

A Fourier Space Time-stepping Approach Applied to Problems in Finance

by

Julian Lippa

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We study the use of a Fourier Space Time-stepping (FST) approach in solving partial integro-differential equations (PIDE) to value contingent claims arising from various financial problems. Under a jump diffusion stochastic model for the underlying asset, a derivation of the FST method is presented to solve the corresponding PIDE. Illustration of the solution method is provided for valuing single- and multi-asset European options under jump diffusion models. The valuation of a Guaranteed Minimum Withdrawal Benefit (GMWB) variable annuity is formulated as a stochastic optimal control problem. We show how the payment of incentive fees inherent in a hedge fund manager's compensation can be formulated as a path-dependent contingent claim. The financial pricing problems are numerically evaluated using the FST method and results are discussed qualitatively.

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Chapter 1

Introduction

A financial derivative is a product that offers a return based on the return of another underlying asset. The value of the derivative is *derived* from the underlying asset. Derivatives can be broadly grouped into two general categories: forward commitments and contingent claims. Forward contracts are an agreement between two parties to transact in the future. In contrast, contingent claims offer payoffs dependent on whether a stated event occurs within the life of the derivative contract.

One of the main purposes of the market for derivatives is for risk management. Through the use of derivative contracts, market participants can effectively hedge various risks inherent in both financial products and general business operations. In other words, derivatives give market participants the ability to customize and create suitable risk profiles for their businesses. Derivatives are traded in various markets including equity, interest rate, credit, currency and commodity markets.

An option, a type of derivative, gives the buyer the right, but not the obligation, to transact in the underlying asset at a fixed price over a stated period of time. Options are sometimes referred to as contingent claims as the buyer will exercise when market conditions are advantageous, and therefore the payoff is dependent on the occurrence of a specified event.

A central concept in derivative pricing is the concept of arbitrage, which occurs when equivalent assets or portfolios of assets have different prices at the same time. In this case, a market participant, known as an arbitrageur, can enjoy risk-free profits without any commitment of capital. An arbitrage-free market is the basic assumption used in derivative pricing in order to compute the no-arbitrage price, which is the value at which buyers and sellers can transact without any price advantages.

The pricing of derivatives requires an assumption of the stochastic price process model for the underlying asset. Subsequently, a dynamic self-financing replicating portfolio is constructed to hedge the derivative. The no-arbitrage price, or fair value, of the contingent claim is the initial cost required to construct the replicating portfolio. Many stochastic models for the underlying asset price have been studied extensively in the quantitative finance literature.

The Black-Scholes-Merton (BSM) model assumes that asset prices follow geometric Brownian motion. It is well known that the constant volatility BSM model suffers from incon-

sistency with observed market prices and does not capture the observed implied volatility smile. There are two broad lines of research which aim to augment and improve the BSM model. The volatility term in the model can be driven by a stochastic process as in Heston (1993), or volatility can undergo regime switches, as in Naik (1993). Another line of research looks to add jumps directly into the asset price process, as in Merton (1976) and Kou (2002).

The jump diffusion stochastic process for the underlying asset price was first suggested by Merton (1976). The jump diffusion model adds a random jump component to geometric Brownian motion. With two independent components driving the change in asset price, Brownian motion and jumps, the price changes through time are thus composed of two different types. The geometric Brownian motion represents “normal” changes in the asset price. For example, these changes can be fluctuations in the supply and demand for the asset or changes in the economic outlook. The second type of change, the “abnormal” changes in the asset price, is represented by the random jump term in the model. Within the equity market, abnormal changes can represent significant new information that causes a drastic change in the asset’s price, such as surprise earnings reports or a failed acquisition. More generally, unexpected macroeconomic information, such as surprise changes to a government’s monetary policy, may cause sudden jumps in prices of various assets, such as bonds, equities, currencies and commodities, simultaneously. Furthermore, abnormal market-wide changes in asset prices can be caused from external market events, such as natural disasters or terrorist attacks.

The standard BSM model, using geometric Brownian motion, constructs a stochastic process with continuous sample paths. To better represent reality, the jump diffusion model augments the BSM model by adding Poisson-driven random jumps, which cause sample paths to be discontinuous. In this model, the jumps in the asset price occur randomly, and the size of the jump is random. The Poisson process models the occurrence of an abnormal event, where occurrences are independently and identically distributed. When a jump occurs, the magnitude of the jump is also random, with a log-normal distribution for the jump size suggested by Merton (1976). The suggestion of the use of a double exponential distribution for the random jump size was made by Kou (2002).

Through the use of a self-financing replicating portfolio for the contingent claim, Black and Scholes (1973) demonstrate that the option pricing problem under the BSM model reduces to solving a second-order partial differential equation (PDE). When the underlying asset follows a jump diffusion model, Merton (1973) shows that the pricing problem reduces to solving a second-order partial integro-differential equation (PIDE).

Throughout this essay, a jump diffusion process is assumed for the underlying asset, which leads to solving a PIDE for the value of several contingent claims. In general, the pricing PIDE is solved by discretizing the space and time domains into a finite mesh of points. The option payoff is used as the terminal condition and the solution is generated at other points in time by moving backwards in time. Most commonly, finite difference approaches are used to solve the pricing PIDE, in which the differential operator is approximated by finite differences.

This essay studies the Fourier Space Time-stepping (FST) approach for solving the pricing PIDE for contingent claims as developed by Jackson, Jaimungal and Surkov (2007). The

FST method uses a continuous Fourier transform to convert the PIDE into Fourier space. The continuous Fourier transform is a linear operator which maps spatial derivatives into multiplications in Fourier space. With these convenient properties, the PIDE reduces to a linear first-order differential equation, which can be solved in closed-form in Fourier space. In this way, the method is able to produce exact pricing results between monitoring dates (if any) of an option, using a continuous domain. In practice, using a discrete computational domain leads to approximations as a discrete Fourier transform is used to approximate the continuous Fourier transform. European options, with no monitoring dates, can thus be priced in one time-step of the algorithm. Other options with conditions applied throughout the life of the option, such as early exercise constraints, require one time-step between discrete monitoring dates. Therefore, unlike finite difference approaches, discretization of the time domain in the FST method is dependent on the type of option, specifically the conditions imposed on the option's value throughout its life. Finite difference approaches require time discretization regardless of the option type and certain schemes impose a time-step restriction for stability. Additionally, the FST method does not require any assumption of the asset price behaviour outside the truncated computational domain, as is required with finite difference methods.

The FST method can be easily applied to asset price models following exponential Lévy processes. The distribution of a Lévy process is described completely by its so called *characteristic function*, which is given in closed-form by the Lévy-Khinchine formula. The FST method admits substantial flexibility as the characteristic exponent of the stochastic process is available in closed-form and can be factored out of the Fourier transform of the PIDE, for all exponential Lévy processes. Therefore, option prices can be computed easily for a variety of stochastic processes without modifications to the algorithm other than changing the characteristic exponent for the desired Lévy process. Popular Lévy processes used in mathematical finance include the Variance Gamma (VG) model of Madan and Seneta (1990), the Normal Inverse Gaussian (NIG) model of Bandorff-Nielsen (1997), and the CGMY model of Carr, Geman, Madan, and Yor (2002). Additionally, the FST method derived in this essay can be extended to regime-switching models to capture stochastic volatility behaviour. Extension of the method to pricing options under mean-reversion models can also be made, as shown in Surkov (2009).

There are many problems in finance with embedded options which can be formulated in terms of contingent claims and valued with the option pricing framework. In this essay, in addition to pricing various types of options, we study two specific problems in finance formulated as contingent claims. The first pricing example is of valuing an insurance product, namely the Guaranteed Minimum Withdrawal Benefit (GMWB) variable annuity. The second pricing problem is a contingent claim inherent in a hedge fund manager's compensation package.

1.1 Guaranteed Minimum Withdrawal Benefits in Variable Annuities

Variable annuities are insurance policies that provide policyholders with tax sheltered growth and carry a variety of embedded options. The policies allow investors to enjoy gains in an underlying portfolio and can be seen as tax-efficient mutual funds, offered as an insurance product. The Guaranteed Minimum Withdrawal Benefit (GMWB) rider in a variable annuity guarantees that the initial investment is returned to the investor over the life of the contract. The variable annuity with the GMWB option does not include any life insurance aspects, so it can be valued as a contingent claim using standard option pricing methods. The pricing problem can be seen as a path-dependent option with conditions applied on discrete monitoring dates. As we apply optimality conditions, the valuation can be regarded as a problem in stochastic optimal control. We use the FST method to solve the pricing PIDE in order to value the GMWB variable annuity. Numerically, we find that GMWB variable annuities are underpriced in the current marketplace.

1.2 Hedge Fund Incentive Fee Valuation

Under a typical hedge fund contract, in addition to a proportional management fee on assets under management, a manager also receives an incentive payment for superior returns. The nonlinear incentive payment is dependent on the value of the hedge fund assets, and thus can be formulated in terms of a contingent claim and valued with the standard option pricing framework. Specifically, because the incentive payment is a function of two additional state variables, the pricing problem is similar to that of a path-dependent option. An investor provides the manager with this option, at no cost, to compensate the manager for his time, skill and efforts, as well as to align the incentives of the manager and the investor. We use the FST method to solve the pricing PIDE in order to value the hedge fund manager's incentive fee option. Numerically, we find that the value of the option provided to the manager is a substantial amount of the investor's initial investment.

1.3 Contents of the Essay

The main contributions of this essay are:

- to derive the Fourier Space Time-stepping (FST) method for solving partial-integro differential equations (PIDE) arising from valuing contingent claims under jump diffusion models, with numerical illustration on single- and multi-asset European options,
- to mathematically formulate the GMWB variable annuity pricing problem and numerically evaluate the fair insurance guarantee fee under a jump diffusion stochastic model using the FST method,

- to mathematically formulate the hedge fund incentive fee contingent claim and numerically evaluate the no-arbitrage value under a jump diffusion stochastic model using the FST method.

The remainder of this essay is structured as follows. Chapter 2 provides a derivation of the Fourier Space Time-stepping (FST) method under jump diffusion processes, with discussion of extensions to other asset models and various types of options. Chapter 3 formulates the pricing problem of GMWBs and provides a numerical approach with appropriate treatment of wrap-around error. Chapter 4 gives a brief introduction on hedge funds and mathematically formulates the incentive fee valuation problem, with a suitable numerical approach. Chapter 5 lists conclusions. The Appendix lists definitions, properties and proofs used throughout the essay.

Chapter 2

Fourier Space Time-stepping Method

2.1 Introduction

In this chapter, we outline a detailed derivation of the Fourier Space Time-stepping (FST) method for option pricing, as developed in Jackson et al. (2007), assuming a jump diffusion stochastic differential equation for the underlying asset price. In this essay, we use the FST method to price various options, but also path-dependent contingent claims where the underlying asset follows a jump diffusion process and the contingent claim value satisfies corresponding partial integro-differential equation (PIDE). Both contingent claims considered in this essay apply to either optimality conditions or jump conditions on specific discrete dates in time. However, since the values of the contingent claims satisfy one-dimensional pricing PIDEs between the specific dates, we can use the FST method to solve the PIDE between dates on which conditions are applied.

The derivation begins by assuming a jump diffusion stochastic process for the underlying asset price. It is well known that the jump diffusion stochastic differential equation leads to a pricing PIDE for the value of the option. With a log transformation, the pricing PIDE is converted into one with constant coefficients and a cross-correlation integral. Applying a continuous Fourier transform to this new PIDE transforms it into a linear ordinary differential equation (ODE), in Fourier space. The ODE can be solved easily, in Fourier space, and then converted back to real space after the time-stepping has been completed. Computationally, the spatial domain is truncated and discretized, and a discrete Fourier transform is used to approximate the continuous Fourier transform.

2.2 Jump Diffusion Processes

2.2.1 Mathematical Model

The sample paths of the asset price S are modeled by a jump diffusion stochastic differential equation

$$\frac{dS}{S} = \mu dt + \sigma dZ + (\eta - 1) dq, \quad (2.1)$$

where

- μ is the drift rate,
- dq is the independent Poisson process, $= \begin{cases} 0 & \text{with probability } 1 - \lambda dt, \\ 1 & \text{with probability } \lambda dt. \end{cases}$
- λ is the mean arrival time of the Poisson process,
- $\eta - 1$ is an impulse function producing a jump from S to $S\eta$,
- σ is the volatility,
- dZ is an increment of the standard Gauss-Wiener process.

Let $\mathcal{V}(S, \tau)$ be the value of a European option. By Ito's Lemma and no-arbitrage arguments, the backward partial integro-differential equation (PIDE) for $\mathcal{V}(S, \tau)$ is [32,28,2]:

$$\mathcal{V}_\tau = \frac{\sigma^2 S^2}{2} \mathcal{V}_{SS} + (r - \lambda \kappa) S \mathcal{V}_S - r \mathcal{V} + \left(\lambda \int_0^\infty \mathcal{V}(S\eta) g(\eta) d\eta - \lambda \mathcal{V} \right) \quad (2.2)$$

where

- T is the expiry date of the option,
- r is the risk free rate, $r \geq 0$,
- $\tau = T - t$ where t is the current time,
- $g(\eta)$ is the probability density function of the jump amplitude η ,
thus for all η , $g(\eta) \geq 0$, and $\int_0^\infty g(\eta) d\eta = 1$.
- κ is $E[\eta - 1]$, with $E[\eta] = \int_0^\infty \eta g(\eta) d\eta$.

The derivation of equation (2.2) has been omitted from this work for brevity, but can be found in [32,28,2]. The PIDE (2.2) can be rewritten as:

$$\mathcal{V}_\tau = \frac{\sigma^2 S^2}{2} \mathcal{V}_{SS} + (r - \lambda \kappa) S \mathcal{V}_S - (r + \lambda) \mathcal{V} + \left(\lambda \int_0^\infty \mathcal{V}(S\eta) g(\eta) d\eta \right). \quad (2.3)$$

A log transformation of equation (2.3) will result in a PIDE with constant coefficients and a cross-correlation integral. It is natural to use a Fourier transform to solve the PIDE due to the properties of the Fourier transform. Definitions and properties used in the following can be seen in detail in Appendix A.

To apply the log transformation, we define $x = \log(S)$ and $v(x, \tau) = \mathcal{V}(e^x, \tau)$. The partial derivatives of the function v with respect to x and τ , in terms of \mathcal{V} are:

$$v_x := \frac{\partial v}{\partial x} = \mathcal{V}_S e^x,$$

$$v_{xx} := \frac{\partial^2 v}{\partial^2 x} = \mathcal{V}_{SS} e^{2x} + \mathcal{V}_S e^x,$$

$$v_\tau := \frac{\partial v}{\partial \tau} = \mathcal{V}_\tau.$$

Solving for $\mathcal{V}_\tau, \mathcal{V}_S$ and \mathcal{V}_{SS} , we get:

$$\mathcal{V}_\tau = v_\tau, \quad (2.4)$$

$$\mathcal{V}_S = \frac{v_x}{e^x}, \quad (2.5)$$

$$\mathcal{V}_{SS} = \frac{v_{xx} - v_x}{e^{2x}}. \quad (2.6)$$

Let $\eta = e^y$, $d\eta = e^y dy$, and note that as $\eta \rightarrow 0$, $y \rightarrow -\infty$. Using this and plugging (2.4), (2.5), (2.6), and $S = e^x$ into the PIDE (2.3) gives us the PIDE in terms of v :

$$v_\tau = \frac{\sigma^2}{2} (v_{xx} - v_x) + (r - \lambda\kappa)v_x - (r + \lambda)v + \lambda \left(\int_{-\infty}^{\infty} v(x+y)g(e^y)e^y dy \right). \quad (2.7)$$

Note that the PIDE (2.7) has constant coefficients. We can represent the integral term as a cross-correlation integral, as defined in Appendix A, by defining $f(x) = g(e^x)e^x$. Substituting $f(x)$ into the PIDE (2.7) gives

$$v_\tau = \frac{\sigma^2}{2} (v_{xx} - v_x) + (r - \lambda\kappa)v_x - (r + \lambda)v + \lambda \left(\int_{-\infty}^{\infty} v(x+y)f(y) dy \right). \quad (2.8)$$

Note that the function f is real-valued and hence the integral term in PIDE (2.8) is a cross-correlation integral. Using this notation, we can rewrite the PIDE (2.8) as

$$v_\tau = \frac{\sigma^2}{2} v_{xx} + \left(r - \lambda\kappa - \frac{\sigma^2}{2} \right) v_x - (r + \lambda)v + \lambda (v(x) \star f(x)). \quad (2.9)$$

Therefore, with a log transformation and a few changes of variables, the original PIDE given by (2.2) can be represented as the PIDE (2.9). The PIDE (2.9) has constant coefficients and a cross-correlation integral term.

2.2.2 Continuous Fourier Transform

Given a PIDE in the form of (2.9), with constant coefficients and a cross-correlation integral term, applying a Fourier transform yields significant simplifications. The continuous Fourier transform \mathcal{F} , a linear operator, is applied on the PIDE (2.9) and gives

$$\begin{aligned} \mathcal{F}[v_\tau](k) &= \mathcal{F}\left[\frac{\sigma^2}{2}v_{xx}\right](k) + \mathcal{F}\left[\left(r - \lambda\kappa - \frac{\sigma^2}{2}\right)v_x\right](k) - \mathcal{F}[(r + \lambda)v](k) \\ &\quad + \mathcal{F}[\lambda v(x) \star f(x)](k). \end{aligned} \quad (2.10)$$

$$\begin{aligned} &= \frac{\sigma^2}{2}\mathcal{F}[v_{xx}](k) + \left(r - \lambda\kappa - \frac{\sigma^2}{2}\right)\mathcal{F}[v_x](k) - (r + \lambda)\mathcal{F}[v](k) \\ &\quad + \lambda\mathcal{F}[v(x) \star f(x)](k), \end{aligned} \quad (2.11)$$

where the transform variable k represents the frequency. The Fourier transform has several important properties that will simplify the PIDE (2.11) significantly. The Fourier transform maps spatial derivatives in real space into multiplications in Fourier space. Specifically, in Fourier space, the first and second derivatives of v with respect to x simply reduce to a scalar value multiplied by the function v . The derivative of v with respect to τ simply becomes the derivative of the Fourier transform of v , with respect to τ , in Fourier space. Additionally, applying a Fourier transform to a cross-correlation integral reduces to multiplying the Fourier transforms of each function. Using properties (A.4), (A.5) and (A.6), as given in Appendix A, we simplify equation (2.11) to:

$$\begin{aligned} \frac{\partial}{\partial\tau}\mathcal{F}[v](k) &= \frac{\sigma^2}{2}(2\pi ik)^2\mathcal{F}[v](k) + \left(r - \lambda\kappa - \frac{\sigma^2}{2}\right)(2\pi ik)\mathcal{F}[v](k) - (r + \lambda)\mathcal{F}[v](k) \\ &\quad + \lambda\mathcal{F}[v](k)\overline{\mathcal{F}[f(x)]}(k), \end{aligned} \quad (2.12)$$

where \bar{z} denotes the complex conjugate of z . Rearranging the terms in PIDE (2.12) for simplicity, we can rewrite it as

$$0 = -\frac{\partial}{\partial\tau}\mathcal{F}[v](k) + \mathcal{F}[v](k) \left(-\frac{\sigma^2}{2}(2\pi k)^2 + \left(r - \lambda\kappa - \frac{\sigma^2}{2}\right)(2\pi ik) - (r + \lambda) + \lambda\overline{\mathcal{F}[f(x)]}(k) \right). \quad (2.13)$$

For ease of exposition, we adopt the notation $V(k) := \mathcal{F}[v](k)$ and $F(k) := \mathcal{F}[f(x)](k)$. With this notation, we denote derivatives as $V_\tau(k) := \frac{\partial}{\partial\tau}\mathcal{F}[v](k)$. Additionally, we define the *characteristic exponent* $\Psi(k)$ to be

$$\Psi(k) := \left(-\frac{\sigma^2}{2}(2\pi k)^2 + \left(r - \lambda\kappa - \frac{\sigma^2}{2}\right)(2\pi ik) - (r + \lambda) + \lambda\overline{F}(k) \right). \quad (2.14)$$

Note that equation (2.14) corresponds to the characteristic exponent for the specific jump diffusion process given by equation (2.1). With the new function definitions, specifically V, F , and Ψ , we can rewrite the PIDE (2.13) as

$$0 = V_\tau(k) - V(k) \cdot \Psi(k). \quad (2.15)$$

Therefore, we can see that by using a number of change of variables and the properties of the Fourier transforms, we have converted the original PIDE, given by (2.2), into a linear ordinary differential equation (ODE) in τ given by (2.15). The ODE can be solved in closed-form in Fourier space. We can then convert the option price from Fourier space back into the real space by taking the inverse Fourier transform.

2.2.3 Solving the Ordinary Differential Equation

The ODE (2.15) can be easily solved by the integrating factor method. Multiplying (2.15) by the integrating factor $e^{-\Psi(k)\tau}$ gives

$$0 = e^{-\Psi(k)\tau} V_\tau(k) - e^{-\Psi(k)\tau} V(k) \cdot \Psi(k). \quad (2.16)$$

Using the derivative product rule, we can express (2.16) as

$$0 = \frac{\partial}{\partial \tau} (V(k) \cdot e^{-\Psi(k)\tau}). \quad (2.17)$$

Integrating both sides of (2.17) with respect to τ gives

$$\begin{aligned} C &= V(k) \cdot e^{-\Psi(k)\tau} \\ V(k) &= C \cdot e^{\Psi(k)\tau}, \end{aligned}$$

where C is a constant.

We define $V^n(k) := \mathcal{F}[v(x, \tau_n)](k)$ to be the Fourier transform of the option price at a time τ_n . We have the result that for any $0 \leq \tau_l \leq \tau_u \leq T$, after computing the Fourier transform of the option price at time τ_l , $V^l(k)$, we can compute the Fourier transform of the option price at time τ_u by

$$V^u(k) = V^l(k) \cdot e^{\Psi(k)(\tau_u - \tau_l)}. \quad (2.18)$$

To recover the option value in real space, we apply an inverse Fourier transform on (2.18):

$$\begin{aligned} \mathcal{F}^{-1}[V^u(k)](x) &= \mathcal{F}^{-1}[V^l(k) \cdot e^{\Psi(k)(\tau_u - \tau_l)}](x) \\ v(x, \tau_u) &= \mathcal{F}^{-1}[V^l(k) \cdot e^{\Psi(k)(\tau_u - \tau_l)}](x). \end{aligned} \quad (2.19)$$

Equation (2.19) shows us how we can time-step, in Fourier space, and then recover the option prices in real space, using a continuous Fourier transform and its inverse.

2.2.4 Discrete Fourier Transform

Computationally, we need to approximate the continuous Fourier transform of the option price $v(x, \tau)$ with a Discrete Fourier Transform (DFT). The DFT effectively approximates the continuous Fourier transform via the midpoint rule. The approximation has error $\mathcal{O}(\Delta x^2)$,

where Δx is the constant spacing in the x direction. We want to find the DFT of $v(x, \tau)$ for a constant time τ . The continuous Fourier transform is

$$V(k) := \int_{-\infty}^{\infty} v(x, \tau) e^{-i2\pi kx} dx. \quad (2.20)$$

We truncate the log stock price from the original domain of $[-\infty, \infty]$ to the new domain $\Omega = [x_{min}, x_{max}]$. The Fourier transform is approximated on this domain as

$$V(k) \approx \int_{x_{min}}^{x_{max}} v(x, \tau) e^{-i2\pi kx} dx. \quad (2.21)$$

We discretize the space and frequency domains as follows. For the log-asset price domain discretization, let $x_n = x_{min} + n \cdot \Delta x$, for $n = 0, 1, \dots, N-1$, and $\Delta x = \frac{x_{max} - x_{min}}{N-1} = \frac{X}{N-1}$, with $X = x_{max} - x_{min}$. Additionally, for the frequency domain discretization, let $k_m = \Delta k \cdot m$, for $m = -\frac{N}{2} + 1, \dots, \frac{N}{2}$, and $\Delta k = \frac{k_{max} - k_{min}}{N-1}$. Using $k_{min} = \Delta k(-\frac{N}{2} + 1)$, we have that $\Delta k = \frac{2k_{max}}{N}$. We use the Nyquist frequency conditions, which imply that $k_{max} = \frac{1}{2\Delta x}$, and hence $\Delta k = \frac{N-1}{XN}$.

The discrete version of the Fourier transform of the option price $v(x, \tau)$ at a constant time τ is

$$\begin{aligned} V(k_m) &\approx \sum_{n=0}^{N-1} v(x_n, \tau) e^{-2\pi i k_m x_n} \Delta x \\ &= \sum_{n=0}^{N-1} v(x_n, \tau) e^{-2\pi i k_m (x_{min} + n\Delta x)} \cdot \Delta x \\ &= e^{-2\pi i k_m x_{min}} \cdot \Delta x \sum_{n=0}^{N-1} v(x_n, \tau) e^{-2\pi i k_m \Delta x \cdot n} \\ &= e^{-2\pi i k_m x_{min}} \cdot \Delta x \sum_{n=0}^{N-1} v(x_n, \tau) e^{-2\pi i (\Delta k \cdot m) \Delta x \cdot n} \\ &= e^{-2\pi i k_m x_{min}} \cdot \Delta x \sum_{n=0}^{N-1} v(x_n, \tau) e^{-2\pi i m (\frac{N-1}{XN}) (\frac{X}{N-1}) n} \\ &= e^{-2\pi i k_m x_{min}} \cdot \Delta x \sum_{n=0}^{N-1} v(x_n, \tau) e^{-2\pi i \frac{m}{N} n} \end{aligned} \quad (2.22)$$

We define $\hat{V}(m) := \sum_{n=0}^{N-1} v(x_n, \tau) e^{-2\pi i \frac{m}{N} n}$ to be the discrete Fourier transform (DFT) of the option price. Similar to the continuous case, we use notation $\hat{V}^l(m) := \sum_{n=0}^{N-1} v(x_n, \tau_l) e^{-2\pi i \frac{m}{N} n}$ to represent the discrete Fourier transform of the option price at a specific time τ_l . With this notation, we can rewrite (2.22) as

$$V(k_m) \approx e^{-2\pi i k_m x_{min}} \cdot \Delta x \cdot \hat{V}(m). \quad (2.23)$$

2.2.5 Fourier Space Time-stepping Method

We have derived a time-marching equation in Fourier space, given by (2.18), which is rewritten here for convenience:

$$V^u(k) = V^l(k) \cdot e^{\Psi(k)(\tau_u - \tau_l)}.$$

Equation (2.18) describes a way to compute the Fourier transform of the option price at a later time τ_u , given that we have computed the values at an earlier time τ_l , for any $0 \leq \tau_l \leq \tau_u \leq T$. Note that equation (2.18) uses a continuous Fourier transform. In practice, we have to approximate the continuous Fourier transform with a discrete Fourier transform (DFT) using equation (2.23). Specifically, we use two approximations in the method:

$$V^u(k_m) \approx e^{-2\pi i k_m x_{min}} \cdot \Delta x \cdot \hat{V}^u(m), \quad (2.24)$$

and

$$V^l(k_m) \approx e^{-2\pi i k_m x_{min}} \cdot \Delta x \cdot \hat{V}^l(m). \quad (2.25)$$

Using approximations (2.24) and (2.25), we approximate equation (2.18) with the equation

$$e^{-2\pi i k_m x_{min}} \cdot \Delta x \cdot \hat{V}^u(m) = e^{-2\pi i k_m x_{min}} \cdot \Delta x \cdot \hat{V}^l(m) \cdot e^{\Psi(k_m)(\tau_u - \tau_l)}. \quad (2.26)$$

The factors of $e^{-2\pi i k_m x_{min}} \Delta x$ in equation (2.26) cancel out so it simply reduces to

$$\hat{V}^u(m) = \hat{V}^l(m) \cdot e^{\Psi(k_m)(\tau_u - \tau_l)}. \quad (2.27)$$

We then can use an inverse discrete Fourier transform (IDFT) to recover the option price in real space for a log stock price x_n , for any $n = 0, 1, \dots, N - 1$, as

$$\begin{aligned} IDFT \left[\hat{V}^u(m) \right] (x_n) &= IDFT \left[\hat{V}^l(m) \cdot e^{\Psi(k_m)(\tau_u - \tau_l)} \right] (x_n) \\ v(x_n, \tau_u) &= IDFT \left[\hat{V}^u(m) \cdot e^{\Psi(k_m)(\tau_u - \tau_l)} \right] (x_n). \end{aligned}$$

The computed value $v(x_n, \tau_u)$ represents the price of the option when the stock price is $S = e^{x_n}$ at a time τ_u . Recalling that $v(x, \tau) = \mathcal{V}(e^x, \tau)$ recovers the original notation for the option price.

2.3 Illustration of Method

2.3.1 Single Asset European Put under Merton Jump Diffusion

This section provides a simple illustration of the Fourier Space Time-stepping method using a European put option with the underlying asset following Merton jump diffusion with

continuous dividends. With the stock price S following Merton jump diffusion, we have a normal distribution for the jump size, $y = \log(\eta)$:

$$f(y) = \frac{1}{\sqrt{2\pi\gamma}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\gamma}\right)^2},$$

with

$$\kappa := E[\eta - 1] = e^{(\mu + \frac{\gamma^2}{2})} - 1.$$

From Appendix B.1, we have derived a closed-form representation for the Fourier transform of the jump size distribution function f , which is given by

$$F(k) := \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\gamma}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\gamma}\right)^2} \cdot e^{-2\pi i k y} dy = e^{-2(\pi i k \mu + (\pi k \gamma)^2)}.$$

Thus, the characteristic exponent for the FST approach is given by:

$$\begin{aligned} \Psi(k) &:= -\frac{\sigma^2}{2}(2\pi k)^2 + \left(r - q - \lambda\kappa - \frac{\sigma^2}{2}\right)(2\pi i k) - (r + \lambda) + \lambda \bar{F}(k) \\ &= -\frac{\sigma^2}{2}(2\pi k)^2 + \left(r - q - \lambda(e^{(\mu + \frac{\gamma^2}{2})} - 1) - \frac{\sigma^2}{2}\right)(2\pi i k) - (r + \lambda) + \lambda e^{2(\pi i k \mu + (\pi k \gamma)^2)}, \end{aligned}$$

where q is the continuous dividend rate.

The payoff of a European put option is

$$\mathcal{V}(S, \tau = 0) = \max(K - S, 0),$$

where τ is time running backwards (so $\tau = 0$ corresponds to $t = T$).

To price the European put, we solve the PIDE given by equation (2.2) numerically with the FST method in one time-step. We discretize the S domain by defining nodes in the S direction denoted as $[S_0, S_1, \dots, S_{i_{max}}]$, which are equally spaced in $\log(S)$. The number of nodes in the S direction is referred to as ‘‘S nodes’’ in Table 2.2. We discretize the time domain according to the time-dependent conditions imposed on the option value. Since this example is a simple European put option, there are no conditions imposed throughout the option’s life. Therefore, we discretize the time domain with $\tau^0 = 0$ and $\tau^1 = T$, noting that $\Delta\tau = T$. Using the FST method, we price European options in one time-step, from $\tau = 0$ to $\tau = T$. The parameters for the European option are given in Table 2.1. The pricing results of the European put option are given in Table 2.2. The pseudocode of the pricing algorithm is given in Appendix E.1. It is important to note that linear interpolation of the solution at the point of interest ($S = 100$) in the spatial domain generates the 2nd-order convergence exhibited in Table 2.2.

Parameter	Value
S	100
K	100
r	0.05
q	0.02
T	10
σ	0.15
λ	0.1
μ	-1.08
γ	0.4

Table 2.1: Parameters for European put option under Merton jump diffusion.

Refinement	S Nodes	Price	Change	Ratio
0	512	18.00676353		
1	1024	18.00441008	0.00235345	
2	2048	18.00382308	0.00058700	4.01
3	4096	18.00367649	0.00014659	4.00
4	8192	18.00363986	0.00003663	4.00

Table 2.2: Pricing results of the European put option with parameters in Table 2.1. Parameters from Andersen and Andreasen (2000), with closed-form reference price of 18.0034. Semi-closed form reference price of 18.00362936 given by Surkov (2009). The order of convergence is 2 in space.

2.4 Extensions

2.4.1 Asset Models

When the underlying asset follows the jump diffusion stochastic process given by equation (2.1), we show in Section 2.2 that the characteristic exponent for the FST implementation is

$$\Psi(k) = \left(-\frac{\sigma^2}{2}(2\pi k)^2 + \left(r - \lambda\kappa - \frac{\sigma^2}{2} \right) (2\pi ik) - (r + \lambda) + \lambda \bar{F}(k) \right),$$

where $F(k)$ represents the continuous Fourier transform of the jump size distribution $f(y)$, with $y = \log(\eta)$. Recall that $\lambda = 0$ reduces the stochastic model to geometric Brownian motion, with no jumps. Note that the stochastic process (2.1) drifts at μ . Throughout the rest of this essay, we will be considering jump diffusion stochastic processes of the form (2.1), but which drift at a rate of $(\mu - \xi)$, where ξ is a generic fee imposed on the asset that varies in different pricing examples. In the previous section, Section 2.3.1, we set $\xi = q$, where q is the continuous dividend rate, to price European options with continuous dividends. In Section 3, we value Guaranteed Minimum Withdrawal Benefit (GMWB) variable annuities, and we set $\xi = \alpha_g$, where α_g represents the insurance guarantee fee. In Section 4, we value hedge fund incentive fees, and we set $\xi = m_{total}$, where m_{total} represents the hedge fund management fee. In either case, when the underlying asset process is of the form (2.1) but drifts at a rate of $(\mu - \xi)$, the characteristic exponent in the FST approach is

$$\Psi(k) = \left(-\frac{\sigma^2}{2}(2\pi k)^2 + \left(r - \xi - \lambda\kappa - \frac{\sigma^2}{2} \right) (2\pi ik) - (r + \lambda) + \lambda \bar{F}(k) \right).$$

Furthermore, different jump diffusion models have been proposed in modeling asset prices in the literature. Merton (1976) proposed using a normal distribution for $f(y)$, where $y = \log(\eta)$, corresponding to a log-normal distribution in η . Kou (2002) suggested a double exponential distribution for $f(y)$. In both Merton Jump Diffusion and Kou Jump Diffusion, the continuous Fourier transform of the jump density $f(y)$ is available in closed-form by solving an integral analytically. Table 2.3 lists functions used for the characteristic exponent in different jump models. A detailed derivation of the results is given in Appendix B. A pricing example of a European call option under Kou Jump Diffusion is given in Appendix C.1.

In this essay, a jump diffusion process is assumed for the underlying asset in a variety of pricing examples. Both geometric Brownian motion and jump diffusion are included in the more general class of models known as Lévy processes. An exponential Lévy model defines the asset price process to be of the form

$$S(t) = S(0)e^{X(t)},$$

where $X(t)$ is a Lévy process. A Lévy process has independent and stationary increments, and is stochastically continuous. Lévy processes may have an infinite number of jumps

	Merton Jump Diffusion	Kou Jump Diffusion
Lévy Density	$f(y) = \frac{1}{\sqrt{2\pi\gamma}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\gamma}\right)^2}$	$f(y) = p\eta_1 e^{-y\eta_1} \cdot \mathbb{1}_{\{y \geq 0\}} + (1-p)\eta_2 e^{y\eta_2} \cdot \mathbb{1}_{\{y \leq 0\}}$
κ	$e^{(\mu + \frac{\gamma^2}{2})} - 1$	$p \frac{\eta_1}{\eta_1 - 1} + (1-p) \frac{\eta_2}{\eta_2 + 1} - 1$
$\bar{F}(k)$	$e^{2(\pi i k \mu + (\pi k \gamma)^2)}$	$\frac{p}{1 - 2\pi i k \left(\frac{1}{\eta_1}\right)} + \frac{1-p}{1 + 2\pi i k \left(\frac{1}{\eta_2}\right)}$

Table 2.3: Jump diffusion process functions for Fourier Space Time-stepping implementation, with derivation given in Appendix B. The parameter p in the Kou model represents the probability of an upward jump.

Model	Lévy Density	Characteristic Exponent
Variance Gamma	$\frac{1}{\tilde{\mu} y } e^{C_1 y - C_2 y }$	$-\frac{1}{\tilde{\mu}} \ln(1 - i\tilde{\gamma}\tilde{\mu}\omega + \frac{\tilde{\sigma}^2 \tilde{\mu}\omega^2}{2})$
Normal Inverse Gaussian	$\frac{C_3}{ y } e^{C_1 y} K_1(C_4 y)$	$-\frac{1}{\tilde{\mu}} (\sqrt{1 - 2i\tilde{\gamma}\tilde{\mu}\omega + \tilde{\sigma}^2 \tilde{\mu}\omega^2} - 1)$
Carr-Geman-Madan-Yor	$\frac{C}{ y ^{1+Y}} (e^{-G y } \mathbb{1}_{y < 0} + e^{-M y } \mathbb{1}_{y > 0})$	$C\Gamma(-Y) [(M - i\omega)^Y - M^Y + (G + i\omega)^Y - G^Y]$

Table 2.4: Lévy densities and characteristic exponents for exponential Lévy processes. In this table, $\tilde{\gamma}$ and $\tilde{\sigma}$ are the drift and volatility of the driving Brownian motion (if applicable). $\omega = 2\pi k$, $C_1 = \frac{\tilde{\gamma}}{\tilde{\sigma}^2}$, $C_2 = \frac{\sqrt{\tilde{\gamma}^2 + 2\tilde{\sigma}^2/\tilde{\mu}}}{\tilde{\sigma}^2}$, $C_3 = \frac{\sqrt{\tilde{\gamma}^2 + \tilde{\sigma}^2/\tilde{\mu}}}{\pi\tilde{\sigma}\sqrt{\tilde{\mu}}}$, $C_4 = \frac{\sqrt{\tilde{\gamma}^2 + \tilde{\sigma}^2/\tilde{\mu}}}{\tilde{\sigma}^2}$ and $K_p(x)$ is the modified Bessel function of the second kind. Table reproduced from Surkov (2009). Interested readers are directed to Surkov (2009) for further details.

on a finite interval, whereas jump models can only have a finite number of jumps on a finite interval. Popular Lévy processes applied to mathematical finance include the Variance Gamma (VG) model of Madan and Seneta (1990), the Normal Inverse Gaussian (NIG) model of Bandorff-Nielsen (1997), and the CGMY model of Carr, Geman, Madan, and Yor (2002).

Although the mathematical derivation of the Fourier Space Time-stepping method in Section 2.2 used a jump diffusion price process, the FST method can be applied to all asset price models following exponential Lévy processes. Furthermore, the method is flexible and generic in that the only necessary modification to the algorithm for pricing contingent claims with different underlying exponential Lévy processes is to change the characteristic exponent, Ψ . By the Lévy-Khintchine representation, the characteristic function is available in closed-form for all Lévy processes. Surkov (2009) lists the characteristic exponents used in the FST implementation for the VG model, NIG model, and CGMY model, which are reproduced for convenience in Table 2.4.

The FST method as derived in this essay requires volatility to be constant. With constant volatility, a log transformation in the standard pricing PIDE transforms it into one with constant coefficients and a cross-correlation integral. Surkov (2009) extends the basic FST method to incorporate regime-switching in order to generate stochastic volatility behaviour using constant volatility models. Regime-switching models, first proposed by Naik (1993), assume asset prices switch between states, with each state having a different level of volatility. Although these models can better fit the implied volatility smile observed in markets, regime-switching models cannot capture the correlation between volatility and the asset price. To improve, local volatility models have been proposed which specify volatility as a function of time and asset price. Additionally, stochastic volatility models, such as the one proposed by Heston (1993), allow the asset spot price to be correlated with the volatility level. In these models, volatility is a continuous stochastic variable, in addition to the underlying asset price. Extending the FST approach to price options under both local volatility models and stochastic volatility models is a relevant open direction for future research.

Outside equity markets, mean-reverting stochastic models are commonly used in modeling asset prices. These models are particularly popular in commodity and interest rate markets. Some models, such as the one proposed by Chewlow and Strickland (2000), allow for mean-reversion in addition to jumps. Surkov (2009) develops a general, multi-factor model with jumps for the FST method.

2.4.2 Options

Bermudan and American Options

The derivation of the FST method assumed the pricing of a standard European option. In Section 2.3.1, we have illustrated the method through the pricing of a European put option. In this example, the FST method computes prices of the European option in one time-step, as there are no conditions imposed on the option value throughout the life of a European option. However, in general, when pricing with different types of options, there may be conditions imposed on the option value. For example, consider a Bermudan option which

can be exercised only on discrete monitoring dates during the life of the option. This leads to imposing early exercise conditions on the monitoring dates when computing a solution. Similarly, the pricing examples which follow in this essay are similar to Bermudan options in that conditions on the contingent claim value are imposed on specified monitoring dates. When applying the FST method to these types of options or contingent claims, we are able to time-step between monitoring dates in only one time-step. Additionally, the FST method can be easily extended to price American options. In this case, the early exercise constraint is applied throughout the life of the option, in real space. Additionally, Surkov (2009) shows that the FST method can be extended, based on the penalty method of Forysth and Vetzal (2002), to obtain quadratic convergence when pricing American options.

Asian options

An Asian option gives the buyer a payoff that depends on the average asset price over some period of time. This is a path-dependent option in which the option price is a function of the underlying asset price as well as the average of the underlying asset price up to that time. Asian options can be either sampled continuously or discretely, corresponding to the times at which the average of the asset price is taken over some period of time.

In the simplified case of a European Asian option sampled discretely, Dewynne and Wilmott (1995) show that the pricing problem reduces to solving one-dimensional decoupled P(I)DEs between sampling dates and applying jump conditions on discrete sampling dates. In this case, as we will see, the pricing problem is very similar to the path-dependent contingent claims considered in this essay. Therefore, simple modifications (changing the jump conditions and contract payoff) can be made to the numerical methods considered in this essay in order to price European Asian options sampled discretely.

More generally, American Asian options under jump diffusion can be valued using a semi-Lagrangian method, as presented in Y. d'Halluin et al. (2003). The authors show that, using this method, the pricing problem of continuously observed American Asian options reduces to solving decoupled sets of one-dimensional PIDEs. Extending the FST method to price American Asian options is a relevant open research direction.

Multi-Dimensional Options

Although the derivation in Section 2.2 used a single-asset European option, the FST method is easily extended to a multi-dimensional framework to price multi-asset options. For example, in a two-dimensional case, the European option value is a function of two stock prices, S_1 and S_2 , as well as time t . We can denote the value of the option as $\mathcal{V}(S_1, S_2, \tau)$. The FST method makes use of a two-dimensional Fourier transform, which is defined on a function $f(x, y)$ as

$$\mathcal{F}[f(x, y)](k_1, k_2) = \int \int_{-\infty}^{\infty} f(x, y) e^{-i2\pi(k_1x + k_2y)} dx dy.$$

The resulting method in the two-dimensional case is very similar to the one-dimensional case, except that a two-dimensional Fourier transform is used in the time-stepping with a two-dimensional characteristic exponent. With both assets following jump diffusion processes,

the two-dimensional characteristic exponent is

$$\begin{aligned} \Psi(k_1, k_2) := & -\frac{\sigma_1^2}{2}(2\pi k_1)^2 - \frac{\sigma_2^2}{2}(2\pi k_2)^2 + \left(r - \lambda_1 \kappa_1 - \frac{\sigma_1^2}{2}\right)(2\pi i k_1) + \left(r - \lambda_2 \kappa_2 - \frac{\sigma_2^2}{2}\right)(2\pi i k_2) \\ & - (r + \lambda_1 + \lambda_2) + \lambda_1 \overline{F}_1(k_1) + \lambda_2 \overline{F}_2(k_2) - \rho \sigma_1 \sigma_2 (2\pi i k_1)(2\pi i k_2). \end{aligned}$$

A pricing example of a two-asset European spread call option, where both assets follow Merton Jump Diffusion, is given in Appendix [C.2](#).

Chapter 3

Guaranteed Minimum Withdrawal Benefits in Variable Annuities

3.1 Introduction

Variable annuity policies are financial contracts between a policyholder and an insurance company, in which cash flows, dependent on the performance of a reference portfolio, are delivered to the policyholder. In exchange, the policyholder delivers a lump-sum premium to the insurance company at initiation of the contract. Advantages of the variable annuity policy are that the policyholder participates in the equity returns of the underlying reference portfolio and that income and investment gains are tax-deferred.

Variable annuities can be issued with a variety of embedded options. One such option is a Guaranteed Minimum Withdrawal Benefit (GMWB) rider. This contract ensures that the entire initial investment is returned to the policyholder throughout the duration of the contract, regardless of the performance of the underlying risky asset portfolio. However, the policyholder still participates in the upside equity returns of the reference portfolio. A contracted withdrawal amount is specified for each withdrawal date, which may be either annually or semi-annually. At each withdrawal date, the policyholder can choose to withdraw up to the contracted amount without penalty, or above the contracted amount subject to a penalty proportional to the excess above contracted.

Since GMWB variable annuities do not carry any life insurance components, pricing of the annuity can be formulated with the mathematical finance framework. There has been some work on pricing variable annuities with the GMWB rider from a financial no-arbitrage standpoint. Work in this area can be divided based on either static or dynamic policyholder withdrawal policies. Kolkiewicz and Liu (2012) assume a constant withdrawal rate and employ Monte Carlo techniques for the variable annuity valuation. Dynamic withdrawal policies are considered in Milevsky and Salisbury (2006), Dai et al. (2007), and Chen and Forsyth (2007). In these works, the worst case hedging cost, from the point of view of the insurance company, is computed. This is also known as a “loss maximizing” policyholder strategy from the insurance company viewpoint. Under this assumption for policyholder

behaviour, the GMWB pricing problem equivocates to an optimal stochastic control problem. Dai et al. (2007) and Chen and Forsyth (2007) consider separate formulations for which withdrawals are continuous and discrete. The continuous case leads to a singular stochastic control model which makes use of a Hamilton-Jacobi-Bellman equation. In the case of discrete withdrawals, the dynamic withdrawal problem can be solved with PDE approaches coupled with optimality conditions at discrete withdrawal dates. Chen and Forsyth (2007) formulate the pricing problem including a fund management fee applied on the balance of the risky asset portfolio, separate from the insurance guarantee fee.

In this chapter, we analyze the GMWB pricing problem with discrete withdrawals assuming a dynamic withdrawal policy. We compute the worst case hedging cost from the insurance company's point of view. Specifically, at each withdrawal date, the policyholder chooses the withdrawal amount as to maximize the cash flow received and the future value of the annuity subsequent to the withdrawal. Fund management fees are excluded from the asset price process, as in Dai et al. (2007). A jump diffusion process is assumed for the underlying reference portfolio, which is a realistic model for long-term contracts. The pricing model does not include mortality effects. We look to analyze the performance of the Fourier Space Time-stepping (FST) method to solve the pricing problem as well as compare the numerical results to fees typically charged in the marketplace.

3.2 Mathematical Model

Let S be the risky asset that is chosen for investment by the policyholder under the variable annuity. The policyholder participates in the gains of the asset S under the GMWB variable annuity contract. In practice, S is usually a mutual fund. In this paper, we assume that the risky asset S follows a jump-diffusion process. Let W be the risky asset account of the variable annuity. Let A be the investor's guarantee account. An insurance guarantee fee α_g is deducted from the risky asset account each year in order to provide compensation to the insurance company for the GMWB contract. Therefore, the risky asset account W follows the stochastic differential equation

$$\begin{aligned} dW &= (\mu - \alpha_g)Wdt + \sigma WdZ + W(\eta - 1)dq, & \text{if } W > 0 \\ dW &= 0, & \text{if } W = 0 \end{aligned} \quad (3.1)$$

where

- μ is the drift rate,
- α_g is the fee paid by the policyholder to the insurer for the embedded guarantee,
- dq is the independent Poisson process, $= \begin{cases} 0 & \text{with probability } 1 - \lambda dt, \\ 1 & \text{with probability } \lambda dt. \end{cases}$
- λ is the mean arrival time of the Poisson process,
- $\eta - 1$ is an impulse function producing a jump from W to $W\eta$,
- σ is the volatility,
- dZ is an increment of the standard Gauss-Wiener process.

In this chapter, we have excluded fees charged by fund management in the asset price process. Formulation of the pricing problem including fund fees is presented in Chen and Forsyth (2007).

Let T represent the expiry date of the contract, and let t be the current time. We define t^k , $k = 1, \dots, K$, to be the k th withdrawal time. There is no withdrawal at time $t^0 = 0$, which represents the initiation of the contract. The control variable $\gamma^k \in [0, A]$ represents the discrete withdrawal amount at time t^k . This is the amount the policyholder chooses to withdraw at a time t^k .

At initiation of the contract, $t^0 = 0$, the policyholder delivers an initial lump-sum investment W_0 to the insurance company. This is the initial value of the W and A accounts:

$$W^0 = A^0 = W_0,$$

where W^0 represents the balance in the W account at time $t^0 = 0$.

Subsequent to the γ^k being withdrawn at time t^k , both the risky asset account W and the guarantee account A are updated to reflect the withdrawal:

$$W^{k+} = \max(W^k - \gamma^k, 0), \quad (3.2)$$

$$A^{k+} = A^k - \gamma^k, \quad (3.3)$$

where W^{k+} represents the balance of the risky asset account infinitesimally after time t^k . The guarantee feature of the GMWB can be noted in equations (3.2) and (3.3). The policyholder can withdraw an amount as long as $A \geq 0$, regardless of the balance in the risky asset account W . This ensures that the policyholder is returned their initial investment at minimum, as W_0 is the initial balance of the guarantee account A .

The contractual variable G^k represents the amount available to withdraw at time t^k . If the withdrawal amount γ^k is less than or equal to G^k , then the policyholder is not penalized. Otherwise, if γ^k is greater than G^k , then a penalty is charged proportional to the amount in excess of the contracted withdrawal amount G^k . Let $\chi = \chi(t)$ represent the deferred surrender charge, which may be a function of time t . The total cash-flow received by the policyholder at a time t^k is given by $f(\gamma^k)$:

$$f(\gamma^k) = \begin{cases} \gamma^k & \text{if } 0 \leq \gamma^k \leq G^k, \\ \gamma^k - \chi(\gamma^k - G^k) & \text{if } \gamma^k > G^k. \end{cases}$$

Let $\bar{\mathcal{V}}(W, A, t)$ denote the no-arbitrage value of the annuity. This is the total cost that the insurance company would incur to fully hedge the GMWB product. Note that $\bar{\mathcal{V}}$ is parameterized with t being the current time. Below, we will define the no-arbitrage value of the annuity $\mathcal{V}(W, A, \tau)$, with $\tau = T - t$ being time running backwards. The only difference between $\bar{\mathcal{V}}$ and \mathcal{V} is the parameterization with respect to time. For simplicity sake, we will use $\bar{\mathcal{V}}$ and time t when discussing the conditions going forwards in time and we will use \mathcal{V} and τ when discussing the conditions going backwards in time t , or equivalently, forwards in τ .

At contract expiration, $t = T$, the policyholder will withdraw as much as possible from the two accounts. Therefore, the terminal condition for the no-arbitrage value is

$$\bar{\mathcal{V}}(W, A, t = T) = \max(W, A(1 - \chi)).$$

In terms of option pricing, this terminal condition can be seen as the “payoff” to the policyholder at contract maturity. The payoff is a function of the risky asset account W and a path-dependent state variable A .

At withdrawal time t^k , the policyholder chooses to withdraw an amount γ^k , and receives an actual cash flow given by $f(\gamma^k)$. We compute the worst case hedging cost from the insurance company’s point of view. Specifically, the policyholder chooses a withdrawal amount γ^k to maximize the cash flow received $f(\gamma^k)$ plus the future value of annuity subsequent to the withdrawal. Therefore, the annuity value \mathcal{V} satisfies the following optimality condition

$$\begin{aligned} \bar{\mathcal{V}}(W, A, t^{k-}) &= \sup_{\gamma^k \in [0, A]} [\bar{\mathcal{V}}(\max(W - \gamma^k, 0), A - \gamma^k, t^k) + f(\gamma^k)], \\ &t = t^k, k = 1, \dots, K, \end{aligned} \quad (3.4)$$

with t^{k-} representing the time infinitesimally before time t^k . The worst case, or largest, hedging cost of the GMWB for the insurance company occurs when the policyholder acts according to optimality condition (3.4). If the policyholder acts sub-optimally, the insurance company can recognize extraordinary gains.

Between withdrawal dates, within the time interval $[t^{k-1}, t^{k-}]$, $k = 1, \dots, K$, the policyholder does not withdraw any amounts and thus, by Ito’s Lemma and no-arbitrage arguments, the annuity value $\bar{\mathcal{V}}(W, A, t)$ solves the following PIDE

$$\begin{aligned} 0 &= \bar{\mathcal{V}}_t + \frac{1}{2} \sigma^2 W^2 \bar{\mathcal{V}}_{WW} + (r - \alpha_g - \lambda \kappa) W \bar{\mathcal{V}}_W - r \bar{\mathcal{V}} + \left(\lambda \int_0^\infty \bar{\mathcal{V}}(W\eta) g(\eta) d\eta - \lambda \bar{\mathcal{V}} \right), \\ &t \in [t^{k-1}, t^{k-}], k = 1, \dots, K, \end{aligned} \quad (3.5)$$

where

- r is the risk free rate, $r \geq 0$,
- $g(\eta)$ is the probability density function of the jump amplitude η ,
thus for all η , $g(\eta) \geq 0$, and $\int_0^\infty g(\eta) d\eta = 1$.
- κ is $E[\eta - 1]$, with $E[\eta] = \int_0^\infty \eta g(\eta) d\eta$.

Computationally, it is convenient to specify the annuity value in terms of τ and work forwards in τ (backwards in time t) when computing a solution. We let $\mathcal{V}(W, A, \tau)$ specify the annuity value parameterized by τ . We will be working forwards in τ . We define τ^k , $k = 0, \dots, K - 1$, to be the k th withdrawal time running backwards, with $\tau^0 = 0$ and $\tau^K = T$. There is no withdrawal at time $\tau^K = T$, which represents the initiation of the contract, $t = 0$.

The annuity value has the following terminal condition

$$\mathcal{V}(W, A, \tau = 0) = \max(W, A(1 - \chi)).$$

On withdrawal dates, $\tau^k, k = 0, \dots, K - 1$, the following optimality conditions are applied

$$\begin{aligned} \mathcal{V}(W, A, \tau^{k+}) &= \sup_{\gamma^k \in [0, A]} [\mathcal{V}(\max(W - \gamma^k, 0), A - \gamma^k, \tau^k) + f(\gamma^k)], \\ \tau &= \tau^k, k = 0, \dots, K - 1, \end{aligned} \quad (3.6)$$

where τ^{k+} is the time infinitesimally after time τ^k (going forwards in τ).

Between withdrawal dates, within the time interval $[\tau^{k+}, \tau^{k+1}]$, $k = 0, \dots, K - 1$, the annuity value $\mathcal{V}(W, A, \tau)$ solves the following PIDE

$$\begin{aligned} \mathcal{V}_\tau &= \frac{1}{2} \sigma^2 W^2 \mathcal{V}_{WW} + (r - \alpha_g - \lambda \kappa) W \mathcal{V}_W - r \mathcal{V} + \left(\lambda \int_0^\infty \mathcal{V}(W \eta) g(\eta) d\eta - \lambda \mathcal{V} \right), \\ \tau &\in [\tau^{k+}, \tau^{k+1}], k = 0, \dots, K - 1. \end{aligned} \quad (3.7)$$

We note that the annuity value $\mathcal{V}(W, A, \tau)$ has no dependence on the withdrawal guarantee account A for $\tau \in [\tau^{k+}, \tau^{k+1}]$, $k = 0, \dots, K - 1$, as A is constant between withdrawal dates and is updated only when a withdrawal occurs.

For the discrete withdrawal GMWB formulation above, the pricing problem can be seen as a path-dependent option where A is a path-dependent state variable which is updated on withdrawal dates. Between withdrawal dates, the annuity value can be solved using PIDE solution techniques. On a withdrawal date τ^k , a withdrawal amount $\gamma^k \in [0, A]$ is chosen, from a finite set of values, to satisfy (3.6). The pricing problem can be regarded as a problem in stochastic optimal control.

Recall that we would like to recover the fee that the insurance company should charge for the variable annuity with a guaranteed minimum withdrawal benefit. As done in Dai et al. (2007) and Chen et al. (2007), we determine the no-arbitrage fee α_g as the rate at which the annuity value \mathcal{V} equals the initial premium W_0 paid by the policyholder at time $\tau = T$ (recall this is at time $t = 0$).

3.3 Numerical Approach

We discretize the W domain by defining nodes in the W direction denoted as $[W_0, W_1, \dots, W_{i_{max}}]$, which are equally spaced in $\log(W)$. A log-transformation in the PIDE (3.7) simplifies to a PIDE having constant coefficients and a cross-correlation integral. We discretize the A domain by defining nodes in the A direction denoted as $[A_0, A_1, \dots, A_{j_{max}}]$. We discretize the time domain according to the contract details regarding withdrawal frequency. We define discrete withdrawal dates denoted as $\tau^k, k = 0, \dots, K - 1$, with $\tau^0 = 0$ and $\tau^K = T$. In order to choose a withdrawal amount on behalf of the policyholder to satisfy the optimality

conditions on withdrawal dates, we construct a finite set of evenly-spaced amounts at time τ^k , for each $A_j, j = 0, \dots, j_{max}$, as $[\gamma_{j,0}^k, \gamma_{j,1}^k, \dots, \gamma_{j,k_{max}}^k]$, with $\gamma_{j,0}^k = 0$ and $\gamma_{j,k_{max}}^k = A_j$.

The PIDE given by equation (3.7) has no dependence on the guarantee account A due to the fact that A is only updated on discrete withdrawal dates and does not change between withdrawal dates. This reduces the pricing problem to solving sets of decoupled problems, or in other words, sets of independent one-dimensional problems.

On withdrawal dates, we solve a set of decoupled PIDEs, one for each $A_j, j = 0, \dots, j_{max}$, as given by (3.7). To solve the one-dimensional PIDE (3.7) between withdrawal dates, we use a Fourier Space Time-stepping (FST) approach which is derived for a jump diffusion stochastic process in Chapter 2. With the FST method, changing the stochastic process of the underlying asset requires a minor change in the time-stepping algorithm. In this case, we change the characteristic exponent to the one derived from the GMWB stochastic process (3.1). We use characteristic exponent $\Psi^{GMWB}(k)$ in the FST implementation, given by

$$\Psi^{GMWB}(k) = \left(-\frac{\sigma^2}{2}(2\pi k)^2 + \left(r - \alpha_g - \lambda\kappa - \frac{\sigma^2}{2} \right) (2\pi i k) - (r + \lambda) + \lambda \bar{F}(k) \right). \quad (3.8)$$

Additionally, the FST approach has the advantage that we do not need to discretize the time domain between withdrawal dates in order to complete the time-stepping. Specifically, with withdrawal dates $\tau^k, k = 0, \dots, K - 1$, we can time-step between withdrawal dates $[\tau^{k+}, \tau^{k+1}]$ in only one time-step.

Note that, when solving the PIDE using the FST method, the concept of *zero padding* is applied to eliminate wrap-around error in the annuity value solution. Wrap-around error and the use of zero padding for elimination of the error is discussed in Section D in the Appendix.

On withdrawal dates, we solve a set of decoupled optimization problems according to (3.6). We fix a $W_i, i = 0, \dots, i_{max}$ and $A_j, j = 0, \dots, j_{max}$, and consider a finite set of withdrawal amounts $\gamma^k \in [0, A_j]$ at a time τ^k . We choose the γ^k to satisfy condition (3.6).

The approach above describes a numerical method for finding $\mathcal{V}(W, A, \tau)$, for $0 \leq \tau \leq T$. Using this, we look to determine the insurance guarantee fee α_g that the insurance company should charge for the product. Recall that α_g is the fee deducted from the risky asset account W each year. Let $\mathcal{V}(\alpha_g, W, A, \tau)$ denote the value of the annuity for a given insurance fee α_g . At the initiation of the contract, we want the value of the annuity to equal the initial lump-sum payment delivered by the policyholder to the insurance company. Therefore, for an initial lump-sum payment of W_0 , we want to determine a α_g such that

$$\mathcal{V}(\alpha_g, W = W_0, A = W_0, \tau = T) = W_0, \quad (3.9)$$

where $\tau = T$ denotes the initiation of the contract ($t = 0$). With this condition and a numerical method for finding $\mathcal{V}(\alpha_g, W, A, \tau = T)$ for any α_g , we find the no-arbitrage fee α_g by using a Newton Iteration. This reduces to choosing an initial fee α_{g_0} and iterating until condition (3.9) is satisfied.

3.4 Numerical Results

The parameters for a typical GMWB variable annuity contract are given in Table 3.1. The underlying asset is assumed to follow a jump diffusion process with jump parameters given in Table 3.2. The convergence of the annuity price is given in Table 3.3. A no-arbitrage fee is not imposed in Table 3.3, corresponding to $\alpha_g = 0$. The order of convergence is 2 in space. Recall that we would like to find the no-arbitrage fee such that the annuity value at initiation of the contract equals the initial lump-sum premium paid by the policyholder. Mathematically, this equivocates to satisfying condition (3.9). In this example, the initial lump-sum $W_0 = 100$. The Newton iterates to find the corresponding α_g , so that equation (3.9) is satisfied, are given in Table 3.4. The fee found by each Newton iteration and the corresponding value are given. The no-arbitrage insurance guarantee fee α_g was found to be 0.04196447.

	Parameter	Value
T	Expiry time	10 years
r	Interest Rate	5%
G	Contract withdrawal amount	10
W_0	Initial lump-sum premium	100
σ	Volatility	0.3
$\Delta\tau^k$	Withdrawal interval	1 year
α_m	Mutual fund fee	0%
χ	Surrender charge	0.1

Table 3.1: Contract parameters for GMWB with underlying asset following Merton Jump Diffusion.

Parameter	Value
λ	0.1
ν	-0.9
ζ	0.45

Table 3.2: GMWB Merton Jump Diffusion parameters.

In a second example, the parameters for the GMWB variable annuity contract are given in Table 3.5. The underlying asset is assumed to follow geometric Brownian motion, which corresponds to $\lambda = 0$ in the jump diffusion model. In this example, the surrender charges are decreasing as a function of time, as given in Table 3.6. The convergence of the annuity price is given in Table 3.7. A no-arbitrage fee of $\alpha_g = 0.0088$ is imposed in Table 3.7, as found in Chen et al. (2007). Therefore, the annuity value should converge to the initial lump-sum premium amount $W_0 = 100$. The order of convergence is 2 in space.

Refinement	W Nodes	A Nodes	Price	Change	Ratio
0	64	64	124.13579290		
1	128	128	123.94373900	0.19205390	
2	256	256	123.89011710	0.05362190	3.58
3	512	512	123.88038840	0.00972870	5.51
4	1024	1024	123.87751209	0.00287631	3.38

Table 3.3: Convergence results for GMWB contract with parameters given in Table 3.1 and jump distribution parameters in Table 3.2. No-arbitrage insurance fee is not imposed, $\alpha_g = 0$. The order of convergence is 2 in space.

Newton Iterate	Fee	Annuity Value
1	0.00000000	23.86712228
2	0.02967984	104.83594324
3	0.03991374	100.66324229
4	0.04186814	100.02964716
5	0.04196448	99.99999732
6	0.04196447	100.00000000

Table 3.4: Newton iterates to find the no-arbitrage insurance fee α_g for GMWB contract in Table 3.5. Tolerance of 10^{-8} . Annuity Value is the GMWB annuity value corresponding to the specific fee used. Reference value of 0.045452043 given in Huang et al. (2012). Note that the reference value is with continuous withdrawals permitted. The discrete withdrawal insurance fee should be less than the continuous withdrawal insurance fee.

	Parameter	Value
T	Expiry time	10 years
r	Interest Rate	5%
G	Contract withdrawal amount	10
W_0	Initial lump-sum premium	100
σ	Volatility	0.15
$\Delta\tau^k$	Withdrawal interval	1 year
α_m	Mutual fund fee	0%

Table 3.5: Contract parameters for GMWB with underlying asset following geometric Brownian motion.

Year	Surrender charge
$0 \leq t < 2$	8%
$2 \leq t < 3$	7%
$3 \leq t < 4$	6%
$4 \leq t < 5$	5%
$5 \leq t < 6$	4%
$6 \leq t < 7$	3%
$t \geq 7$	0%

Table 3.6: Decreasing surrender charge schedule for contract given in Table 3.5.

Refinement	W Nodes	A Nodes	Price	Change	Ratio
0	128	128	100.85406466		
1	256	256	100.19725079	0.65681387	
2	512	512	100.04037291	0.15687789	4.19
3	1024	1024	100.00130127	0.03907164	4.02

Table 3.7: Convergence results for GMWB contract with parameters given in Table 3.5. Note this contract uses a decreasing fee structure given in Table 3.6. There are no jumps in the underlying asset model, $\lambda = 0$, reducing the model to geometric Brownian motion. No-arbitrage insurance fee of $\alpha_g = 0.0088$ imposed. Annuity value should converge to $W_0 = 100$ as shown in Chen et al. (2007). The order of convergence is 2 in space.

Chapter 4

Hedge Fund Incentive Fee Valuation

4.1 Introduction

There has been rapid growth in the hedge fund industry. Hedge fund industry assets surged by \$122 billion to \$2.375 trillion in the first quarter of 2013, which marks a record for total industry capital¹. New net capital allocated to hedge funds in the first quarter was \$15.2 billion, with the largest inflow of \$9.6 billion to Fixed-Income Relative Value Arbitrage funds². In 2000, total hedge fund industry assets were approximately \$500 billion³.

Hedge funds are open-ended and actively managed investment pools or investment vehicles. There are three main elements which distinguish a hedge fund from their traditional counterpart, a mutual fund. A hedge fund (1) is privately organized, (2) usually offers incentive fees to managers based on performance, and (3) can invest in nontraditional investments [7].

Hedge funds are privately organized investment vehicles available to sophisticated investors. The privately organized and unlisted nature allows hedge funds to be typically less regulated than public investment vehicles open to the public. This allows hedge funds to offer opportunities outside those offered by traditional, regulated investment pools. Since investments in a hedge fund are limited to sophisticated investors, the raising of capital is completed privately and is limited to a narrower range of investors, mainly foundations, endowments, family offices, and pension funds.

The ability of the hedge fund to offer sophisticated strategies and nontraditional investment opportunities stems from the fact that the hedge fund industry provides performance-based fees to top fund managers. The incentive fees are to compensate managers for their skill and expertise, as well as to align the interests of investors and managers.

The less regulated nature of hedge funds allow them to carry greater investment flexibility than the traditional mutual fund. In terms of investment assets, hedge funds have scope to

¹Source: Hedge Fund Research, Inc.

²Source: Hedge Fund Research, Inc.

³Source: Chartered Alternative Investment Analyst (CAIA) Association.

invest in nonpublic and unlisted securities. With respect to investment products, hedge funds can take long and short positions in underlying assets, as well as use derivative strategies and invest in structured products. Additionally, a hedge fund can often employ a significant amount of leverage to its positions.

Compensation to hedge fund managers typically arises from a management fee and a performance-based incentive fee. Management fees are yearly fees proportional to the net asset value (NAV) of the fund. Incentive fees are a form of compensation for superior fund performance. Both types of fees are calculated and paid to managers on discrete valuation dates, typically either yearly or quarterly. The performance-based incentive fees paid to fund managers are computed based on the concept of a high-water mark (HWM). The HWM is defined as the highest NAV recorded on valuation dates when incentive fees (if any) are paid. Incentive fees are paid to managers when the NAV, after management fees are deducted, surpasses the current HWM. In some hedge fund contracts, a hurdle rate, or minimum required rate of return is set such that incentive fees are not paid until the hurdle rate is achieved and the HWM is surpassed. When this occurs, the manager receives a percentage, known as the performance rate, of the fund's profits as an incentive payment. After the incentive fees are paid to managers, a new HWM is set. The main idea is that incentive fees are paid only once for cumulative net profits to the investors, and not paid to managers when previous losses are recouped [7].

The incentive fees paid to managers are a direct transfer of wealth from the investors to the fund managers. While the investor has a symmetric payoff with regards to the performance of the fund, the manager has an asymmetric payoff. The manager has unlimited upside potential stemming from incentive fees for superior performance, but limited downside risk as no action is taken for poor performance (except the possibility of losing the job as fund manager). Therefore, the fund manager's payoff can be formulated as an option or contingent claim on the NAV of the fund, with additional state variables.

It is well known that the typical hedge fund compensation structure is a so-called "2/20" structure, which represents a 2% management fee with a 20% performance rate. However, exceptionally successful funds are able to command much larger compensation. For example, Renaissance Technologies' Medallion Fund, which averaged a 35% net yearly return after fees since 1989, currently operates under a 5% management fee with a 44% performance rate⁴.

In recent years, a new type of compensation structure for hedge funds has been gaining popularity. The scheme is known as first-loss capital, in which hedge fund managers put up a percentage, usually 10%, of their own capital. In exchange, managers can command a much larger performance rate on fund profits. The manager's deposit is drawn down if the fund experiences losses, which effectively puts managers at a "first-loss" position⁵. He and Kou (2011) investigate this new compensation structure and compare it to the traditional structure. Via a behavioral finance approach, the authors solve a utility maximization problem. They show that the first-loss scheme can, in most cases, improve the satisfaction of regulators, fund managers, and fund investors, simultaneously, as compared to the traditional compensation structure.

⁴Sources: Bloomberg Businessweek, Insider Monkey, Wall Street Journal

⁵Source: Wall Street Journal's Market Watch

In a similar approach, the traditional compensation structure is studied in a variety of papers. In each, the manager maximizes a utility function and manager behaviour is analyzed. Carpenter (2000) solves the maximization problem analytically and concludes that increasing the performance rate causes managers to reduce volatility. Hodder and Jackwerth (2007) solve the discrete time maximization problem numerically, with the inclusion of a liquidation barrier. They showed that volatility increases when the assets are near a level that would pay a performance fee to the manager. Kouwenberg and Ziemba (2007) included managerial ownership in the model and showed that increasing the performance rate will increase asset volatility. He and Kou (2011) include both manager co-investment and a liquidation barrier to compare both traditional and first-loss schemes.

Goetzmann et. al (2003) model hedge fund assets directly and use option pricing techniques to value the incentive fee claim as a path-dependent option. The high-water mark changes continuously and is thus simply the maximum asset value over the life of the hedge fund. Under a geometric Brownian motion process for the fund's assets, the authors find a closed-form solution for the value of the claim. The model does not incorporate hurdle rates, which are common in some hedge fund contracts. Additionally, manager co-investing is not considered in the model.

This essay develops a mathematical model for valuing hedge fund incentive fee contingent claims in the traditional compensation structure. The framework is similar to that in Xiao (2006). The manager's option is a function of the current NAV as well as the HWM and the last recorded NAV. The formulation is analogous to that of a path-dependent option with two state variables, with the state variables changing discretely. We assume that the hedge fund assets follow a jump diffusion process. A jump diffusion model is appropriate for hedge fund assets as they are able to trade in derivative strategies and structured products as well as employ high degrees of leverage. Additionally, manager co-investing is not considered in this model, but may be an interesting way to extend the model presented.

4.2 Mathematical Model

Assume that the investment in the hedge fund is given by a individual investor and let S denote the balance in the investor's account from the hedge fund. The sample paths of the hedge fund balance S can be modeled by a jump diffusion stochastic differential equation:

$$\frac{dS}{S} = (\mu - m_{total}) dt + \sigma dZ + (\eta - 1) dq, \quad (4.1)$$

where

- μ is the drift rate,
- m_{total} is the rate at which management fees are deducted from the investor's account,
- dq is the independent Poisson process, $= \begin{cases} 0 & \text{with probability } 1 - \lambda dt, \\ 1 & \text{with probability } \lambda dt, \end{cases}$
- λ is the mean arrival time of the Poisson process,
- $\eta - 1$ is an impulse function producing a jump from S to $S\eta$,
- σ is the volatility,
- dZ is an increment of the standard Gauss-Wiener process.

The hedge fund contract between the manager and the investor specifies certain variables necessary in calculating the manager's incentive payment. We denote the manager's performance rate as p , which represents the percentage of profits that the manager receives as an incentive payment. The hurdle rate, h , represents the minimum required rate of return that the investor must receive before the manager is paid an incentive payment. Additionally, we define valuation dates, typically quarterly or yearly, which reflect dates at which incentive fees are paid. We assume that the time interval between valuation dates is constant.

Let T represent the hedge fund "expiry" date, and let t be the current time. We specify the valuation dates as $t^k, k = 1, \dots, K$, to be the k th valuation date, with $t^0 = 0$ and $t^K = T$. There is no valuation at initiation of hedge fund operations at time $t^0 = 0$.

When the hedge fund begins operations, we assume that the investor delivers an initial investment amount of S_0 to the hedge fund. We define two state variables, S_{old} and H , in order to calculate the value of the manager's option. S_{old} represents the value of the investor's account on the previous valuation date, after fees have been paid. H denotes the high-water mark (HWM), which is highest asset value after fees have been paid, as recorded on valuation dates, so far over the life of the hedge fund. Upon beginning operations, at time $t^0 = 0$, the value of the S, S_{old} and H variables are simply equal to the initial investment S_0 :

$$S^0 = S_{old}^0 = H^0 = S_0, \quad (4.2)$$

where S^0 represents the balance in the S account at time $t^0 = 0$.

Note that between valuation dates, the balance of the investor's account S follows the stochastic jump diffusion process (4.1). The state variables S_{old} and H do not change between valuation dates, and are updated only on discrete valuation dates after incentive fees have been calculated and paid.

At valuation dates $t^k, k = 1, \dots, K$, hedge fund returns and incentive fees are calculated. The incentive fee paid to the hedge fund manager at time t^k is

$$P(S, S_{old}, H, t^k) = p \cdot \max [S - \max [S_{old}(1 + h\Delta t), H], 0]. \quad (4.3)$$

Intuitively, the hedge fund manager receives a percentage, p , of the hedge fund profits after a specified minimum required return, h , is achieved on the previous asset balance S_{old} , if this is greater than the high-water mark H .

Subsequent to the incentive fee being calculated, both the investor's account balance S as well as state variables S_{old} and H have to be updated to reflect the payment to the manager. The updating rules are

$$S^{k+} = S^k - P(S^k, S_{old}^k, H^k, t^k), \quad (4.4)$$

$$S_{old}^{k+} = S^k - P(S^k, S_{old}^k, H^k, t^k), \quad (4.5)$$

$$H^{k+} = \max(S^k - P(S^k, S_{old}^k, H^k, t^k), H^k), \quad (4.6)$$

where S^{k+} denotes the balance in the S account instantaneously after time t^k . Updating rule (4.4) shows that the incentive fee or performance payment P is a direct transfer of wealth from the investor to the hedge fund manager.

Let $\bar{\mathcal{V}}(S, S_{old}, H, t)$ denote the no-arbitrage value of the manager's option. Section 4.5 gives a detailed explanation of the intuition behind the no-arbitrage value in this pricing example. Note that $\bar{\mathcal{V}}$ is parameterized with t being the current time. Below, we will define the no-arbitrage value of the annuity $\mathcal{V}(S, S_{old}, H, \tau)$, with $\tau = T - t$ being time running backwards. The only difference between $\bar{\mathcal{V}}$ and \mathcal{V} is the parameterization with respect to time. For simplicity sake, we will use $\bar{\mathcal{V}}$ and time t when discussing the conditions going forwards in time and we will use \mathcal{V} and τ when discussing the conditions going backwards in time t , or equivalently, forwards in τ .

For pricing purposes, we assume that the hedge fund "expires" or closes at time $t = T$, and that no incentive fees are paid afterwards. Therefore, the terminal condition for the no-arbitrage value is

$$\bar{\mathcal{V}}(S, S_{old}, H, t = T) = p \cdot \max[S - \max[S_{old}(1 + h\Delta t), H], 0]. \quad (4.7)$$

On valuation dates $t^k, k = 1, \dots, K$, the manager's option satisfies the following jump condition:

$$\begin{aligned} \bar{\mathcal{V}}(S, S_{old}, H, t^{k-}) &= \bar{\mathcal{V}}(S - \mathcal{P}, S - \mathcal{P}, \max(S - \mathcal{P}, H), t^k) + \mathcal{P}, \\ t &= t^k, k = 1, \dots, K, \end{aligned} \quad (4.8)$$

where

$$\mathcal{P} := P(S, S_{old}, H, t^k) = p \cdot \max[S - \max[S_{old}(1 + h\Delta t), H], 0]. \quad (4.9)$$

Between valuation dates, within the time interval $[t^{k-1}, t^{k-}]$, $k = 1, \dots, K$, by Ito's Lemma and no-arbitrage arguments, the manager's option value $\bar{\mathcal{V}}(S, S_{old}, H, \tau)$ solves the following PIDE

$$\begin{aligned} 0 &= \bar{\mathcal{V}}_t + \frac{\sigma^2 S^2}{2} \bar{\mathcal{V}}_{SS} + (r - m_{total} - \lambda \kappa) S \bar{\mathcal{V}}_S - r \bar{\mathcal{V}} + \left(\lambda \int_0^\infty \bar{\mathcal{V}}(S\eta) g(\eta) d\eta - \lambda \bar{\mathcal{V}} \right), \\ t &\in [t^{k-1}, t^{k-}], k = 1, \dots, K, \end{aligned} \quad (4.10)$$

where

$$\begin{aligned}
r & \text{ is the risk free rate, } r \geq 0, \\
g(\eta) & \text{ is the probability density function of the jump amplitude } \eta, \\
& \text{ thus for all } \eta, g(\eta) \geq 0, \text{ and } \int_0^\infty g(\eta) d\eta = 1. \\
\kappa & \text{ is } E[\eta - 1], \text{ with } E[\eta] = \int_0^\infty \eta g(\eta) d\eta.
\end{aligned}$$

Computationally, it is convenient to specify the manager's option value in terms of τ and work forwards in τ (backwards in time t) when computing a solution. We let $\mathcal{V}(S, S_{old}, H, \tau)$ specify the annuity value parameterized by τ . We will be working forwards in τ . We specify the valuation dates as $\tau^k, k = 0, \dots, K - 1$, to be the k th valuation date, with $\tau^0 = 0$ and $\tau^K = T$. Note that $\Delta t = \Delta \tau$.

The terminal condition for the no-arbitrage value is

$$\mathcal{V}(S, S_{old}, H, \tau = 0) = p \cdot \max[S - \max[S_{old}(1 + h\Delta\tau), H], 0]. \quad (4.11)$$

On valuation dates $\tau^k, k = 0, \dots, K - 1$, the manager's option satisfies the following jump condition:

$$\begin{aligned}
\mathcal{V}(S, S_{old}, H, \tau^{k+}) &= \mathcal{V}(S - \mathcal{P}, S - \mathcal{P}, \max(S - \mathcal{P}, H), \tau^k) + \mathcal{P}, \\
\tau &= \tau^k, k = 0, \dots, K - 1,
\end{aligned} \quad (4.12)$$

where τ^{k+} is the time infinitesimally after time τ^k (going forwards in τ) and

$$\mathcal{P} := P(S, S_{old}, H, \tau^k) = p \cdot \max[S - \max[S_{old}(1 + h\Delta\tau), H], 0]. \quad (4.13)$$

Between valuation dates, within the time interval $[\tau^{k+}, \tau^{k+1}]$, $k = 0, \dots, K - 1$, by Ito's Lemma and no-arbitrage arguments, the manager's option value $\mathcal{V}(S, S_{old}, H, \tau)$ solves the following PIDE

$$\begin{aligned}
\mathcal{V}_\tau &= \frac{\sigma^2 S^2}{2} \mathcal{V}_{SS} + (r - m_{total} - \lambda\kappa) S \mathcal{V}_S - r \mathcal{V} + \left(\lambda \int_0^\infty \mathcal{V}(S\eta) g(\eta) d\eta - \lambda \mathcal{V} \right), \\
\tau &\in [\tau^{k+}, \tau^{k+1}], k = 0, \dots, K - 1,
\end{aligned} \quad (4.14)$$

We note that the manager's option value $\mathcal{V}(S, S_{old}, H, \tau)$ has no dependence on the asset value at the previous valuation date S_{old} or the high-water mark H for $\tau \in [\tau^{k+}, \tau^{k+1}]$, $k = 0, \dots, K - 1$, as S_{old} and H are constant between valuation dates and are updated on valuation dates when incentive fees are paid.

4.3 Similarity Reduction

In this framework, the pricing of the manager's incentive fee contingent claim is a three-dimensional problem. Fortunately, due to the characteristics of the contract payoff, there is a method to reduce the dimensionality of the problem and improve speed when computing a numerical solution. With the definition of a homogeneous function, we are able to show that the value of the manager's option is homogeneous of degree 1, leading to a similarity reduction.

Definition 1. A function $f(x_1, x_2, \dots, x_n)$ is homogeneous of degree m in the variables x_1, x_2, \dots, x_n if

$$f(cx_1, cx_2, \dots, cx_n) = c^m f(x_1, x_2, \dots, x_n) \quad (4.15)$$

for a constant c .

Theorem 1. Suppose a function $F(S, x_1, x_2, \tau)$ satisfies the PIDE (4.14). If

1. the contract payoff $F(S, x_1, x_2, \tau = 0)$ is homogeneous of degree 1 in variables (S, x_1, x_2) , and
2. the jump conditions $F(S, x_1, x_2, \tau^k), k = 1, \dots, K - 1$ are homogeneous of degree 1 in variables (S, x_1, x_2) ,

then $F(S, x_1, x_2, \tau)$ is homogeneous of degree one, for $0 \leq \tau \leq T$, in variables (S, x_1, x_2) .

Proof. First, assume that the contract payoff is homogeneous of degree 1 in variables (S, x_1, x_2) . That is, we can write

$$c \cdot F(S, x_1, x_2, \tau = 0) = F(c \cdot S, c \cdot x_1, c \cdot x_2, \tau = 0) \quad (4.16)$$

for a constant c .

We define a new function \hat{F} , and we want to show that

$$F(S, x_1, x_2, \tau) = \frac{1}{c} \hat{F}(c \cdot S, c \cdot x_1, c \cdot x_2, \tau) \quad (4.17)$$

for $0 \leq \tau \leq T$. Clearly this is true for $\tau = 0$ by equation (4.16). We want to show that (4.17) holds for all $0 \leq \tau \leq T$. We must show that \hat{F} also satisfies PIDE (4.14). Let $\hat{S} = c \cdot S, \hat{x}_1 = c \cdot x_1, \hat{x}_2 = c \cdot x_2$. We have that

$$F_\tau = \frac{1}{c} \hat{F}_\tau \quad (4.18)$$

$$F_{SS} = c \cdot \hat{F}_{\hat{S}\hat{S}} \quad (4.19)$$

$$F_S = \hat{F}_{\hat{S}} \quad (4.20)$$

$$F = \frac{1}{c} \hat{F} \quad (4.21)$$

Plugging (4.18), (4.19), (4.20), (4.21) into the PIDE (4.14) gives

$$\begin{aligned}\frac{1}{c}\hat{F}_\tau &= \frac{\sigma^2(\frac{\hat{S}}{c})^2}{2}c\hat{F}_{\hat{S}\hat{S}} + (r - m_{total} - \lambda\kappa)\frac{\hat{S}}{c}\hat{F}_{\hat{S}} - r\frac{1}{c}\hat{F} + \frac{1}{c}\left(\lambda\int_0^\infty \hat{F}(\hat{S}\eta)g(\eta) d\eta - \lambda\hat{F}\right) \\ \frac{1}{c}\hat{F}_\tau &= \frac{1}{c}\left[\frac{\sigma^2\hat{S}^2}{2}\hat{F}_{\hat{S}\hat{S}} + (r - m_{total} - \lambda\kappa)\hat{F}_{\hat{S}} - r\hat{F} + \left(\lambda\int_0^\infty \hat{F}(\hat{S}\eta)g(\eta) d\eta - \lambda\hat{F}\right)\right] \\ \hat{F}_\tau &= \frac{\sigma^2\hat{S}^2}{2}\hat{F}_{\hat{S}\hat{S}} + (r - m_{total} - \lambda\kappa)\hat{F}_{\hat{S}} - r\hat{F} + \left(\lambda\int_0^\infty \hat{F}(\hat{S}\eta)g(\eta) d\eta - \lambda\hat{F}\right)\end{aligned}$$

This shows that $F(S, x_1, x_2, \tau)$ is homogeneous of degree one, for $[\tau^{k+}, \tau^{k+1}]$, $k = 1, \dots, K-1$, in variables (S, x_1, x_2) . Now, we also know that the jump conditions $F(S, x_1, x_2, \tau^k)$, $k = 1, \dots, K-1$ are homogeneous of degree 1 in variables (S, x_1, x_2) . Therefore, $F(S, x_1, x_2, \tau)$ is homogeneous of degree one, for $0 \leq \tau \leq T$, in variables (S, x_1, x_2) and we can write

$$c \cdot F(S, x_1, x_2, \tau) = F(c \cdot S, c \cdot x_1, c \cdot x_2, \tau) \quad (4.22)$$

for $0 \leq \tau \leq T$. □

Using result (4.22), we can use a similarity reduction to reduce the dimensionality of the hedge fund fee pricing problem.

Firstly, we note

$$\begin{aligned}c \cdot P(S, S_{old}, H, \tau^k) &= c \cdot (p \cdot \max[S - \max[S_{old}(1 + h\Delta\tau), H], 0]) \\ &= p \cdot \max[cS - \max[cS_{old}(1 + h\Delta\tau), cH], 0] \\ &= P(c \cdot S, c \cdot S_{old}, c \cdot H, \tau^k),\end{aligned}$$

for a constant c , and so the incentive fee function P is homogeneous of degree 1 in variables (S, x_1, x_2) .

It is clear that the contract payoff or terminal condition given by (4.11), is homogeneous of degree 1 in variables (S, S_{old}, H) :

$$\begin{aligned}c \cdot \mathcal{V}(S, S_{old}, H, \tau = 0) &= c \cdot P(S, S_{old}, H, \tau = 0) \\ &= P(c \cdot S, c \cdot S_{old}, c \cdot H, \tau = 0) \\ &= \mathcal{V}(c \cdot S, c \cdot S_{old}, c \cdot H, \tau = 0).\end{aligned}$$

Similarly, the jump conditions, given by (4.12), are also homogeneous of degree 1 in variables (S, S_{old}, H) . Therefore, the option value, $\mathcal{V}(S, S_{old}, H, \tau)$, is homogeneous of degree 1 in variables (S, S_{old}, H) , and we can write

$$\mathcal{V}(c \cdot S, c \cdot S_{old}, c \cdot H, \tau) = c \cdot \mathcal{V}(S, S_{old}, H, \tau), \quad (4.23)$$

for $0 \leq \tau \leq T$.

There are several ways to perform the similarity reduction, corresponding to different choices of c . We consider the choices of $c = \frac{H^*}{H}$ and $c = \frac{S_{old}^*}{S_{old}}$, for a constant H^* and S_{old}^* , respectively. A suitable choice of c should only require interpolation information within a bounded computational domain.

For ease of exposition, let us define: $S^+ = S - \mathcal{P}$, $S_{old}^+ = S - \mathcal{P} = S^+$, $H^+ = \max(S - \mathcal{P}, H) = \max(S^+, H)$, where \mathcal{P} is the incentive fee payment as defined in (4.13).

Recall that to apply the jump conditions (4.12) at a time τ^k , interpolation is required at the point $\mathcal{V}(S^+, S_{old}^+, H^+, \tau^k)$. Choosing $c = \frac{H^*}{H}$ and using equation (4.23) with the definitions of S_{old}^+ and H^+ , we can write

$$\begin{aligned} \frac{H^*}{H^+} \cdot \mathcal{V}(S^+, S_{old}^+, H^+, \tau^k) &= \mathcal{V}\left(\frac{H^*}{H^+} \cdot S^+, \frac{H^*}{H^+} \cdot S_{old}^+, \frac{H^*}{H^+} \cdot H^+, \tau^k\right) \\ &= \mathcal{V}\left(\frac{S^+}{\max(S^+, H)} \cdot H^*, \frac{S^+}{\max(S^+, H)} \cdot H^*, H^*, \tau^k\right). \end{aligned} \quad (4.24)$$

Now, we have that

$$\frac{S^+}{\max(S^+, H)} \leq 1 \implies \frac{S^+}{\max(S^+, H)} \cdot H^* \leq H^*.$$

So interpolated information in the S and S_{old} direction is only needed for

$$\begin{aligned} 0 &\leq S \leq H^* \\ 0 &\leq S_{old} \leq H^*, \end{aligned}$$

for a constant H^* . This allows the interpolation to occur within a bounded domain, which is computationally convenient. As shown in [33], a choice of $c = \frac{S_{old}^*}{S_{old}}$, results in interpolation outside the computational domain, and thus is not considered in this essay.

4.4 Numerical Approach

Using the similarity reduction in Section 4.3, we only need to specify a two-dimensional mesh in the S and S_{old} directions. We discretize the S domain by defining nodes in the S direction denoted as $[S_0, S_1, \dots, S_{i_{max}}]$, which are equally spaced in $\log(S)$. A log-transformation in the PIDE (4.14) simplifies to a PIDE having constant coefficients and a cross-correlation integral. We discretize the S_{old} domain by defining nodes in the S_{old} direction denoted as $[S_{old_0}, S_{old_1}, \dots, S_{old_{j_{max}}}]$. Additionally, we discretize the time domain according to the specified hedge fund valuation dates. We define discrete valuation dates denoted as τ^k , $k = 0, \dots, K - 1$, with $\tau^0 = 0$ and $\tau^K = T$. Recall that $\tau = T - t$ is time running backwards.

The PIDE (4.14) has no dependence on the old asset level S_{old} or on the high-water mark H , due to the fact that S_{old} and H are only updated on discrete valuation dates and do not change between valuation dates. This reduces the pricing problem to solving sets of decoupled problems, or in other words, sets of independent one-dimensional problems. With

the similarity reduction in Section 4.3, we fix a constant H^* and reduce the pricing problem to two dimensions. We solve a set of decoupled PIDEs in S , one for each $S_{old_j}, j = 0, \dots, j_{max}$. To solve the one-dimensional PIDE (4.14) between withdrawal dates, we use a Fourier Space Time-stepping (FST) approach. This method, as in Jackson et al. (2007), is derived with a jump diffusion stochastic process for the underlying asset in Chapter 2. With the FST method, changing the stochastic process of the underlying asset requires a minor change in the time-stepping algorithm. We change the characteristic exponent to the one derived from the hedge fund stochastic process (4.1). We use the characteristic exponent $\Psi^{HF}(k)$ in the FST implementation, with

$$\Psi^{HF}(k) = \left(-\frac{\sigma^2}{2}(2\pi k)^2 + \left(r - m_{total} - \lambda\kappa - \frac{\sigma^2}{2} \right) (2\pi i k) - (r + \lambda) + \lambda \bar{F}(k) \right). \quad (4.25)$$

Additionally, the FST approach has the advantage that we do not need to discretize the time domain between valuation dates in order to complete the time-stepping. Specifically, with valuation dates $\tau^k, k = 0, \dots, K - 1$, we can time-step between valuation dates $[\tau^{k+}, \tau^{k+1}]$ in only one time-step.

Note that, when solving the PIDE using the FST method, the concept of *zero padding* is applied to eliminate wrap-around error. Wrap-around error and the use of zero padding for elimination of the error is discussed in Section D in the Appendix.

On valuation dates, we apply jump conditions according to equations (4.12). We fix a $S_l, l = 0, \dots, l_{max}$ and $S_{old_j}, j = 0, \dots, j_{max}$ and calculate the incentive payment $P(S, S_{old}, H^*, \tau^k)$. Subsequently, we apply the similarity reduction equation (4.24) and jump conditions (4.12).

4.5 No-Arbitrage Value

This section gives an intuitive interpretation of the no-arbitrage value of the manager's option computed in this framework. As a preliminary, recall that in the Black-Scholes framework, the no-arbitrage value of an option is found as the initial cost of a self-financing replicating portfolio of the claim. An investor is indifferent between buying the option and investing in the replicating portfolio. Of course, this depends on the assumptions made in Black and Scholes (1973), including frictionless markets and the ability to buy and sell the underlying asset limitlessly.

Similarly, with the current pricing problem, we compute the no-arbitrage value of the manager's option. In theory, this is the initial cost of a self-financing replicating portfolio of the manager's contingent claim. The manager can devise a replicating strategy to hedge this claim. However, the ability to devise a replicating portfolio depends on the extent to which the hedge fund's investment strategy can be replicated by the manager. These considerations include, but are not limited to, whether the assets are publicly traded and liquid, the ability to take both long and short exposure in the assets, and the strategy's overall complexity in terms of duration and frequency of trades.

For illustrative purposes, we assume that the hedge fund's strategy can be replicated by the manager. We also assume that the manager has complete knowledge of the hedge fund's

investment strategy in order to replicate it. In this way, the no-arbitrage value computed is the amount at which the manager can realize regardless of the performance of the hedge fund, through a replicating strategy. The option is given, for free, by the investor to the manager for compensation. In this way, the no-arbitrage value can be seen as a direct transfer of wealth from the investor to the hedge fund manager at the initiation of the hedge fund contract.

4.6 Manager Behaviour

The two main reasons for incentive fees in the hedge fund industry is (1) to compensate fund managers for their time, effort, skills and expertise and (2) to align manager and investor interests. That is, to encourage managers to generate superior returns without taking excessive risk. Fund managers and investors have different interests corresponding to different payoffs from the returns to the fund's assets. In addition to incentive fees, manager co-investing can be used to align interests and to provide "skin-in-the-game". However, this may lead to extreme conservatism on behalf of the manager as both total wealth and total income become highly dependent on the performance of the fund [7].

Incentive fees may cause a perverse incentive in that it may motivate the manager to act in a way opposite to the interests of the investors providing the incentive. The formulation of manager incentive fees as an option or contingent claim results two important conclusions. First, all else equal, increasing the volatility of the fund's assets will increase the value of the manager's incentive fee option significantly, as shown in Table 4.6. A manager will prefer higher volatility to increase the value of the option, whereas the investor generally does not. It is in itself ironic that the investor is the one that provides this incentive to increase volatility. Additionally, the higher the performance rate, p , the larger the incentive to increase the volatility of the fund's assets. Secondly, the manager has an interesting dilemma when the option is out-of-the-money, which corresponds to the fund's assets being less than the current high-water mark. The high-water mark can be viewed as the strike price of the incentive fee option. To increase the value of the option when the option is out-of-the-money, the manager can either (1) increase the volatility of fund assets or (2) pursue repricing of the option [7]. Specifically, repricing of the option corresponds to either resetting the high-water mark (very unlikely to happen) or closing the fund. Closing the fund and opening a new hedge fund effectively reprices the manager's option as the new high-water mark will equal the new fund's assets, or in other words, the option will be at-the-money.

However, it should be noted that although increasing volatility can increase the value of the manager's option, it also increases the probability of negative fund performance, which can be hurtful to manager welfare for several reasons, as outlined in [7]. Firstly, the manager may also have an investment in the fund. Secondly, it increases the probability that the fund's assets will drop below the high-water mark or further below. Thirdly, negative performance may cause investors to redeem capital and reduce the management fees earned by the manager. Lastly, negative performance can hurt the reputation of the manager and can deter investors from committing new capital to the fund.

4.7 Numerical Results

4.7.1 Base Case

The numerical analysis of hedge fund fees begins with a base case example. The base case hedge fund contract details are given in Table 4.1. Parameters regarding the hedge fund assets are given in Table 4.2 with jump diffusion parameters given in Table 4.3.

	Parameter	Value
T	Expiry time	5 years
$\Delta\tau$	Valuation interval	1 year
p	Performance rate	0.2
h	Hurdle rate	0.05
m_{total}	Management rate	0.01

Table 4.1: Hedge fund base case contract example.

Parameter	Value
S_0	100
σ	0.25
r	0.05

Table 4.2: Parameters for base case hedge fund assets.

Parameter	Value
λ	0.1
μ	-0.9
γ	0.45

Table 4.3: Hedge fund base case Merton jump diffusion parameters.

The results from the base case pricing example are given in Table 4.4. The value of the manager's option in the base case is approximately \$9.01. Thus, the investor parts with approximately 9.01% of her wealth when investing in a hedge fund with these assumed characteristics. Recall that the investor provides the manager with this option, for free, in exchange for the skill and expertise of the manager to generate superior returns. As detailed in Section 4.5, the hedge fund manager can realize this value through a replicating strategy, regardless of the performance of the fund. Hence, there is a 9.01% transfer of wealth between the investor and the manager, at the initiation of the hedge fund contract.

Note that the ratios in Table 4.4 demonstrate non-smooth convergence to the solution. This is due to linear interpolation in the jump conditions between mesh nodes, caused by

a non-zero hurdle rate, h . Table 4.5 shows the convergence results with the base case, augmented with $h = 0$. It is clear that in this example, smooth convergence to the solution is exhibited. The order of convergence is 1 in space. The reduction in the order of convergence in comparison to previous examples is due to the linear interpolation in the jump conditions. Using other interpolation methods to achieve second order convergence in this pricing example is an area of further research.

Refinement	S Nodes	S_{old} Nodes	Price	Change	Ratio
0	128	128	9.13671296		
1	256	256	9.19849437	0.06178141	
2	512	512	9.00741584	0.19107852	0.32
3	1024	1024	9.00522475	0.00219109	87.21
4	2048	2048	9.00396506	0.00125969	1.74
5	4096	4096	9.00696846	0.00300340	0.42

Table 4.4: Numerical convergence in the hedge fund base case contract example with parameters in Table 4.1. Reference price of 9.010072595 given in [33]. Non-smooth convergence caused by the non-zero hurdle rate h .

Refinement	S Nodes	S_{old} Nodes	Price	Change	Ratio
0	128	128	9.48905810		
1	256	256	9.92282004	0.43376195	
2	512	512	10.19008776	0.26726771	1.62
3	1024	1024	10.33504247	0.14495471	1.84
4	2048	2048	10.41037269	0.07533022	1.92
5	4096	4096	10.44871495	0.03834226	1.96

Table 4.5: Numerical convergence in the base case example, augmented with $h = 0$. Setting the hurdle rate $h = 0$ causes smooth convergence to the solution. The order of convergence is 1 in space.

4.7.2 Volatility and Jumps

The effect on the manager's option due to changing the volatility and occurrence of jumps is given in Table 4.6. A standard assumption in modeling equities with jump diffusion models is that $\lambda = 0.1$, which corresponds to expecting a jump or sudden market crash once in ten years. However, when modeling hedge funds, which are able to employ high degrees of leverage and invest in risky derivative strategies and structured products, it may be more appropriate to assume that jumps occur more often. A value of $\lambda = 0.2$ corresponds to expecting a jump in the asset price once in five years. With this value of λ and a volatility of $\sigma = 0.25$, with other parameters as given in the base case, the value of the manager's option is approximately \$10.81. This is approximately a 20% increase over the base case

option value. It is clear that increasing the occurrence of jumps to a more appropriate level for hedge funds, the value of the manager’s option increases significantly.

		λ				
		0.00	0.05	0.10	0.15	0.20
σ	0.10	3.06	4.23	5.47	6.72	7.97
	0.15	4.45	5.53	6.64	7.75	8.83
	0.20	5.82	6.81	7.81	8.81	9.79
	0.25	7.18	8.09	9.01	9.92	10.81
	0.30	8.55	9.39	10.23	11.06	11.88

Table 4.6: The effect of changing the volatility and jump occurrence on hedge fund fee values.

4.7.3 Compensation Structure

The compensation structure of the hedge fund is composed of the performance rate, p , and the management fee, m_{total} . The effect on the manager’s option due to changing the compensation structure is given in Table 4.7. The typical compensation structure used in the hedge fund industry is 2/20 or “2 and 20”. This corresponds to a 2% management fee and a 20% performance rate. With this specific compensation structure, the value of the manager’s option is \$8.31. Recall that the option values computed are strictly the value of the option generated by incentive fees, and not the total compensation to the hedge fund manager. In this way, it is seen in Table 4.7 that increasing m_{total} decreases the value of the manager’s option, even though it increases the total compensation to the manager. This is because the management fee is deducted from the NAV before the incentive fee is calculated. Therefore, a higher management fee corresponds to a smaller incentive fee paid to the manager, which decreases the value of the manager’s option due to the performance-based fees.

		p				
		0.0%	10.0%	20.0%	30.0%	40.0%
m_{total}	0.0%	0.00	4.99	9.75	14.29	18.61
	1.0%	0.00	4.61	9.01	13.22	17.24
	2.0%	0.00	4.24	8.31	12.21	15.94
	3.0%	0.00	3.91	7.66	11.26	14.72
	4.0%	0.00	3.59	7.05	10.37	13.58

Table 4.7: The effect of changing the compensation structure (performance rate and management fee) on hedge fund fee values.

4.7.4 Rates

The effect on the manager's option due to changing the risk-free rate and hurdle rate is given in Table 4.8. As we would expect, there is an inverse relationship between the value of the manager's option and the hurdle rate h , as incentive fees are earned on returns in excess of the hurdle rate.

		r				
		1.0%	2.0%	3.0%	4.0%	5.0%
h	0.0%	8.45	8.94	9.43	9.94	10.45
	2.5%	7.83	8.29	8.77	9.25	9.74
	5.0%	7.21	7.64	8.09	8.54	9.01
	7.5%	6.61	7.01	7.43	7.86	8.30
	10.0%	6.03	6.41	6.80	7.21	7.62

Table 4.8: The effect of changing rates (risk-free rate and hurdle rate) on hedge fund fee values.

4.7.5 Another Example

A last example shows the value of the manager's option with the risk-free rate $r = 1\%$, a hurdle rate of $h = 0\%$, $\lambda = 0.2$, and a $2/20$ compensation structure, and other parameters as in the base case. This is a reasonable example for a typical hedge fund in current market conditions. Table 4.9 shows that the no-arbitrage value of the manager's option in this case is \$9.59.

Refinement	S Nodes	S_{old} Nodes	Price	Change	Ratio
0	128	128	8.73835683		
1	256	256	9.12016754	0.38181070	
2	512	512	9.35775321	0.23758567	1.61
3	1024	1024	9.48713410	0.12938089	1.84
4	2048	2048	9.55449093	0.06735682	1.92
5	4096	4096	9.58880432	0.03431339	1.96

Table 4.9: Base Case with $r = 0.01$, $h = 0$, $m_{total} = 0.02$, $\lambda = 0.2$.

Chapter 5

Conclusions

With a jump diffusion stochastic process as the underlying asset price model, the pricing of contingent claims leads to solving a PIDE. The Fourier Space Time-stepping (FST) method is studied for solving pricing PIDEs arising from jump diffusion processes. An in-depth derivation of the FST method to price European options under a jump diffusion stochastic process is presented. The FST method uses properties of the Fourier transform to convert the PIDE into a linear first-order differential equation. An advantage of using the FST method over finite difference methods is that time-stepping can be completed in one time-step between discrete monitoring dates where conditions are applied. Furthermore, the FST method is applicable to all Lévy processes, and the algorithm can be modified easily when considering different Lévy processes as the underlying asset price model. The method is easily extended to price multi-asset European options. Single- and multi-asset European option pricing examples exhibit 2nd-order convergence to the solution. In the presented framework, a variety of options can be valued, such as European, Bermudan, American, as well as some types of Asian options and path-dependent contingent claims. The method presented can be extended to incorporate regime-switching. An extension can be made which allows the method to value options under mean-reverting asset price models.

A popular insurance product, the Guaranteed Minimum Withdrawal Benefit (GMWB) variable annuity, can be valued using standard financial option pricing methods as life insurance aspects are not included in the product. A mathematical formulation for valuing GMWB variable annuity contracts is presented in this essay. The worst-case hedging cost is computed from the insurance company's point of view, and thus specific policyholder behaviour is assumed on discrete withdrawal dates. Quantitatively, the variable annuity value is found by solving sets of decoupled PIDEs and applying optimality conditions. The yearly insurance guarantee fee charged on the policyholder by the insurance company is found using a Newton iteration. In a standard contract, the insurance guarantee fee is found to be 420 basis points. The GMWB variable annuities are underpriced in the current marketplace. Prices observed in the market can only be obtained by assuming unrealistic modeling parameters or by assuming sub-optimal policyholder behaviour.

A mathematical formulation has been presented to value a hedge fund manager's incentive fee contingent claim inherent in the traditional hedge fund compensation structure. The

pricing problem is three-dimensional, with inclusion of high-water marks and hurdle rates. However, a similarity reduction can reduce the problem to two dimensions. The numerical approach is similar to that of a path-dependent option with state variables changing on discrete monitoring dates. Quantitatively, the no-arbitrage value of the manager's option is found by solving sets of decoupled PIDEs and applying jump conditions. The incentive fee option is provided by the investor, for free, to the manager for compensation. The no-arbitrage value represents the direct transfer of wealth from the investor to the hedge fund manager. The manager is able to construct a replicating strategy to hedge the contingent claim and realize the value of the option at initiation of the hedge fund contract. In reasonable market conditions, the value of the manager's option is found to be \$9.59 from a \$100 investment from the investor.

As discussed in the essay, further research can include:

- Extending the FST method to value options under stochastic volatility and local volatility asset price models
- Extending the FST method to price all types of Asian options under jump models
- Quantitatively analyzing the first-loss hedge fund compensation scheme under a jump diffusion model with comparison to the traditional compensation structure

APPENDICES

Appendix A

Definitions and Properties

Definition 2. The Fourier transform of a function $f(s)$ is defined as:

$$\mathcal{F}[f(s)](k) = \int_{-\infty}^{\infty} f(s)e^{-i2\pi ks} ds.$$

Definition 3. The discrete Fourier transform of $f(x)$ is defined as

$$DFT[f(x)](m) = \sum_{n=0}^{N-1} f(x_n)e^{-2\pi i(\frac{m}{N})n}.$$

Definition 4. The convolution of functions f and g , denoted by $[f \otimes g](x)$, is defined as

$$[f \otimes g](x) = \int_{-\infty}^{\infty} f(y)g(x-y) dy.$$

Definition 5. The cross-correlation of functions f and g , denoted by $[f \star g](x)$, is defined as

$$[f \star g](x) = \int_{-\infty}^{\infty} \overline{f(y)}g(x+y) dy,$$

where \bar{z} denotes the complex conjugate of z .

Property 1. Convolution: The Fourier transform of $[f \otimes g](x)$ is the product of the Fourier transforms of f and g

$$\begin{aligned} \mathcal{F}[[f \otimes g](x)](k) &= \int_{-\infty}^{\infty} [f \otimes g](x)e^{-i2\pi kx} dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y)g(x-y) dy \right) e^{-i2\pi kx} dx \\ &= \int_{-\infty}^{\infty} f(y) \left(\int_{-\infty}^{\infty} g(x-y)e^{-i2\pi kx} dx \right) dy. \end{aligned}$$

Let $u = x - y$, so that $x = u + y$, and

$$\begin{aligned}
\mathcal{F}[[f \otimes g](x)](k) &= \int_{-\infty}^{\infty} f(y) \left(\int_{-\infty}^{\infty} g(u) e^{-i2\pi k(u+y)} du \right) dy \\
&= \int_{-\infty}^{\infty} f(y) \left(\int_{-\infty}^{\infty} g(u) e^{-i2\pi ku} e^{-i2\pi ky} du \right) dy \\
&= \int_{-\infty}^{\infty} f(y) e^{-i2\pi ky} dy \left(\int_{-\infty}^{\infty} g(u) e^{-i2\pi ku} du \right) \\
&= \mathcal{F}[f(x)](k) \cdot \mathcal{F}[g(x)](k).
\end{aligned} \tag{A.1}$$

Property 2. Cross-correlation: The cross-correlation of two functions f and g written as a convolution of two functions is

$$[f \star g](x) = \int_{-\infty}^{\infty} \overline{f(y)} g(x + y) dy$$

Let $u = -y$, $du = -dy$, and note that as $y \rightarrow -\infty, u \rightarrow \infty$, so

$$\begin{aligned}
[f \star g](x) &= \int_{\infty}^{-\infty} \overline{f(-u)} g(x - u) (-du) \\
&= \int_{-\infty}^{\infty} \overline{f(-u)} g(x - u) du \\
&= \overline{f(-x)} \otimes g(x).
\end{aligned}$$

The Fourier transform of the cross-correlation of two functions f and g is

$$\begin{aligned}
\mathcal{F}[[f \star g](x)](k) &= \mathcal{F}[\overline{f(-x)} \otimes g(x)](k) \\
&= \mathcal{F}[\overline{f(-x)}](k) \cdot \mathcal{F}[g(x)](k).
\end{aligned} \tag{A.2}$$

In the case that the function f is real, we can use the fact that $\overline{f(x)} = f(x)$ and that $\mathcal{F}[f(-x)](k) = \overline{\mathcal{F}[f(x)](k)}$ to simplify equation (A.2) to

$$\mathcal{F}[[f \star g](x)](k) = \overline{\mathcal{F}[f(x)](k)} \cdot \mathcal{F}[g(x)](k). \tag{A.3}$$

For ease of notation, we can adopt the convention that $F(k) := \mathcal{F}[f(x)](k)$ and $G(k) := \mathcal{F}[g(x)](k)$ and rewrite (A.3) as

$$\mathcal{F}[[f \star g](x)](k) = \overline{F(k)} \cdot G(k). \tag{A.4}$$

Property 3. Fourier Transform:

a) The Fourier transform of the the partial derivative of a function v with respect to t can be represented as

$$\mathcal{F} \left[\frac{\partial}{\partial t} v(x, t) \right] (k) = \frac{\partial}{\partial t} \mathcal{F} [v(x, t)] (k). \tag{A.5}$$

b) The Fourier transform of the the partial derivative of a function v with respect to x can be represented as

$$\mathcal{F}\left[\frac{\partial^n}{\partial x^n}v(x,t)\right](k) = (2\pi ik)^n \mathcal{F}[v(x,t)](k). \quad (\text{A.6})$$

Appendix B

Fourier Transforms of Distributions

B.1 Fourier Transform of Normal Distribution

Let us find an expression for the Fourier transform of $f(y)$, where $f(y)$ is given by

$$f(y) = \frac{1}{\sqrt{2\pi\gamma}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\gamma}\right)^2}.$$

We define the Fourier transform of $f(y)$ as

$$F(k) := \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\gamma}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\gamma}\right)^2} \cdot e^{-2\pi i k y} dy.$$

Let $y' = y - \mu$, $dy' = dy$, so that

$$\begin{aligned} F(k) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\gamma}} e^{-\frac{1}{2}\left(\frac{y'}{\gamma}\right)^2} \cdot e^{-2\pi i k (y'+\mu)} dy' \\ &= \frac{e^{-2\pi i k \mu}}{\sqrt{2\pi\gamma}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{y'}{\gamma}\right)^2} \cdot e^{-2\pi i k y'} dy'. \end{aligned}$$

By Euler, we know $e^{-i\theta} = \cos \theta - i \sin \theta$, which then gives

$$\begin{aligned} F(k) &= \frac{e^{-2\pi i k \mu}}{\sqrt{2\pi\gamma}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{y'}{\gamma}\right)^2} (\cos(2\pi k y') - i \sin(2\pi k y')) dy' \\ &= \frac{e^{-2\pi i k \mu}}{\sqrt{2\pi\gamma}} \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{y'}{\gamma}\right)^2} \cos(2\pi k y') dy' - i \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{y'}{\gamma}\right)^2} \sin(2\pi k y') dy' \right). \end{aligned}$$

Since \sin is an odd function, $\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{y'}{\gamma}\right)^2} \sin(2\pi k y') dy' = 0$. Simplifying, we get

$$F(k) = \frac{e^{-2\pi i k \mu}}{\sqrt{2\pi\gamma}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{y'}{\gamma}\right)^2} \cos(2\pi k y') dy'. \quad (\text{B.1})$$

We know from [1], that $\int_0^\infty e^{-at^2} \cos(2xt) dt = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-\frac{x^2}{a}}$. We can express (B.1) as

$$F(k) = \frac{e^{-2\pi ik\mu}}{\sqrt{2\pi\gamma}} 2 \cdot \int_0^\infty e^{-ay'^2} \cos(2xy') dy',$$

with $a = \frac{1}{2\gamma^2}$ and $x = \pi k$. Using this, we have

$$\begin{aligned} F(k) &= \frac{e^{-2\pi ik\mu}}{\sqrt{2\pi\gamma}} 2 \left(\frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-\frac{x^2}{a}} \right) \\ &= \frac{e^{-2\pi ik\mu}}{\sqrt{2\gamma}} \sqrt{\frac{1}{a}} e^{-\frac{x^2}{a}} \\ &= \frac{e^{-2\pi ik\mu}}{\sqrt{2\gamma}} \sqrt{2\gamma^2} e^{-(\pi k)^2 2\gamma^2} \\ &= e^{-2\pi ik\mu} e^{-2(\pi k\gamma)^2} \\ &= e^{-2(\pi ik\mu + (\pi k\gamma)^2)}. \end{aligned}$$

Therefore, $F(k) = e^{-2(\pi ik\mu + (\pi k\gamma)^2)}$ and $\overline{F}(k) = e^{2(\pi ik\mu + (\pi k\gamma)^2)}$.

B.2 Fourier Transform of Double Exponential Distribution

Let us find an expression for the Fourier transform of $f(y)$, where $f(y)$ is given by

$$f(y) = p\eta_1 e^{-y\eta_1} \cdot 1_{\{y \geq 0\}} + (1-p)\eta_2 e^{y\eta_2} \cdot 1_{\{y \leq 0\}}.$$

We define the Fourier transform of $f(y)$ as

$$F(k) := \int_{-\infty}^{\infty} (p\eta_1 e^{-y\eta_1} \cdot 1_{\{y \geq 0\}} + (1-p)\eta_2 e^{y\eta_2} \cdot 1_{\{y \leq 0\}}) e^{-2\pi iky} dy.$$

We can simplify this easily as follows

$$\begin{aligned}
F(k) &= \int_{-\infty}^{\infty} p\eta_1 e^{-y\eta_1} \cdot 1_{\{y \geq 0\}} e^{-2\pi iky} dy + \int_{-\infty}^{\infty} (1-p)\eta_2 e^{y\eta_2} \cdot 1_{\{y \leq 0\}} e^{-2\pi iky} dy \\
&= \int_0^{\infty} p\eta_1 e^{-y\eta_1} \cdot e^{-2\pi iky} dy + \int_{-\infty}^0 (1-p)\eta_2 e^{y\eta_2} \cdot e^{-2\pi iky} dy \\
&= p\eta_1 \int_0^{\infty} e^{-y(\eta_1 + 2\pi ik)} dy + (1-p)\eta_2 \int_{-\infty}^0 e^{y(\eta_2 - 2\pi ik)} dy \\
&= p\eta_1 \int_0^{\infty} e^{-y(\eta_1 + 2\pi ik)} dy + (1-p)\eta_2 \int_{-\infty}^0 e^{y(\eta_2 - 2\pi ik)} dy \\
&= p\eta_1 \left[\frac{-1}{\eta_1 + 2\pi ik} e^{-y(\eta_1 + 2\pi ik)} \right]_0^{\infty} + (1-p)\eta_2 \left[\frac{1}{\eta_2 - 2\pi ik} e^{y(\eta_2 - 2\pi ik)} \right]_{-\infty}^0 \\
&= p\eta_1 \left(\frac{1}{\eta_1 + 2\pi ik} \right) + (1-p)\eta_2 \left(\frac{1}{\eta_2 - 2\pi ik} \right) \\
&= \frac{p}{1 + 2\pi ik \left(\frac{1}{\eta_1} \right)} + \frac{1-p}{1 - 2\pi ik \left(\frac{1}{\eta_2} \right)}.
\end{aligned}$$

Therefore, $F(k) = \frac{p}{1 + 2\pi ik \left(\frac{1}{\eta_1} \right)} + \frac{1-p}{1 - 2\pi ik \left(\frac{1}{\eta_2} \right)}$ and $\bar{F}(k) = \frac{p}{1 - 2\pi ik \left(\frac{1}{\eta_1} \right)} + \frac{1-p}{1 + 2\pi ik \left(\frac{1}{\eta_2} \right)}$.

Appendix C

Numerical Results

C.1 Single Asset European Options under Kou Jump Diffusion

Assuming the stock price S follows a jump diffusion stochastic process with Kou jump density, we have a double exponential distribution for the jump size, $y = \log(\eta)$:

$$f(y) = p\eta_1 e^{-y\eta_1} \cdot \mathbb{1}_{\{y \geq 0\}} + (1-p)\eta_2 e^{y\eta_2} \cdot \mathbb{1}_{\{y \leq 0\}},$$

with

$$\kappa := E[\eta - 1] = p \frac{n_1}{n_1 - 1} + (1-p) \frac{n_2}{n_2 + 1} - 1.$$

From the Appendix, we also have a closed-form representation for the Fourier transform of the jump size distribution function f :

$$F(k) := \int_{-\infty}^{\infty} (p\eta_1 e^{-y\eta_1} \cdot \mathbb{1}_{\{y \geq 0\}} + (1-p)\eta_2 e^{y\eta_2} \cdot \mathbb{1}_{\{y \leq 0\}}) e^{-2\pi i k y} dy = \frac{p}{1 + 2\pi i k (\frac{1}{\eta_1})} + \frac{1-p}{1 - 2\pi i k (\frac{1}{\eta_2})}.$$

This gives the following characteristic exponent for the FST implementation

$$\begin{aligned} \Psi(k) &:= -\frac{\sigma^2}{2}(2\pi k)^2 + \left(r - \lambda\kappa - \frac{\sigma^2}{2} \right) (2\pi i k) - (r + \lambda) + \lambda \bar{F}(k) \\ &= -\frac{\sigma^2}{2}(2\pi k)^2 + \left(r - \lambda \left(p \frac{n_1}{n_1 - 1} + (1-p) \frac{n_2}{n_2 + 1} - 1 \right) - \frac{\sigma^2}{2} \right) (2\pi i k) \\ &\quad - (r + \lambda) + \lambda \frac{p}{1 + 2\pi i k (\frac{1}{\eta_1})} + \frac{1-p}{1 - 2\pi i k (\frac{1}{\eta_2})}. \end{aligned}$$

The payoff of a European call option is

$$\mathcal{V}(S, \tau = 0) = \max(S - K, 0).$$

The parameters for the European option are given in Table [C.1](#). The pricing results of the European call option are given in Table [C.2](#).

Parameter	Value
S	100
K	110
r	0
T	1
σ	0.2
λ	0.2
p	0.5
η_1	3
η_2	2

Table C.1: Parameters for European call option under Kou jump diffusion.

Refinement	S Nodes	Price	Change	Ratio
0	512	7.34936037		
1	1024	7.29721584	0.05214453	
2	2048	7.28419741	0.01301843	4.01
3	4096	7.28094329	0.00325412	4.00
4	8192	7.28016823	0.00077506	4.20

Table C.2: Pricing results of the European call option with parameters in Table C.1. Parameters and a reference price of 7.27993383 computed using Fourier quadrature method given in Surkov (2009). The order of convergence is 2 in space.

C.2 Multiple Asset European Options under Merton Jump Diffusion

This section prices a two-dimensional European option under Merton Jump Diffusion. With both assets following jump diffusion processes, the two-dimensional characteristic exponent is

$$\begin{aligned} \Psi(k_1, k_2) := & -\frac{\sigma_1^2}{2}(2\pi k_1)^2 - \frac{\sigma_2^2}{2}(2\pi k_2)^2 + \left(r - \lambda_1 \kappa_1 - \frac{\sigma_1^2}{2}\right)(2\pi i k_1) + \left(r - \lambda_2 \kappa_2 - \frac{\sigma_2^2}{2}\right)(2\pi i k_2) \\ & - (r + \lambda_1 + \lambda_2) + \lambda_1 \overline{F_1}(k_1) + \lambda_2 \overline{F_2}(k_2) - \rho \sigma_1 \sigma_2 (2\pi i k_1)(2\pi i k_2). \end{aligned}$$

With Merton jump diffusion as the jump density for both assets, we assume the jump sizes $y_1 = \log(\eta_1)$ and $y_2 = \log(\eta_2)$ have normal distributions:

$$\begin{aligned} f_1(y_1) &= \frac{1}{\sqrt{2\pi\gamma_1}} e^{-\frac{1}{2}\left(\frac{y_1 - \mu_1}{\gamma_1}\right)^2}, \\ f_2(y_2) &= \frac{1}{\sqrt{2\pi\gamma_2}} e^{-\frac{1}{2}\left(\frac{y_2 - \mu_2}{\gamma_2}\right)^2}, \end{aligned}$$

with

$$\begin{aligned} \kappa_1 &:= E[\eta_1 - 1] = e^{(\mu_1 + \frac{\gamma_1^2}{2})} - 1, \\ \kappa_2 &:= E[\eta_2 - 1] = e^{(\mu_2 + \frac{\gamma_2^2}{2})} - 1. \end{aligned}$$

From the Appendix, we have derived a closed-form representation for the Fourier transforms of the jump size distribution functions f_1 and f_2 , which are given by

$$\begin{aligned} F_1(k_1) &:= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\gamma}} e^{-\frac{1}{2}\left(\frac{y_1 - \mu}{\gamma}\right)^2} \cdot e^{-2\pi i k_1 y_1} dy_1 = e^{-2(\pi i k_1 \mu_1 + (\pi k_1 \gamma_1)^2)}, \\ F_2(k_2) &:= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\gamma}} e^{-\frac{1}{2}\left(\frac{y_2 - \mu}{\gamma}\right)^2} \cdot e^{-2\pi i k_2 y_2} dy_2 = e^{-2(\pi i k_2 \mu_2 + (\pi k_2 \gamma_2)^2)}. \end{aligned}$$

Spread options are multi-asset options that can be viewed as options on the difference of two assets. The payoff of a spread call option at maturity is

$$\mathcal{V}(S_1, S_2, \tau = 0) = \max(B_2 S_2 - B_1 S_1 - K, 0).$$

The parameters of the two securities are listed in Table C.3 and the option contract details are listed in Table C.4. The spread call option values are given in Table C.5.

	Security 1	Security 2
S	100	100
σ	0.1	0.2
λ	0.25	0.5
μ	-0.13	0.11
γ	0.37	0.41

Table C.3: Security Parameters for European Spread Call Option under Merton jump diffusion.

Parameter	Value
B_1	1
B_2	1
K	2
T	1
r	0.1
ρ	0.5

Table C.4: Parameters for European Spread Call Option under Merton jump diffusion.

Refinement	S_1 Nodes	S_2 Nodes	Price	Change	Ratio
0	512	512	13.78687374		
1	1024	1024	13.72948026	0.05739348	
2	2048	2048	13.71793892	0.01154134	4.97
3	4096	4096	13.71452513	0.00341379	3.38
4	8192	8192	13.71378645	0.00073868	4.62

Table C.5: Pricing results of the European spread call option with parameters in Table C.3 and Table C.4. Parameters and a reference price of 13.714948858 computed using Kirk's approximation formula given in Surkov (2009). The order of convergence is approximately 2 in space.

Appendix D

Wrap-Around Error

In the implementation of the Fourier Space Time-stepping method used to solve the PIDE arising from the contingent claims, a fast Fourier transform (FFT) is used. Let us look at the GMWB valuation as an example here, with PIDE given by (3.7). The FFT is used to convert the PIDE into Fourier space in order to complete the time-stepping. Applying the FFT causes wrap-around error in the annuity value solution. The annuity value \mathcal{V} is an aperiodic function numerically represented by a finite and discrete grid. When the FFT is applied, \mathcal{V} is essentially represented by a periodic function with a period equal to the finite grid size. The periodic assumption implicit in the FFT causes annuity values on the ends of the grid to be “wrapped around” to the other side of the grid. Specifically, wrap-around error at $W \gg W_{max}$ produces spurious annuity values at $W \simeq W_{max}$. Similarly, wrap-around error at $W \ll W_{min}$ produces spurious annuity values at $W \simeq W_{min}$.

A simple cure to remedy the wrap-around error is to use *zero padding*. Let the value of the annuity at the node $W_i, i = 0, \dots, i_{max}$, and $A_j, j = 0, \dots, j_{max}$, at time τ^n be denoted as $\mathcal{V}(W_i, A_j, \tau^n) = \mathcal{V}_{i,j}^n$.

Without zero padding, for a constant A_j , we apply a FFT to the vector \mathcal{V}_j^n :

$$\mathcal{V}_j^n = \underbrace{[\mathcal{V}_{0,j}^n, \mathcal{V}_{1,j}^n, \dots, \mathcal{V}_{N-1,j}^n]}_N,$$

where N is the number of nodes in the W direction.

Using zero padding in the implementation amounts to adding nodes to the computational grid in the W direction. These nodes have 0 value. Specifically, for a constant A_j , we construct the vector $\hat{\mathcal{V}}_j^n$:

$$\hat{\mathcal{V}}_j^n = \underbrace{[\mathcal{V}_{0,j}^n, \mathcal{V}_{1,j}^n, \dots, \mathcal{V}_{N-1,j}^n]}_N, \underbrace{[0, 0, \dots, 0]}_N.$$

The FFT is then applied on $\hat{\mathcal{V}}_j^n$, with size $2N$. After completing the time-stepping in Fourier space, an inverse FFT is used and the added zero value nodes are deleted to retrieve the option values at a time τ^{n+1} , represented by \mathcal{V}_j^{n+1} .

The effect of the wrap-around error and the reduction in wrap-around error by use of the zero padding method on a European call is shown in Table D.2. Note that the grid in this case is centered (in the log-asset price domain) at $S = 100$. The option value at the node $S = 100$ without padding is very close to the closed-form solution. Hence, wrap-around error plays an inconsequential role for European options where information is only required at one node (at the center) of the finite grid. This is shown in Table D.2.

Parameter	Value
S	100
K	100
r	0.05
q	0.0
T	1
σ	0.3
λ	0.1
μ	-0.9
γ	0.45

Table D.1: Parameters for Wrap-Around Analysis

S	No padding	Padding	Closed-form solution
0.001	1826.57433595	-0.00000002	0.00000000
0.01	1.47814527	-0.00000001	0.00000000
0.1	0.00078225	0.00000001	0.00000000
1	0.00000029	0.00000000	0.00000000
10	0.00000001	0.00000001	0.00000000
100	16.39941093	16.39941094	16.39939280
1000	904.91306656	904.87793530	904.91306653
10000	9904.87710579	9904.17442654	9904.87710577
100000	99904.87705758	99894.33705407	99904.87705759

Table D.2: Effect of wrap-around error on the value of a European call option. Parameters given in Table D.1. With no zero padding applied, wrap-around error creates spurious option values at the ends of the grid. With zero padding, the wrap-around error is eliminated without a major reduction in accuracy of option values in the middle of the grid, as compared with the closed-form solution. When only pricing a European option with $S = 100$, wrap-around error does not affect the solution.

However, in the contingent claim pricing examples, either jump conditions or optimality conditions are applied on discrete monitoring dates, which causes the annuity value information to be used across the entire grid. We will investigate the GMWB variable annuity valuation here as an example. In the case of pricing GMWBs, due to the optimality conditions applied on withdrawal dates, the annuity value information is used across the

entire grid on withdrawal dates. When zero padding is not applied, the optimality conditions cause the wrap-around error at the ends of the grid to affect the annuity value across the whole grid. For convenience, the optimality conditions (3.6) are rewritten here:

$$\mathcal{V}(W, A, \tau^{k+}) = \sup_{\gamma^k \in [0, A]} [\mathcal{V}(\max(W - \gamma^k, 0), A - \gamma^k, \tau^k) + f(\gamma^k)], \quad \tau = \tau^k, k = 0, \dots, K-1.$$

The annuity value satisfies these optimality conditions at withdrawal time τ^k . When the ends of the grids have spurious annuity values caused by wrap-around error, a spurious γ^k is chosen, which pollutes the annuity values across the entire grid. The use of zero padding to eliminate wrap-around error is essential when pricing GMWBs with FST.

Appendix E

Pseudocode

E.1 European Options under Merton Jump Diffusion

Inputs: $S, K, r, T, \sigma, \lambda, \mu, \gamma$

Set $N = 8192$

Set $x_{max} = 7.5, x_{min} = -7.5$

Discretize the log stock price domain:

Let $\Delta x = \frac{x_{max} - x_{min}}{N-1}$

Let $\mathbf{x} = x_{min}, x_{min} + \Delta x, \dots, x_{max} - \Delta x, x_{max}$

Discretize the frequency domain:

Let $\Delta k = \frac{N-1}{(x_{max} - x_{min}) \cdot N}$

Let $\mathbf{m} = 0, 1, \dots, \frac{N}{2}, -\frac{N}{2} + 1, \dots, -1$

Let $\mathbf{k} = \Delta k \cdot \mathbf{m}$

Characteristic Exponent:

Let $\kappa = \exp(\mu + \frac{\gamma^2}{2}) - 1$

Let $\Psi = \left(-\frac{\sigma^2}{2} (2\pi\mathbf{k})^2 + \left(r - \lambda\kappa - \frac{\sigma^2}{2} \right) (2\pi i\mathbf{k}) - (r + \lambda) + \lambda e^{2(\pi i\mathbf{k}\mu + (\pi\mathbf{k}\gamma)^2)} \right)$

European call option:

Let $\mathbf{callvalue}_{\tau_u} = \max(S \cdot \exp(\mathbf{x}) - K, 0)$

$\mathbf{callvalue}_{\tau_u} = IFFT(FFT(\mathbf{callvalue}_{\tau_u}) \cdot \exp(\Psi \cdot T))$

Interpolate $\mathbf{callvalue}_{\tau_u}(x)$ at $x = 0$ to retrieve the *callprice*.

European put option:

Let $\mathbf{putvalue}_{\tau_u} = \max(K - S \cdot \exp(\mathbf{x}), 0)$

$\mathbf{putvalue}_{\tau_u} = IFFT(FFT(\mathbf{putvalue}_{\tau_u}) \cdot \exp(\Psi \cdot T))$

Linearly interpolate $\mathbf{putvalue}_{\tau_u}(x)$ at $x = 0$ to retrieve the *putprice*.

Return: *callprice, putprice*.

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