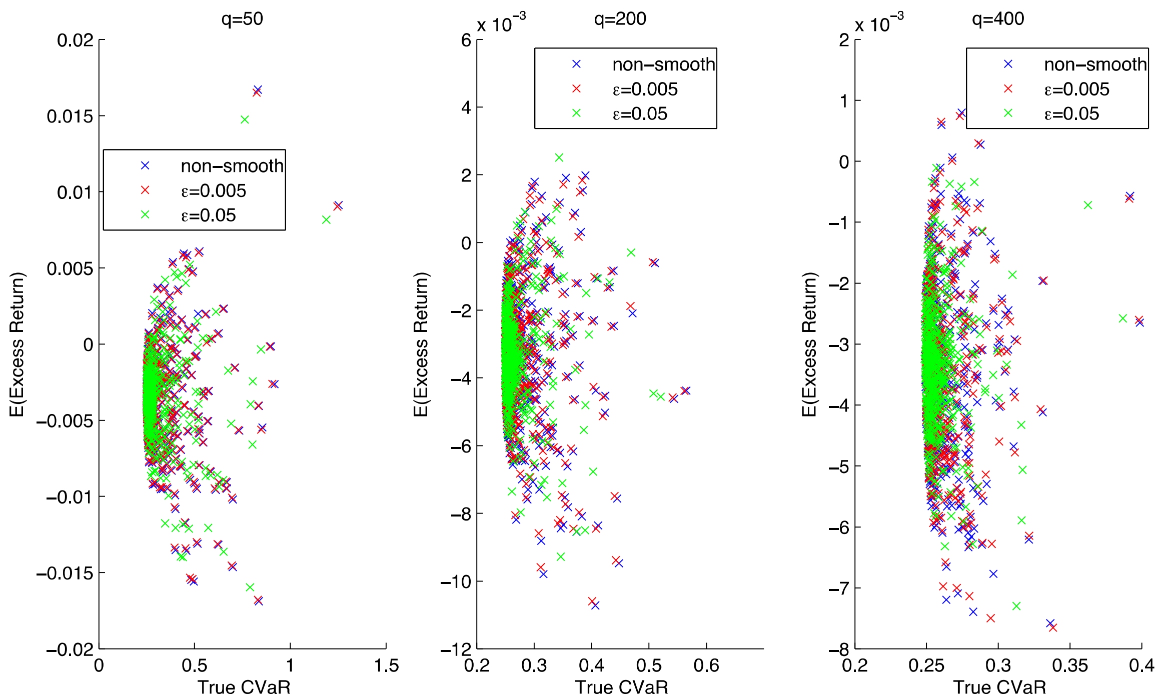

Figure 7: Experiment 2: Scatter plot of true CVaRs and minimum expected means in Normal+Exponential distribution.

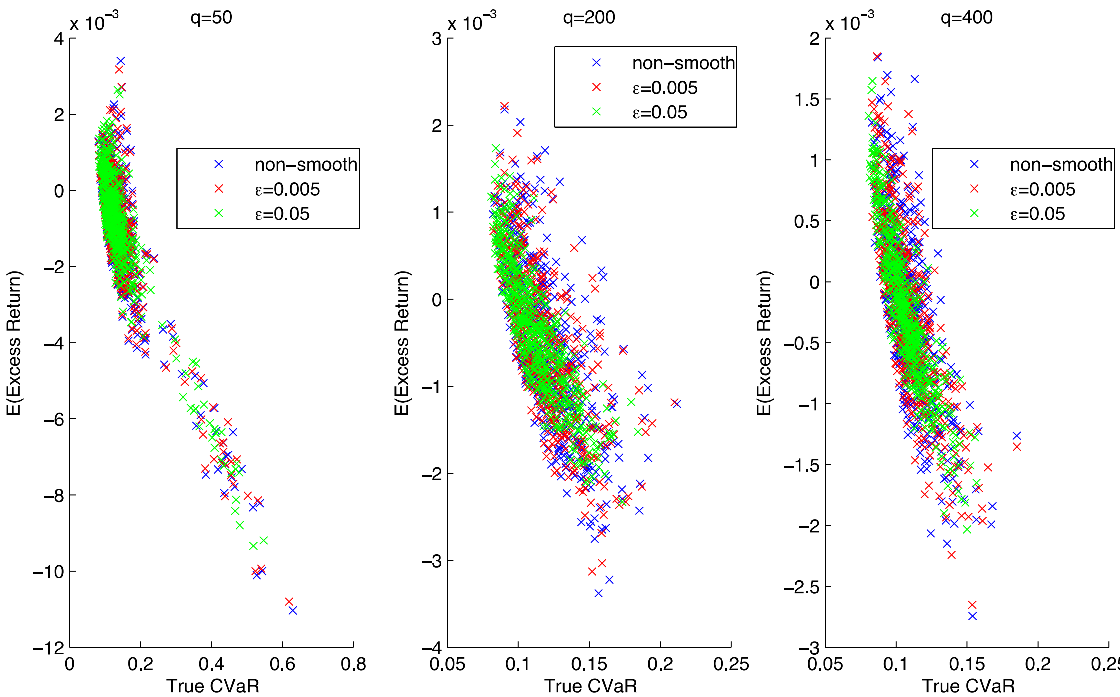

Figure 8: Experiment 2: Scatter plot of true CVaRs and minimum expected means in Normal+Power distribution.

6.3 Experiment 3

From this experiment on, we will focus on the situations where only few scenarios are available because this is the case where the original CVaR formulation is fragile. For consistency with the previous experiments we will run the experiments with only 50 scenarios. In this experiment we will compare the performance of a range of smoothing parameter ϵ on the variation of the minimum CVaR results in the TMEC setting. We want to test if the reduction in variation will be greater with a higher smoothing resolution.

From Table 7, 8 and 9, we observe a consistency among all three distributions that as the smoothing resolution ϵ increases, the standard deviation of the minimum CVaR decreases. With $\epsilon = 0.01$, the smoothed version reduces about 10% of standard deviation over the original formulation in the Normal and the mixture of the Normal and Power distributions. However, in the mixture of the Normal and Exponential distributions, the smoothing resolution has to be increased to $\epsilon = 0.05$ to produce a similar reduction.

6.4 Experiment 4

In this experiment we repeat the experiment 3 to compare the performance of different smoothing parameters in the GMC setting. The results are shown in Table 10, 11 and 12.

In the GMC setting, we continue to observe that higher smoothing resolution helps reducing standard deviation of the minimum CVaR. The results in the Normal and the mixture of Normal and Exponential distributions are similar to the TMEC setting. However, in the mixture of Normal and Power distribution, the reduction is much less than in the TMEC, although it still reduces the deviation consistently. Similar observation is made in Experiment 2 where even with high smoothing resolution, the reduction in deviation is not significant.

6.5 Experiment 5

The rate of the exponential tail loss is controlled by the λ parameter in the mixture of Normal and Exponential distributions. As λ increases, the exponential loss occurs more frequently, as illustrated in Figure 9. In this experiment we will compare the performance of variation reduction by the smoothing technique in various λ setting. The results are shown in Table 13.

The reduction of the standard deviation of minimum CVaR by the smoothing technique is more significant as exponential loss happens more frequently (as λ increases). With

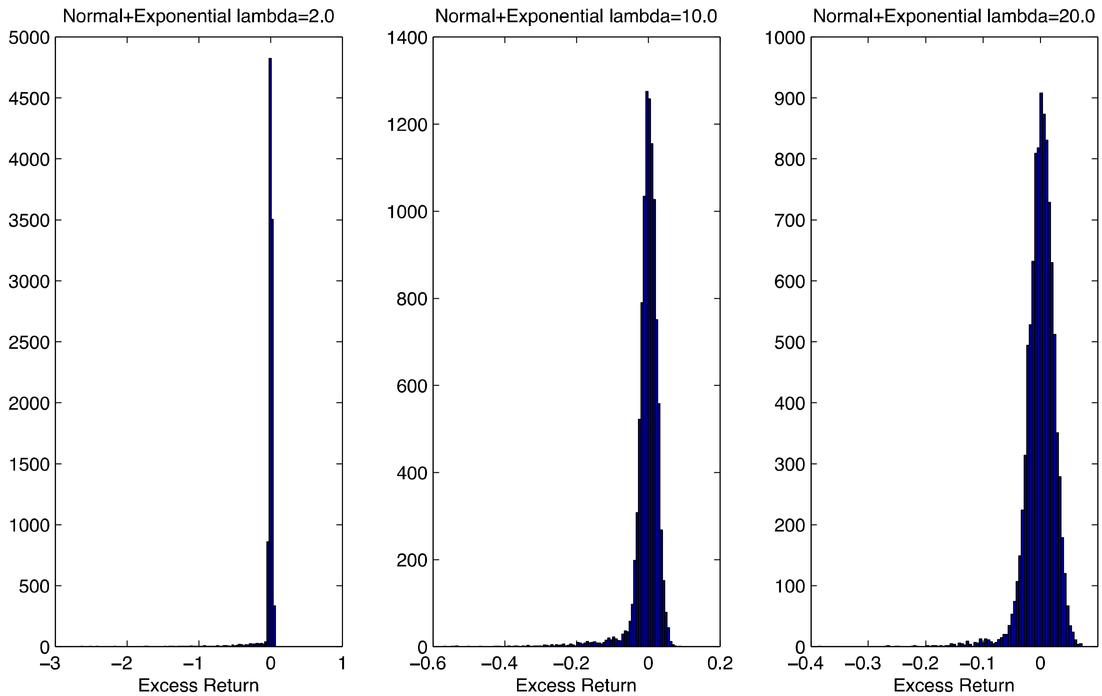

Figure 9: Histogram of 10,000 samples of the Normal+Exponential distribution with various λ.

$\lambda = 2.0$, the rate of event occurrence is lower. As observed in Figure 9, most of the occurrences are concentrated in the center, around 0. In this distribution the excess returns are likely to happen within the center. In this situation, we observe that the smoothing technique does not reduce the variation by much. However, on the other extreme, when $\lambda = 20.0$, the excess returns are more likely to happen further from the center. In this case, the smoothing technique reduces much more variation.

6.6 Experiment 6

For the mixture of the Normal and one-sided Power distributions, as illustrated in Figure 10, there is a jump in the frequency of the tail loss that results in a hump in the histogram, as the parameter γ increases. In this experiment we will compare the performance of variation reduction by the smoothing technique in various γ setting. The results are shown in Table 14.

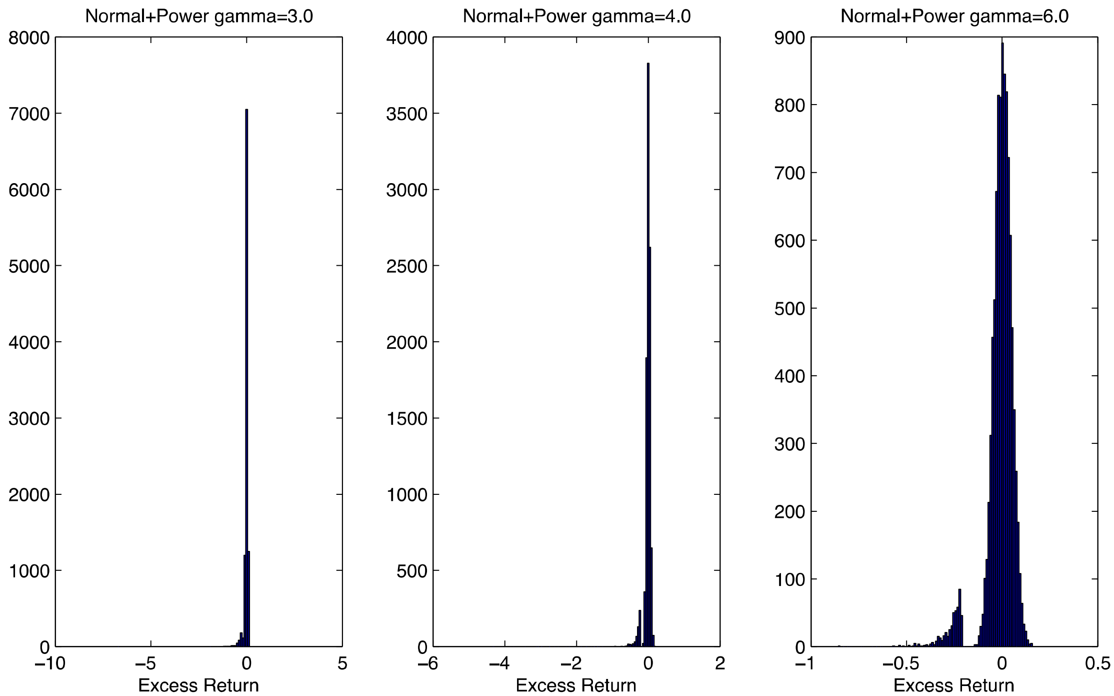

Figure 10: Histogram of 10,000 samples of the Normal+Power distribution with various γ.

The reduction in the standard deviation of minimum CVaR by the smoothing technique is more effective as the γ parameter of the Power distribution increases. When γ is as low as 3.0, most of the excess returns are concentrated in the center, which is 0. In this case, the smoothing technique is not very effective in reducing the standard deviation. However, when γ is increased to 6.0, the excess returns are no longer concentrated around 0 but a good portion are negative returns further away from 0. In this case, the smoothing technique is effective at reducing the standard deviation.

7 Conclusion

In this paper, we compare the minimum CVaR computed for the optimal portfolios which are solved by the original CVaR optimization formulation proposed by Rockafellar and Uryasev and by the smoothed version by Alexander, Coleman and Li. The experiments are run in both the TMEC setting where the true mean of the underlying distribution is known and a minimum expected return is guaranteed, and the GMC where the global minimum CVaR is sought. In the experiments where only 50 scenarios are available, the smoothed version always produces minimum CVaR with a lower standard deviation among the samples than the original formulation. In the paper by Lim, Shanthikumar and Vahn, the optimal portfolios solved by the original formulation were shown to be unreliable. In this paper, we observe that the smoothing technique can improve the reliability of the optimal portfolios.

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Appendix

Experiment 1

Table 1: Experiment 1. Comparison of variation reduction of different values of smoothing parameter ϵ (0.0, 0.005, 0.05). $\epsilon = 0.0$ means no smoothing. q (50, 200, 400) mean-CVaR frontiers are generated with range of expected means $[2.5 : 7.5] * 10^{-3}$. Portfolio losses are simulated under Normal distribution. Comparison is shown by the minimum, maximum and standard deviations of CVaRs at different level of expected means.

	$\epsilon = 0.0$	$\epsilon = 0.005$	$\epsilon = 0.05$
mean=0.0030			
q=50	min: 0.050379 max: 0.081689 std: 0.005937	min: 0.050523 max: 0.078693 std: 0.005577 (-6.08 %)	min: 0.050540 max: 0.064479 std: 0.003982 (-32.93 %)
q=200	min: 0.050294 max: 0.062970 std: 0.002749	min: 0.050393 max: 0.063121 std: 0.002437 (-11.33 %)	min: 0.050249 max: 0.055630 std: 0.001166 (-57.56 %)
q=400	min: 0.050076 max: 0.058771 std: 0.001771	min: 0.050061 max: 0.056835 std: 0.001442 (-18.59 %)	min: 0.050117 max: 0.053348 std: 0.000696 (-60.73 %)
mean=0.0050			
q=50	min: 0.079099 max: 0.130318 std: 0.009690	min: 0.078406 max: 0.127671 std: 0.009127 (-5.81 %)	min: 0.078564 max: 0.109421 std: 0.005994 (-38.14 %)
q=200	min: 0.078071 max: 0.103711 std: 0.005265	min: 0.078148 max: 0.103454 std: 0.005040 (-4.26 %)	min: 0.078300 max: 0.093084 std: 0.002879 (-45.32 %)
q=400	min: 0.078433 max: 0.094300 std: 0.002951	min: 0.078141 max: 0.092386 std: 0.002706 (-8.30 %)	min: 0.077987 max: 0.083424 std: 0.001227 (-58.43 %)
mean=0.0070			
q=50	min: 0.118340 max: 0.185188 std: 0.015255	min: 0.118166 max: 0.183027 std: 0.014459 (-5.21 %)	min: 0.118491 max: 0.170168 std: 0.010606 (-30.47 %)
q=200	min: 0.118332 max: 0.163814 std: 0.008799	min: 0.118431 max: 0.154259 std: 0.007943 (-9.73 %)	min: 0.118421 max: 0.140171 std: 0.004704 (-46.55 %)
q=400	min: 0.118443 max: 0.138215 std: 0.004822	min: 0.117990 max: 0.136426 std: 0.004346 (-9.86 %)	min: 0.118258 max: 0.128975 std: 0.002227 (-53.81 %)

Table 2: Experiment 1. Comparison of variation reduction of different values of smoothing parameter ϵ (0.0, 0.005, 0.05). $\epsilon = 0.0$ means no smoothing. q (50, 200, 400) mean-CVaR frontiers are generated with range of expected means $[2.5 : 7.5] * 10^{-3}$. Portfolio losses are simulated under Normal+Exponential distribution. Comparison is shown by the minimum, maximum and standard deviations of CVaRs at different level of expected means.

	$\epsilon = 0.0$	$\epsilon = 0.005$	$\epsilon = 0.05$
mean=0.0030			
q=50	min: 0.290935 max: 0.569548 std: 0.062397	min: 0.290924 max: 0.567099 std: 0.061425 (-1.56 %)	min: 0.291236 max: 0.544886 std: 0.053392 (-14.43 %)
q=200	min: 0.291010 max: 0.551235 std: 0.048730	min: 0.290898 max: 0.546080 std: 0.047628 (-2.26 %)	min: 0.290962 max: 0.501605 std: 0.038447 (-21.10 %)
q=400	min: 0.290766 max: 0.340432 std: 0.012726	min: 0.290720 max: 0.337131 std: 0.011968 (-5.96 %)	min: 0.291625 max: 0.320204 std: 0.007584 (-40.40 %)
mean=0.0050			
q=50	min: 0.314748 max: 0.598246 std: 0.065743	min: 0.314730 max: 0.595365 std: 0.064890 (-1.30 %)	min: 0.315441 max: 0.570174 std: 0.057438 (-12.63 %)
q=200	min: 0.315173 max: 0.584825 std: 0.046576	min: 0.315139 max: 0.580367 std: 0.045770 (-1.73 %)	min: 0.315255 max: 0.540768 std: 0.038417 (-17.52 %)
q=400	min: 0.315231 max: 0.370463 std: 0.013692	min: 0.315047 max: 0.368413 std: 0.012892 (-5.84 %)	min: 0.315243 max: 0.346632 std: 0.009010 (-34.20 %)
mean=0.0070			
q=50	min: 0.347001 max: 0.746492 std: 0.082613	min: 0.346987 max: 0.744864 std: 0.081940 (-0.81 %)	min: 0.347033 max: 0.695279 std: 0.073013 (-11.62 %)
q=200	min: 0.347274 max: 0.617886 std: 0.043564	min: 0.346830 max: 0.613657 std: 0.042809 (-1.73 %)	min: 0.348362 max: 0.558782 std: 0.034512 (-20.78 %)
q=400	min: 0.346862 max: 0.401421 std: 0.013464	min: 0.346978 max: 0.405863 std: 0.013383 (-0.61 %)	min: 0.347691 max: 0.391903 std: 0.010507 (-21.96 %)

Table 3: Experiment 1. Comparison of variation reduction of different values of smoothing parameter ϵ (0.0, 0.005, 0.05). $\epsilon = 0.0$ means no smoothing. q (50, 200, 400) mean-CVaR frontiers are generated with range of expected means $[2.5 : 7.5] * 10^{-3}$. Portfolio losses are simulated under Normal+Power distribution. Comparison is shown by the minimum, maximum and standard deviations of CVaRs at different level of expected means.

	$\epsilon = 0.0$	$\epsilon = 0.005$	$\epsilon = 0.05$
mean=0.0030			
q=50	min: 0.095627 max: 0.193767 std: 0.017950	min: 0.095369 max: 0.186906 std: 0.016805 (-6.38 %)	min: 0.094800 max: 0.130766 std: 0.010095 (-43.76 %)
q=200	min: 0.094132 max: 0.116334 std: 0.005071	min: 0.093813 max: 0.112235 std: 0.004880 (-3.77 %)	min: 0.093007 max: 0.105978 std: 0.003119 (-38.50 %)
q=400	min: 0.093116 max: 0.112565 std: 0.004416	min: 0.093013 max: 0.112559 std: 0.003603 (-18.40 %)	min: 0.093103 max: 0.105366 std: 0.002020 (-54.26 %)
mean=0.0050			
q=50	min: 0.115257 max: 0.211919 std: 0.018210	min: 0.115394 max: 0.205191 std: 0.017270 (-5.16 %)	min: 0.114351 max: 0.159090 std: 0.011399 (-37.40 %)
q=200	min: 0.113585 max: 0.135498 std: 0.006151	min: 0.113263 max: 0.135705 std: 0.005619 (-8.64 %)	min: 0.112452 max: 0.129885 std: 0.003864 (-37.19 %)
q=400	min: 0.112489 max: 0.134715 std: 0.004893	min: 0.112355 max: 0.136442 std: 0.004557 (-6.88 %)	min: 0.112596 max: 0.127714 std: 0.002569 (-47.50 %)
mean=0.0070			
q=50	min: 0.133948 max: 0.230796 std: 0.020309	min: 0.133865 max: 0.224270 std: 0.019119 (-5.86 %)	min: 0.135386 max: 0.189839 std: 0.013650 (-32.79 %)
q=200	min: 0.133818 max: 0.161453 std: 0.006984	min: 0.133044 max: 0.163203 std: 0.006633 (-5.02 %)	min: 0.132399 max: 0.154623 std: 0.004764 (-31.79 %)
q=400	min: 0.132371 max: 0.157507 std: 0.005770	min: 0.132385 max: 0.160408 std: 0.005673 (-1.66 %)	min: 0.132655 max: 0.150413 std: 0.003207 (-44.42 %)

Experiment 2

Table 4: Experiment 2. Comparison of variation reduction of different values of smoothing parameter ϵ (0.0, 0.005, 0.05). $\epsilon = 0.0$ means no smoothing. q (50, 200, 400) mean-CVaR frontiers are generated. Each frontier is the global minimum CVaR and its corresponding mean. Portfolio losses are simulated under Normal distribution. Comparison is shown by the minimum, maximum and standard deviations of CVaRs.

	$\epsilon = 0.0$	$\epsilon = 0.005$	$\epsilon = 0.05$
q=50	min: 0.046555 max: 0.098756 std: 0.008730	min: 0.046561 max: 0.094785 std: 0.008100 (-7.22 %)	min: 0.046718 max: 0.087207 std: 0.005676 (-34.99 %)
q=200	min: 0.046710 max: 0.069287 std: 0.004150	min: 0.046603 max: 0.063710 std: 0.003269 (-21.24 %)	min: 0.046470 max: 0.057349 std: 0.001424 (-65.68 %)
q=400	min: 0.046498 max: 0.062158 std: 0.002471	min: 0.046530 max: 0.058613 std: 0.001988 (-19.54 %)	min: 0.046483 max: 0.050916 std: 0.000814 (-67.06 %)

Table 5: Experiment 2. Comparison of variation reduction of different values of smoothing parameter ϵ (0.0, 0.005, 0.05). $\epsilon = 0.0$ means no smoothing. q (50, 200, 400) mean-CVaR frontiers are generated. Each frontier is the global minimum CVaR and its corresponding mean. Portfolio losses are simulated under Normal+Exponential distribution. Comparison is shown by the minimum, maximum and standard deviations of CVaRs.

	$\epsilon = 0.0$	$\epsilon = 0.005$	$\epsilon = 0.05$
q=50	min: 0.249265 max: 1.252156 std: 0.114992	min: 0.249306 max: 1.245693 std: 0.113694 (-1.13 %)	min: 0.249428 max: 1.187703 std: 0.100706 (-12.42 %)
q=200	min: 0.249321 max: 0.564790 std: 0.043689	min: 0.249342 max: 0.561450 std: 0.042587 (-2.52 %)	min: 0.249122 max: 0.520145 std: 0.032587 (-25.41 %)
q=400	min: 0.249055 max: 0.398648 std: 0.016373	min: 0.249006 max: 0.397963 std: 0.016446 (0.45 %)	min: 0.248685 max: 0.386792 std: 0.012788 (-21.89 %)

Table 6: Experiment 2. Comparison of variation reduction of different values of smoothing parameter ϵ (0.0, 0.005, 0.05). $\epsilon = 0.0$ means no smoothing. q (50, 200, 400) mean-CVaR frontiers are generated. Each frontier is the global minimum CVaR and its corresponding mean. Portfolio losses are simulated under Normal+Power distribution. Comparison is shown by the minimum, maximum and standard deviations of CVaRs.

	$\epsilon = 0.0$	$\epsilon = 0.005$	$\epsilon = 0.05$
q=50	min: 0.081961 max: 0.628890 std: 0.073601	min: 0.081775 max: 0.618426 std: 0.073326 (-0.37 %)	min: 0.083450 max: 0.547177 std: 0.072056 (-2.10 %)
q=200	min: 0.082165 max: 0.211775 std: 0.021527	min: 0.082640 max: 0.210037 std: 0.021281 (-1.14 %)	min: 0.080936 max: 0.184371 std: 0.019375 (-10.00 %)
q=400	min: 0.082918 max: 0.185060 std: 0.015028	min: 0.082943 max: 0.185157 std: 0.015557 (3.52 %)	min: 0.080609 max: 0.156789 std: 0.014616 (-2.74 %)

Experiment 3

Table 7: Experiment 3. Extension of Experiement 1. Comparison of variation reduction of different values of smoothing parameter ϵ (0.0, 0.001, 0.005, 0.01, 0.05). 50 mean-CVaR frontiers are generated with range of expected means $[2.5 : 7.5] * 10^{-3}$. Portfolio losses are simulated under Normal distribution. Comparison is shown by the minimum, maximum and standard deviations of CVaRs at different level of expected means.

$\epsilon = 0.0$	$\epsilon = 0.001$	$\epsilon = 0.005$	$\epsilon = 0.01$	$\epsilon = 0.05$
		mean=0.0030		
min: 0.050379 max: 0.081689 std: 0.005937	min: 0.050408 max: 0.081345 std: 0.005914 (-0.40 %)	min: 0.050523 max: 0.078693 std: 0.005577 (-6.08 %)	min: 0.050692 max: 0.074084 std: 0.005120 (-13.77 %)	min: 0.050540 max: 0.064479 std: 0.003982 (-32.93 %)
		mean=0.0050		
min: 0.079099 max: 0.130318 std: 0.009690	min: 0.079106 max: 0.129802 std: 0.009549 (-1.45 %)	min: 0.078406 max: 0.127671 std: 0.009127 (-5.81 %)	min: 0.078109 max: 0.125049 std: 0.008637 (-10.87 %)	min: 0.078564 max: 0.109421 std: 0.005994 (-38.14 %)
		mean=0.0070		
min: 0.118340 max: 0.185188 std: 0.015255	min: 0.118290 max: 0.184233 std: 0.015035 (-1.44 %)	min: 0.118166 max: 0.183027 std: 0.014459 (-5.21 %)	min: 0.118191 max: 0.181561 std: 0.013810 (-9.47 %)	min: 0.118491 max: 0.170168 std: 0.010606 (-30.47 %)

Table 8: Experiment 3. Extension of Experiement 1. Comparison of variation reduction of different values of smoothing parameter ϵ (0.0, 0.001, 0.005, 0.01, 0.05). 50 mean-CVaR frontiers are generated with range of expected means $[2.5 : 7.5] * 10^{-3}$. Portfolio losses are simulated under Normal+Exponential distribution. Comparison is shown by the minimum, maximum and standard deviations of CVaRs at different level of expected means.

$\epsilon = 0.0$	$\epsilon = 0.001$	$\epsilon = 0.005$	$\epsilon = 0.01$	$\epsilon = 0.05$
mean=0.0030				
min: 0.290935 max: 0.569548 std: 0.062397	min: 0.290933 max: 0.569055 std: 0.062202 (-0.31 %)	min: 0.290924 max: 0.567099 std: 0.061425 (-1.56 %)	min: 0.290916 max: 0.564650 std: 0.060399 (-3.20 %)	min: 0.291236 max: 0.544886 std: 0.053392 (-14.43 %)
mean=0.0050				
min: 0.314748 max: 0.598246 std: 0.065743	min: 0.314751 max: 0.597672 std: 0.065571 (-0.26 %)	min: 0.314730 max: 0.595365 std: 0.064890 (-1.30 %)	min: 0.314775 max: 0.592486 std: 0.064051 (-2.57 %)	min: 0.315441 max: 0.570174 std: 0.057438 (-12.63 %)
mean=0.0070				
min: 0.347001 max: 0.746492 std: 0.082613	min: 0.346999 max: 0.746169 std: 0.082476 (-0.17 %)	min: 0.346987 max: 0.744864 std: 0.081940 (-0.81 %)	min: 0.346989 max: 0.743247 std: 0.081277 (-1.62 %)	min: 0.347033 max: 0.695279 std: 0.073013 (-11.62 %)

Table 9: Experiment 3. Extension of Experiement 1. Comparison of variation reduction of different values of smoothing parameter ϵ (0.0, 0.001, 0.005, 0.01, 0.05). 50 mean-CVaR frontiers are generated with range of expected means $[2.5 : 7.5] * 10^{-3}$. Portfolio losses are simulated under Normal+Power distribution. Comparison is shown by the minimum, maximum and standard deviations of CVaRs at different level of expected means.

$\epsilon = 0.0$	$\epsilon = 0.001$	$\epsilon = 0.005$	$\epsilon = 0.01$	$\epsilon = 0.05$
mean=0.0030				
min: 0.095627 max: 0.193767 std: 0.017950	min: 0.095577 max: 0.192396 std: 0.017716 (-1.31 %)	min: 0.095369 max: 0.186906 std: 0.016805 (-6.38 %)	min: 0.095117 max: 0.180133 std: 0.015775 (-12.12 %)	min: 0.094800 max: 0.130766 std: 0.010095 (-43.76 %)
mean=0.0050				
min: 0.115257 max: 0.211919 std: 0.018210	min: 0.115279 max: 0.210575 std: 0.018094 (-0.63 %)	min: 0.115394 max: 0.205191 std: 0.017270 (-5.16 %)	min: 0.115624 max: 0.198555 std: 0.016315 (-10.40 %)	min: 0.114351 max: 0.159090 std: 0.011399 (-37.40 %)
mean=0.0070				
min: 0.133948 max: 0.230796 std: 0.020309	min: 0.133920 max: 0.229484 std: 0.020058 (-1.24 %)	min: 0.133865 max: 0.224270 std: 0.019119 (-5.86 %)	min: 0.133892 max: 0.217949 std: 0.018024 (-11.25 %)	min: 0.135386 max: 0.189839 std: 0.013650 (-32.79 %)

Experiment 4

Table 10: Experiment 4. Extension of Experiment 2. Comparison of variation reduction of different values of smoothing parameter ϵ (0.0, 0.001, 0.005, 0.01, 0.05). 50 mean-CVaR frontiers are generated. Each frontier is the global minimum CVaR and its corresponding mean. Portfolio losses are simulated under Normal distribution. Comparison is shown by the minimum, maximum and standard deviations of CVaRs.

$\epsilon = 0.0$	$\epsilon = 0.001$	$\epsilon = 0.005$	$\epsilon = 0.01$	$\epsilon = 0.05$
min: 0.046555 max: 0.098756 std: 0.008730	min: 0.046541 max: 0.097955 std: 0.008590 (-1.61 %)	min: 0.046561 max: 0.094785 std: 0.008100 (-7.22 %)	min: 0.046712 max: 0.090903 std: 0.007589 (-13.07 %)	min: 0.046718 max: 0.087207 std: 0.005676 (-34.99 %)

Table 11: Experiment 4. Extension of Experiment 2. Comparison of variation reduction of different values of smoothing parameter ϵ (0.0, 0.001, 0.005, 0.01, 0.05). 50 mean-CVaR frontiers are generated. Each frontier is the global minimum CVaR and its corresponding mean. Portfolio losses are simulated under Normal+Exponential distribution. Comparison is shown by the minimum, maximum and standard deviations of CVaRs.

$\epsilon = 0.0$	$\epsilon = 0.001$	$\epsilon = 0.005$	$\epsilon = 0.01$	$\epsilon = 0.05$
min: 0.249265 max: 1.252156 std: 0.114992	min: 0.249261 max: 1.250862 std: 0.114881 (-0.10 %)	min: 0.249306 max: 1.245693 std: 0.113694 (-1.13 %)	min: 0.249329 max: 1.239234 std: 0.112222 (-2.41 %)	min: 0.249428 max: 1.187703 std: 0.100706 (-12.42 %)

Table 12: Experiment 4. Extension of Experiment 2. Comparison of variation reduction of different values of smoothing parameter ϵ (0.0, 0.001, 0.005, 0.01, 0.05). 50 mean-CVaR frontiers are generated. Each frontier is the global minimum CVaR and its corresponding mean. Portfolio losses are simulated under Normal+Power distribution. Comparison is shown by the minimum, maximum and standard deviations of CVaRs.

$\epsilon = 0.0$	$\epsilon = 0.001$	$\epsilon = 0.005$	$\epsilon = 0.01$	$\epsilon = 0.05$
min: 0.081961 max: 0.628890 std: 0.073601	min: 0.081866 max: 0.626797 std: 0.073539 (-0.08 %)	min: 0.081775 max: 0.618426 std: 0.073326 (-0.37 %)	min: 0.081916 max: 0.607961 std: 0.073000 (-0.82 %)	min: 0.083450 max: 0.547177 std: 0.072056 (-2.10 %)

Experiment 5

Table 13: Experiment 5. Comparison of variation reduction of different values of smoothing parameter ϵ (0.0, 0.001, 0.005, 0.01, 0.05). 50 mean-CVaR frontiers are generated. Each frontier is the global minimum CVaR and its corresponding mean. Portfolio losses are simulated under Normal+Exponential distribution with various λ. Comparison is shown by the minimum, maximum and standard deviations of CVaRs.

$\epsilon = 0.0$	$\epsilon = 0.001$	$\epsilon = 0.005$	$\epsilon = 0.01$	$\epsilon = 0.05$
λ=2.0000				
min: 1.192987 max: 6.218746 std: 0.568375	min: 1.193002 max: 6.049199 std: 0.564152 (-0.74 %)	min: 1.193078 max: 6.212247 std: 0.566893 (-0.26 %)	min: 1.193173 max: 6.205749 std: 0.565412 (-0.52 %)	min: 1.193234 max: 6.153777 std: 0.553605 (-2.60 %)
λ=4.0000				
min: 0.602995 max: 3.114036 std: 0.284501	min: 0.602988 max: 3.112739 std: 0.284547 (0.02 %)	min: 0.602974 max: 3.107551 std: 0.283364 (-0.40 %)	min: 0.602885 max: 3.101066 std: 0.281887 (-0.92 %)	min: 0.602290 max: 3.049187 std: 0.270300 (-4.99 %)
λ=8.0000				
min: 0.308058 max: 1.562323 std: 0.143229	min: 0.308065 max: 1.561029 std: 0.143158 (-0.05 %)	min: 0.307923 max: 1.555856 std: 0.142007 (-0.85 %)	min: 0.307762 max: 1.549390 std: 0.140499 (-1.91 %)	min: 0.307963 max: 1.497660 std: 0.128973 (-9.95 %)
λ=12.0000				
min: 0.210193 max: 1.045680 std: 0.096150	min: 0.210178 max: 1.044395 std: 0.096020 (-0.14 %)	min: 0.210186 max: 1.039253 std: 0.094852 (-1.35 %)	min: 0.210224 max: 1.032826 std: 0.093365 (-2.90 %)	min: 0.210160 max: 0.981408 std: 0.081989 (-14.73 %)
λ=16.0000				
min: 0.161665 max: 0.788010 std: 0.072551	min: 0.161628 max: 0.786728 std: 0.072439 (-0.15 %)	min: 0.161548 max: 0.781600 std: 0.071279 (-1.75 %)	min: 0.161549 max: 0.775190 std: 0.069837 (-3.74 %)	min: 0.161581 max: 0.723931 std: 0.058528 (-19.33 %)
λ=20.0000				
min: 0.132692 max: 0.633805 std: 0.058122	min: 0.132669 max: 0.632527 std: 0.057981 (-0.24 %)	min: 0.132682 max: 0.627413 std: 0.056807 (-2.26 %)	min: 0.132814 max: 0.621021 std: 0.055354 (-4.76 %)	min: 0.132892 max: 0.569994 std: 0.044368 (-23.66 %)

Experiment 6

Table 14: Experiment 6. Comparison of variation reduction of different values of smoothing parameter ϵ (0.0, 0.001, 0.005, 0.01, 0.05). 50 mean-CVaR frontiers are generated. Each frontier is the global minimum CVaR and its corresponding mean. Portfolio losses are simulated under Normal+Power distribution with various γ. Comparison is shown by the minimum, maximum and standard deviations of CVaRs.

$\epsilon = 0.0$	$\epsilon = 0.001$	$\epsilon = 0.005$	$\epsilon = 0.01$	$\epsilon = 0.05$
γ=3.0000				
min: 0.084543 max: 1.376237 std: 0.179863	min: 0.084419 max: 1.371657 std: 0.179755 (-0.06 %)	min: 0.084178 max: 1.353337 std: 0.179407 (-0.25 %)	min: 0.084246 max: 1.330436 std: 0.178879 (-0.55 %)	min: 0.085450 max: 1.197420 std: 0.177764 (-1.17 %)
γ=3.5000				
min: 0.081961 max: 0.628890 std: 0.073601	min: 0.081866 max: 0.626797 std: 0.073539 (-0.08 %)	min: 0.081775 max: 0.618426 std: 0.073326 (-0.37 %)	min: 0.081916 max: 0.607961 std: 0.073000 (-0.82 %)	min: 0.083450 max: 0.547177 std: 0.072056 (-2.10 %)
γ=4.0000				
min: 0.079868 max: 0.425216 std: 0.045845	min: 0.079969 max: 0.423801 std: 0.045789 (-0.12 %)	min: 0.080183 max: 0.418141 std: 0.045591 (-0.55 %)	min: 0.080268 max: 0.411065 std: 0.045298 (-1.19 %)	min: 0.079662 max: 0.369967 std: 0.044299 (-3.37 %)
γ=4.5000				
min: 0.075782 max: 0.338837 std: 0.034577	min: 0.075779 max: 0.337709 std: 0.034520 (-0.16 %)	min: 0.075778 max: 0.333198 std: 0.034305 (-0.79 %)	min: 0.075771 max: 0.327560 std: 0.034022 (-1.61 %)	min: 0.076008 max: 0.294811 std: 0.032955 (-4.69 %)
γ=5.0000				
min: 0.072532 max: 0.292190 std: 0.028730	min: 0.072571 max: 0.291218 std: 0.028664 (-0.23 %)	min: 0.072622 max: 0.287328 std: 0.028436 (-1.03 %)	min: 0.072564 max: 0.282466 std: 0.028155 (-2.00 %)	min: 0.072383 max: 0.254225 std: 0.027030 (-5.92 %)
γ=5.5000				
min: 0.069516 max: 0.263120 std: 0.025199	min: 0.069528 max: 0.262245 std: 0.025134 (-0.26 %)	min: 0.069485 max: 0.258742 std: 0.024894 (-1.21 %)	min: 0.069687 max: 0.254364 std: 0.024609 (-2.34 %)	min: 0.069331 max: 0.228933 std: 0.023439 (-6.99 %)
γ=6.0000				
min: 0.067398 max: 0.243266 std: 0.022863	min: 0.067416 max: 0.242456 std: 0.022790 (-0.32 %)	min: 0.067436 max: 0.239218 std: 0.022541 (-1.41 %)	min: 0.067541 max: 0.235170 std: 0.022250 (-2.68 %)	min: 0.067365 max: 0.211658 std: 0.021046 (-7.95 %)