Option pricing using TR-BDF2 time stepping method

by

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I hereby declare that I am the sole author of this report. This is a true copy of the report, including any required final revisions, as accepted by my examiners.

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Abstract

The Trapezoidal Rule with second order Backward Difference Formula (TR-BDF2) time stepping method was applied to the Black-Scholes PDE for option pricing. It is proved that TR-BDF2 time stepping method is unconditionally stable, and compared to the usual Crank-Nicolson time stepping method, the TR-BDF2 shows fewer oscillations when computing the derivatives of the solution, which are important hedging parameters.

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Dedication

This is dedicated to the one I love.

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Chapter 1

Introduction

By holding an option, the holder obtains the right but not the obligation to enter into a transaction involving an underlying asset at a predetermined price at a specific date .

The predetermined price is known as the strike price and the specified date is known as the maturity or expiry date of that option. There are different types of options. A call option gives the holder the right to buy an underlying asset while a put option gives the holder the right to sell the asset. European options can only be exercised at maturity whereas American options may be exercised any time by the expiry date of the option.

Regardless of the different types of options, valuation and hedging of this type of financial contracts are always of importance. Different numerical methods can be used to calculate the price of the option. For example, the valuation of different types of options can be modelled as calculating the numerical solutions to corresponding partial differential equations (PDE). By assuming the price of the underlying asset follows a Geometric Brownian Motion, it was shown by Black and Scholes that the valuation of options can be done by solving a second order PDE with time and price of the underlying asset as two independent variables [3]. While the Black-Scholes equation is able to provide a closed form solution for pricing Europeans options, numerical methods are required for the case of American options. The PDEs are discretized and solutions are determined using a discrete set of time steps.

When using numerical methods, it is always possible to have inaccuracies in the solutions, particularly if there are discontinuities in the payoff of the option, or its derivative. As

an example, when using the Crank-Nicolson time stepping method to solve the discretized system, one often encounters spurious oscillations in the Greeks (i.e. the approximate values of the first and second order derivatives of the option prices). Though Rannacher smoothing [13] can be adopted to reduce the oscillations for European options case, it does not work well for American options.

The Trapezoidal Rule with second order Backward Difference Formula (TR-BDF2) can be classified as a fully implicit Runge Kutta method with second order accuracy. It has a wide range of applications in many different areas such as electronics [8], biology [14], mechanical engineering [2] and electrical engineering [1].

Since the TR-BDF2 method is mathematically L-stable (see appendix A for the definition of L-stability and A-stability), it has stronger stability properties than the Crank-Nicolson time stepping method which is only A-stable [12], we will use this method as the time stepping method to derive the solution to the option pricing problems under the Black-Scholes model. In this way, we expect oscillations in Greeks to be damped for both European and American options.

The principle aims of this paper are as follows:

- Use the TR-BDF2 time stepping method to derive the solution to option pricing problems under the Black-Scholes model and thus price European and American options.

- Use Von Neumann stability analysis to analyse the stability properties of the TR-BDF2 time stepping method.

- Compare the results obtained from the TR-BDF2 time stepping method and the Crank-Nicolson time stepping method with Rannacher smoothing in terms of stability of Greeks and rate of convergence.

Chapter 2

Formulation

2.1 Basic background

2.1.1 European options

A European call option is the most basic example of financial derivatives. By holding a European call option, one has the right, but not the obligation to buy an underlying asset at a specific maturity or expiry time T in the future at a specific strike price K. On the other hand, by holding a European put option, instead of buying, one has the right, but not the obligation to sell an underlying asset at the specific expiry day and strike price.

The payoff of a European option can be written mathematically in the following form:

$$\text{Option Payoff} = \begin{cases} \max(K - S, 0), & \text{for put options} \\ \max(S - K, 0), & \text{for call options} \end{cases} \tag{2.1}$$

where S denotes the price of the underlying asset.

2.1.2 American options

The key difference between an American option and a European option is that an American option can be exercised at any time before its maturity or expiry date. When exercising the

American option, the payoff is the same as the European option. However, due to American option's early exercise feature, it is always priced no less than a European option with the same expiry date and strike price, otherwise an arbitrage opportunity is created.

2.2 Black-Scholes model

In the year of 1973, for the purpose of pricing financial derivatives accurately, a partial differential equation was derived by Black and Scholes [3]. This equation is now referred to as the Black-Scholes equation, which is the most fundamental equation of the current mathematical finance studies. The following assumptions should be kept in mind when using the Black-Scholes equation:

- The price of the underlying asset follows geometric Brownian motion with constant drift and volatility.
- The risk-free rate of return is a constant and cash can be borrowed or lent at this rate.
- There are no arbitrage opportunities existing in the market.
- There are no transaction costs when purchasing or selling the underlying assets.
- Short selling is permitted in the market.

2.2.1 European options pricing

A PDE for pricing European options was derived by Black and Scholes:

Consider an underlying asset with price S and assume the price follows the log-normal stochastic process

$$dS = \mu S dt + \sigma S dZ, \tag{2.2}$$

where μ is the drift rate, σ is volatility, and dZ is the increment of a Wiener Process which is defined as:

$$dZ = \phi\sqrt{dt} \tag{2.3}$$

where $\phi \sim \mathcal{N}(0,1)$ follows the standard normal distribution and dt is defined as the increment of time.

Suppose we construct a hedging portfolio which consists of a long position in one option whose value is given by V and a short position in a number of (α shares) underlying asset. Then the value of the portfolio is given by:

$$P = V - \alpha S \tag{2.4}$$

In a small time dt, $P \to P + dP$, we have:

$$dP = dV - (\alpha)dS \tag{2.5}$$

Considering that α actually depends on S, if we take the true differential of P, we obtain:

$$dP = dV - (\alpha)dS - Sd(\alpha). \tag{2.6}$$

Since we are not allowed to peek into the future, so α can not contain any information about the future asset price movements. As a result, Ito's lemma is used here, and we have:

$$dV = (\mu S\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t})dt + \sigma S\frac{\partial V}{\partial S}dZ. \tag{2.7}$$

Substituting equation (2.2) and (2.7) into (2.6), we obtain:

$$dP = \sigma S(\frac{\partial V}{\partial S} - \alpha)dZ + (\mu S\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \alpha\mu S)dt \tag{2.8}$$

If we let $\alpha = \frac{\partial V}{\partial S}$, the risk which arise from the randomness of the price of the underlying asset can be fully hedged. As a result, we can make this portfolio risk-less over the time interval dt and the change of the value of the portfolio is deterministic. Therefore, by holding the portfolio, according to the no-arbitrage principle, a risk-free rate of return should be obtained:

$$dP = rPdt \tag{2.9}$$

where r is the risk-free interest rate. As a result, the following equation is obtained:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{2.10}$$

which is the Black-Scholes equation.

Define $\mathcal{L}$ as:

$$\mathcal{L}V = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV, \tag{2.11}$$

and

τ as:

$$\tau = T - t \tag{2.12}$$

where T is the expiry date, so τ is the time variable running backwards.

Then we can rewrite the Black-Scholes equation as:

$$V_\tau = \mathcal{L}V. \tag{2.13}$$

2.2.2 Boundary conditions

The usual boundary conditions are typically:

$$\text{put} \qquad S \to \infty \quad V \to 0 \tag{2.14}$$

$$\text{call} \qquad S \to \infty \quad V \to S \tag{2.15}$$

and, as $S \to 0$, the Black-Scholes PDE reduces to the ODE:

$$-\frac{\partial V}{\partial \tau} - rV = 0. \tag{2.16}$$

We can simply solve this ODE at $S = 0$.

2.2.3 American options pricing

Since American options have the feature of early exercise, which means the holder of an American option can choose to exercise at any time by the expiry date of the option and receive a payoff:

$$\text{Payoff} = P(S, \tau) \tag{2.17}$$

the American option pricing problem can be viewed as a linear complementarity problems (LCP) [7].

The payoff of an American option can be denoted as:

$$\text{Option Payoff} = \begin{cases} V(S, \tau = 0) = \max(K - S, 0), & \text{for put options} \\ V(S, \tau = 0) = \max(S - K, 0), & \text{for call options} \end{cases} \tag{2.18}$$

where K is the strike price at which the transaction is carried out.

The price of an American option cannot be less than its payoff, otherwise there is an arbitrage opportunity existing in the market. In addition, because the American option may not be exercised at the optimal time by the holder, the value of the portfolio created may not be able to increase at the risk-free rate of return.

With the two constraints above, the linear complementarity problem can be stated as:

$$\min[V_\tau - (\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV), V - P] \geq 0 \tag{2.19}$$

Chapter 3

Discretization

There is no analytical solution to the linear complementarity problem in equation (2.17), so in order to price American option, numerical techniques are required.

3.1 Semi-discretization in time

First, we define the discretization of time as: $(\tau_n)_{n\in\{0,\dots,N\}}$, and set $\Delta\tau = \tau_{n-1} - \tau_n$ where we have τ_0 as the time of option expiry and τ_N is the valuation time.

We will first semi-discretize the V_τ term in equation (2.17). For example, the Crank-Nicolson time stepping method would result in:

$$V^{n+1} = V^n + \frac{\Delta\tau}{2}\big(\mathcal{L}(V^n) + \mathcal{L}(V^{n+1})\big), \tag{3.1}$$

However, this method is not a monotone scheme, hence it may be prone to oscillations when computing the Greeks.

Recently, the TR-BDF2 method has been proposed to alleviate this problem. The TR-BDF2 algorithm uses the following time semi-discretization:

For the TR-BDF2 time stepping method, there are two stages at each time step: the first stage is the Trapezoidal method and the second stage is the Backward Difference Formula method which is applied to the first stage output and initial value at τ_n. So we can

obtain the value at τ_{n+1}.

By using the TR-BDF2 time stepping method to discretize the V_τ term, we have

$$V^{n+1} = \frac{1}{(2-\alpha)}\big(\frac{1}{\alpha}V^* - \frac{(1-\alpha)^2}{\alpha}V^n + (1-\alpha)\Delta\tau\mathcal{L}(V^{n+1})\big), \tag{3.2}$$

where

$$V^* = V^n + \frac{\alpha\Delta\tau}{2}\big(\mathcal{L}(V^n) + \mathcal{L}(V^*)\big) \tag{3.3}$$

and $0 < \alpha < 1$.

If we choose α to be 1, equation (3.3) will become the Crank-Nicolson time stepping method.

Now we define the local truncation error (LTE) as:

$$l_{n+1} = y_{n+1} - y(t_{n+1}) \tag{3.4}$$

assuming the value y_n computed in previous step is equal to the exact solution for y at time $t = t_n$.

If we choose α to be $2 - \sqrt{2}$, then the local truncation error is minimized [1], because the divided-difference estimate of the local truncation error is:

$$LTE^{n+1} = C\Delta\tau_n^3 V^{(3)} \tag{3.5}$$

where

$$C = \frac{-3\alpha^2 + 4\alpha - 2}{12(2-\alpha)}. \tag{3.6}$$

Although the TR-BDF2 method has two stages, it is still a one step method.

3.2 Spatial discretization

Define the discretization of the price of underlying asset by $(S_j)_{j\in\{0,\dots,m\}}$, $h_j = S_j - S_{j-1}$. Using central, forward and backward difference method, we have:

$$\left(\frac{\partial V}{\partial S}\right)_j^n = \frac{V_{j+1}^n - V_{j-1}^n}{h_{j+1} + h_j} \tag{3.7}$$

for central difference,

$$\left(\frac{\partial V}{\partial S}\right)_j^n = \frac{V_{j+1}^n - V_j^n}{h_{j+1}} \tag{3.8}$$

for forward difference, and

$$\left(\frac{\partial V}{\partial S}\right)_j^n = \frac{V_j^n - V_{j-1}^n}{h_j} \tag{3.9}$$

for backward difference. In addition, we have

$$\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n = 2\frac{h_j V_{j+1}^n - (h_{j+1} + h_j)V_j^n + h_{j+1}V_{j-1}^n}{h_j h_{j+1}(h_{j+1} + h_j)}. \tag{3.10}$$

Substituting these discrete approximations into the Trapezoidal stage as well as the BDF2 stage, we can get tridiagonal linear systems for the unknown values V_j^* and V_j^{n+1}.

Define the vectors $V^{n+1} = [V_0^{n+1} V_1^{n+1} \dots V_m^{n+1}]'$, $V^n = [V_0^n V_1^n \dots V_m^n]'$, $V^* = [V_0^* V_1^* \dots V_m^*]'$ and let tridiagonal matrices $\hat{M}$ and $\hat{N}$ be defined so that for row j, we have:

$$[\hat{M}V^n]_j = -\frac{\alpha\Delta\tau a_j}{2}V_{j-1}^n + \frac{\alpha\Delta\tau(a_j + b_j + r)}{2}V_j^n - \frac{\alpha\Delta\tau b_j}{2}V_{j+1}^n, \tag{3.11}$$

and

$$[\hat{N}V^n]_j = -\frac{(1-\alpha)\Delta\tau a_j}{2-\alpha}V_{j-1}^n + \frac{(1-\alpha)\Delta\tau(a_j + b_j + r)}{2-\alpha}V_j^n - \frac{(1-\alpha)\Delta\tau b_j}{2-\alpha}V_{j+1}^n. \tag{3.12}$$

The a_j and b_j are defined as:

$$a_j^{central} = \Big[\frac{\sigma^2 S_j^2}{(S_j - S_{j-1})(S_{j+1} - S_{j-1})} - \frac{rS_j}{S_{j+1} - S_{j-1}}\Big] \tag{3.13}$$

$$b_j^{central} = \Big[\frac{\sigma^2 S_j^2}{(S_{j+1} - S_j)(S_{j+1} - S_{j-1})} + \frac{rS_j}{S_{j+1} - S_{j-1}}\Big] \tag{3.14}$$

in the case of central differencing,

$$a_j^{forward} = \Big[\frac{\sigma^2 S_j^2}{(S_j - S_{j-1})(S_{j+1} - S_{j-1})}\Big] \tag{3.15}$$

$$b_j^{forward} = \Big[\frac{\sigma^2 S_j^2}{(S_{j+1} - S_j)(S_{j+1} - S_{j-1})} + \frac{rS_j}{S_{j+1} - S_j}\Big] \tag{3.16}$$

in the case of forward differencing, or

$$a_j^{backward} = \Big[\frac{\sigma^2 S_j^2}{(S_j - S_{j-1})(S_{j+1} - S_{j-1})} - \frac{rS_j}{S_j - S_{j-1}}\Big] \tag{3.17}$$

$$b_j^{backward} = \Big[\frac{\sigma^2 S_j^2}{(S_{j+1} - S_j)(S_{j+1} - S_{j-1})}\Big] \tag{3.18}$$

in the case of backwards differencing.

It is important to ensure that all a_j and b_j are positive for stability reasons. The following algorithm in Table (3.1) is adopted to decide between central or upstream (forward or backward) discretization at each node.

Thus, for the Trapezoidal stage we have

$$[I + \hat{M}]V^* = [I - \hat{M}]V^n, \tag{3.19}$$

and for the BDF2 stage we have

$$[I + \hat{N}]V^{n+1} = [\frac{1}{\alpha(2-\alpha)}]V^* - [\frac{(1-\alpha)^2}{\alpha(2-\alpha)}]V^n. \tag{3.20}$$

The boundary conditions will be considered in next section.

For j=0,...,m
  If($a_j^{central} \geq 0$ and $b_j^{central} \geq 0$)**then**
    $a_j = a_j^{central}$
    $b_j = b_j^{central}$
  ElseIf($a_j^{forward} \geq 0$ and $b_j^{forward} f \geq 0$)**then**
    $a_j = a_j^{forward}$
    $b_j = b_j^{forward}$
  Else
    $a_j = a_j^{backward}$
    $b_j = b_j^{backward}$
  EndIf
EndFor

Table 3.1: Positive coefficient algorithm

3.3 Penalty method for American option pricing

In order to solve the American option pricing problem, which is also a linear complementarity problem, we can rewrite equation (2.17) as a single equation with a non-linear penalty term $Q(V, P)$, where $P(S, \tau)$ is the payoff of an American option:

$$V_\tau = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + Q(V, P), \tag{3.21}$$

where the penalty term $Q(V, P)$ is defined as:

$$Q(V, P) = \rho \max(P - V, 0). \tag{3.22}$$

Here ρ is chosen to be large enough so that:

$$V_\tau = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \quad \text{if} \quad V > P, \tag{3.23}$$

or

$$V_\tau > \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \quad \text{if} \quad V = P - \epsilon. \tag{3.24}$$

Therefore, $Q(V, P) = \rho\epsilon$, where $\epsilon = Q(V, P)/\rho$, assuming Q is bounded.

For the TR-BDF2 time stepping method, the discrete penalized equations are:

$$\Big[\frac{V_j^* - V_j^n}{\alpha\Delta\tau}\Big] = \frac{1}{2}[(\mathcal{L}V)_j^* + (\mathcal{L}V)_j^n] + q_j^* \tag{3.25}$$

$$\Big[\frac{(2-\alpha)V_j^{n+1} - \frac{1}{\alpha}V_j^* + \frac{(1-\alpha)^2}{\alpha}V_j^n}{(1-\alpha)\Delta\tau}\Big] = (\mathcal{L}V)_j^{n+1} + p_j^{n+1} \tag{3.26}$$

where the penalty term q_j^* and p_j^{n+1} are defined as:

$$q_j^* = \begin{cases} \frac{1}{\alpha\Delta\tau}(P_j - V_j^*)Large, & \text{if } V_j^* < P_j \\ 0, & \text{otherwise} \end{cases} \tag{3.27}$$

and

$$p_j^{n+1} = \begin{cases} \frac{(2-\alpha)}{(1-\alpha)\Delta\tau}(P_j - V_j^{n+1})Large, & \text{if } V_j^{n+1} < P_j \\ 0, & \text{otherwise} \end{cases} \tag{3.28}$$

where $Large$ is a large number and P_j is the payoff at j^{th} node.

To solve those non-linear equations, we define diagonal matrices $\bar{Q}$ and $\bar{P}$ as:

$$\bar{Q}(V^*)_{ij} = \begin{cases} Large, & \text{if } i = j \text{ and } V_j^* < P_j \\ 0, & \text{otherwise} \end{cases} \tag{3.29}$$

and

$$\bar{P}(V^{n+1})_{ij} = \begin{cases} (2-\alpha)Large, & \text{if } i = j \text{ and } V_j^{n+1} < P_j \\ 0, & \text{otherwise} \end{cases} \tag{3.30}$$

Then we are able to rewrite the non-linear equations as:

$$[I + \hat{M} + \bar{Q}(V^*)]V^* = [I - \hat{M}]V^n + [\bar{Q}(V^*)]P, \tag{3.31}$$

and

$$[I + \hat{N} + \bar{P}(V^{n+1})]V^{n+1} = [\frac{1}{\alpha(2-\alpha)}]V^* - [\frac{(1-\alpha)^2}{\alpha(2-\alpha)}]V^n + [\bar{P}(V^{n+1})]P. \tag{3.32}$$

Next, we will use the following algorithm to price American options with variable timesteps.

Let $(V^*)^k$ and $(V^{n+1})^k$ be the k^{th} estimate for V^* and V^{n+1} respectively. Let $(V^*)^0 = V^n$ and $(V^{n+1})^0 = V^n$. The algorithm for pricing an American option with variable timesteps can be stated as:

While $\tau < T$

For $k = 0, ...$until convergence

$[I + \hat{M} + \bar{Q}\big((V^*)^k\big)](V^*)^{k+1} = [I - \hat{M}]V^n + \bar{Q}\big((V^*)^k\big)P$

If $\max_j \frac{|(V_j^*)^{k+1} - (V_j^*)^k|}{\max(1, |(V_j^*)^{k+1}|)} < tol$ **quit**

EndFor

For $l = 0, ...$until convergence

$[I + \hat{N} + \bar{P}\big((V^{n+1})^l\big)](V^{n+1})^{l+1} = [\frac{1}{\alpha(2-\alpha)}]V^* - [\frac{(1-\alpha)^2}{\alpha(2-\alpha)}]V^n + [\bar{P}\big((V^{n+1})^l\big)]P$

If $\max_j \frac{|(V_j^{n+1})^{l+1} - (V_j^{n+1})^l|}{\max(1, |(V_j^{n+1})^{l+1}|)} < tol$ **quit**

EndFor

$\tau = \tau + \Delta\tau$

$\text{MaxRelChange} = \max_j \left[\frac{|V^{n+1} - V^n|}{\max(1, |V^{n+1}|, |V^n|)}\right]$

$\Delta\tau = \left[\frac{\text{dnorm}}{\text{MaxRelChange}}\right]\Delta\tau$

$V^n = V^{n+1}$

EndWhile

Table 3.2: Penalty method for American option pricing

In the algorithm, T is time to maturity, $dnorm$ is timestep size control parameter, tol is tolerance value and $Large$ is defined to be $\frac{1}{tol}$.

3.4 The Crank-Nicolson time stepping method and Rannacher smoothing

As mentioned above, if we choose the value of α to be 1, then the Crank-Nicolson time stepping method can be written as the first stage of the TR-BDF2 time stepping method.[12].

The Crank-Nicolson time stepping is known to be only A-stable [12]. As a result, spurious oscillations in the Greeks can be introduced [9]. Rannacher smoothing, which adds two backward Euler steps before the Crank-Nicolson, is used to smooth off the payoff at maturity and reduce the oscillation problem [13]. We will illustrate that, though Rannacher smoothing is effective for European options, it does not work as well for American options.

Chapter 4

Von Neumann stability analysis of the TR-BDF2 time stepping method: European case

It has been shown that the Crank-Nicolson time stepping method is unconditionally stable [10]. Here, we would also like to study the stability properties of the TR-BDF2 time stepping method. In this chapter, Von Neumann stability analysis is carried out for the TR-BDF2 time stepping method. The coefficients are assumed to be constant and the grid to be equally spaced in $\log S$ coordinates.

By using the change of variable:

$$x = \log S, S = \exp(x) \tag{4.1}$$

we can change the Black-Scholes equation

$$V_\tau = \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV \tag{4.2}$$

into the form of:

$$\overline{V}_\tau = \frac{1}{2}\sigma^2 \overline{V}_{xx} + (r - \frac{1}{2}\sigma^2)\overline{V}_x - r\overline{V} \tag{4.3}$$

where $\overline{V}(x,\tau) = V\big(\exp(x),\tau\big)$.

The trapezoidal stage and second order backward difference stage of the TR-BDF2 time stepping method can be written as:

$$\begin{aligned}\frac{\overline{V}_j^* - \overline{V}_j^n}{\alpha\Delta\tau} &= \frac{1}{2}[\frac{1}{2}\sigma^2(\frac{\overline{V}_{j+1}^* - 2\overline{V}_j^* + \overline{V}_{j-1}^*}{\Delta x^2}) + (r - \frac{1}{2}\sigma^2)\frac{\overline{V}_{j+1}^* - \overline{V}_{j-1}^*}{2\Delta x} - r\overline{V}_j^*] \\ &+\frac{1}{2}[\frac{1}{2}\sigma^2(\frac{\overline{V}_{j+1}^n - 2\overline{V}_j^n + \overline{V}_{j-1}^n}{\Delta x^2}) + (r - \frac{1}{2}\sigma^2)\frac{\overline{V}_{j+1}^n - \overline{V}_{j-1}^n}{2\Delta x} - r\overline{V}_j^n],\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\overline{V}_j^{n+1} = \frac{1}{2-\alpha}\{\frac{1}{\alpha}\overline{V}_j^* - \frac{(1-\alpha)^2}{\alpha}\overline{V}_j^n + (1-\alpha)\Delta\tau[\frac{1}{2}\sigma^2(\frac{\overline{V}_{j+1}^{n+1} - 2\overline{V}_j^{n+1} + \overline{V}_{j-1}^{n+1}}{\Delta x^2}) \\ +(r - \frac{1}{2}\sigma^2)\frac{\overline{V}_{j+1}^{n+1} - \overline{V}_{j-1}^{n+1}}{2\Delta x} - r\overline{V}_j^{n+1}]\}.\end{aligned} \tag{4.5}$$

If we let:

$$\begin{aligned}a &= \frac{1}{2}\sigma^2\frac{1}{\Delta x^2} - (r - \frac{1}{2}\sigma^2)\frac{1}{2\Delta x}, \\ b &= \frac{1}{2}\sigma^2\frac{1}{\Delta x^2} + (r - \frac{1}{2}\sigma^2)\frac{1}{2\Delta x},\end{aligned} \tag{4.6}$$

then the equations above can be rewritten as:

$$\overline{V}_j^*[1 + (a+b+r)\alpha\tfrac{\Delta\tau}{2}] - \alpha\tfrac{\Delta\tau}{2}b\overline{V}_{j+1}^* - \alpha\tfrac{\Delta\tau}{2}a\overline{V}_{j-1}^*$$

$$= \overline{V}_j^n[1 - (a+b+r)\alpha\frac{\Delta\tau}{2}] + \alpha\frac{\Delta\tau}{2}b\overline{V}_{j+1}^n + \alpha\frac{\Delta\tau}{2}a\overline{V}_{j-1}^n, \tag{4.7}$$

and

$$\overline{V}_j^{n+1}[1 + (a+b+r)\tfrac{1-\alpha}{2-\alpha}\Delta\tau] - \tfrac{1-\alpha}{2-\alpha}\Delta\tau b\overline{V}_{j+1}^{n+1} - \tfrac{1-\alpha}{2-\alpha}\Delta\tau a\overline{V}_{j-1}^{n+1}$$

$$= \frac{1}{\alpha(2-\alpha)}\overline{V}_j^* - \frac{(1-\alpha)^2}{\alpha(2-\alpha)}\overline{V}_j^n. \tag{4.8}$$

Let $V^n = [\overline{V}_0^n, \overline{V}_1^n, ..., \overline{V}_M^n]'$ be the discrete solution vector to equation (4.4) and (4.5). Assume the initial solution vector is perturbed by:

$$\widehat{V}^0 = V^0 + E^0 \tag{4.9}$$

where $E^n = [E_0^n, ..., E_M^n]'$ is the perturbation vector. Since $\widehat{V}$ satisfies the equations:

$$\widehat{V}_j^*[1 + (a+b+r)\alpha\tfrac{\Delta\tau}{2}] - \alpha\tfrac{\Delta\tau}{2}b\widehat{V}_{j+1}^* - \alpha\tfrac{\Delta\tau}{2}a\widehat{V}_{j-1}^*$$

$$= \widehat{V}_j^n[1 - (a+b+r)\alpha\frac{\Delta\tau}{2}] + \alpha\frac{\Delta\tau}{2}b\widehat{V}_{j+1}^n + \alpha\frac{\Delta\tau}{2}a\widehat{V}_{j-1}^n, \tag{4.10}$$

and

$$\begin{aligned}\widehat{V}_j^{n+1}[1 + (a+b+r)\frac{1-\alpha}{2-\alpha}\Delta\tau] - \frac{1-\alpha}{2-\alpha}\Delta\tau b\widehat{V}_{j+1}^{n+1} - \frac{1-\alpha}{2-\alpha}\Delta\tau a\widehat{V}_{j-1}^{n+1} \\ = \frac{1}{\alpha(2-\alpha)}\widehat{V}_j^* - \frac{(1-\alpha)^2}{\alpha(2-\alpha)}\widehat{V}_j^n,\end{aligned} \tag{4.11}$$

by subtracting equations (4.10) and (4.7), (4.11) and (4.8) accordingly, we have:

$$E_j^*[1 + (a+b+r)\alpha\tfrac{\Delta\tau}{2}] - \alpha\tfrac{\Delta\tau}{2}bE_{j+1}^* - \alpha\tfrac{\Delta\tau}{2}aE_{j-1}^*$$

$$= E_j^n[1 - (a+b+r)\alpha\frac{\Delta\tau}{2}] + \alpha\frac{\Delta\tau}{2}bE_{j+1}^n + \alpha\frac{\Delta\tau}{2}aE_{j-1}^n, \tag{4.12}$$

and

$E_j^{n+1}[1+(a+b+r)\frac{1-\alpha}{2-\alpha}\Delta\tau] - \frac{1-\alpha}{2-\alpha}\Delta\tau b E_{j+1}^{n+1} - \frac{1-\alpha}{2-\alpha}\Delta\tau a E_{j-1}^{n+1}$

$$= \frac{1}{\alpha(2-\alpha)}E_j^* - \frac{(1-\alpha)^2}{\alpha(2-\alpha)}E_j^n. \tag{4.13}$$

Here, by using Von Neumann stability analysis approach, we want to prove that the initial perturbation is bounded when the number of steps becomes large.

In order to use the Fourier transform method, we assume that the boundary conditions can be replaced by periodic conditions. As a result, the inverse discrete Fourier transform of E_j^n is defined as:

$$E_j^n = \frac{1}{X_N}\sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} (C_k)^n \exp\,(i\frac{2\pi}{N}jk), \tag{4.14}$$

where C_k is the discrete Fourier coefficient of E, i is $\sqrt{-1}$ and the width of the domain is defined as $X_N{=}x_{\frac{N}{2}} - x_{\frac{-N}{2}+1}$. Let $W = \exp\,(\frac{2\pi}{N}i)$ so that we can write E_j^n as:

$$E_j^n = \frac{1}{X_N}\sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} (C_k)^n W^{jk}. \tag{4.15}$$

The inverse discrete Fourier transform of E_j^{n+1} and E_j^* are then given by:

$$E_j^{n+1} = \frac{1}{X_N}\sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} (C_k)^{n+1} W^{jk}, \tag{4.16}$$

and

$$E_j^* = \frac{1}{X_N}\sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} C_k^* W^{jk}. \tag{4.17}$$

Now we substitute (4.15-4.17) into (4.12) and (4.13). For equation (4.12) we get:

$$\begin{aligned}
&\frac{1}{X_N}\sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} C_k^* W^{jk}[1+(a+b+r)\alpha\frac{\Delta\tau}{2}] - \alpha\frac{\Delta\tau}{2}b\frac{1}{X_N}\sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} C_k^* W^{(j+1)k} \\
&\qquad -\alpha\frac{\Delta\tau}{2}a\frac{1}{X_N}\sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} C_k^* W^{(j-1)k} \\
&= \frac{1}{X_N}\sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} (C_k)^n W^{jk}[1-(a+b+r)\alpha\frac{\Delta\tau}{2}] + \alpha\frac{\Delta\tau}{2}b\frac{1}{X_N}\sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} (C_k)^n W^{(j+1)k} \\
&\qquad +\alpha\frac{\Delta\tau}{2}a\frac{1}{X_N}\sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} (C_k)^n W^{(j-1)k},
\end{aligned} \tag{4.18}$$

while for equation (4.13) we have:

$$\begin{aligned}
&\frac{1}{X_N}\sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} (C_k)^{n+1} W^{jk}[1+(a+b+r)\frac{1-\alpha}{2-\alpha}\Delta\tau] - \frac{1-\alpha}{2-\alpha}\Delta\tau b\frac{1}{X_N}\sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} (C_k)^{n+1} W^{(j+1)k} \\
&\qquad -\frac{1-\alpha}{2-\alpha}\Delta\tau a\frac{1}{X_N}\sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} (C_k)^{n+1} W^{(j-1)k} \\
&= \frac{1}{\alpha(2-\alpha)}\frac{1}{X_N}\sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} C_k^* W^{jk} - \frac{(1-\alpha)^2}{\alpha(2-\alpha)}\frac{1}{X_N}\sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} (C_k)^n W^{jk}.
\end{aligned} \tag{4.19}$$

For all Fourier components C_k, we will look at them separately. For equation (4.18), for each k we have:

$$\begin{aligned}
&C_k^* W^{jk}[1+(a+b+r)\alpha\tfrac{\Delta\tau}{2}] - \alpha\tfrac{\Delta\tau}{2}bC_k^* W^{(j+1)k} - \alpha\tfrac{\Delta\tau}{2}aC_k^* W^{(j-1)k} \\
&= (C_k)^n W^{jk}[1-(a+b+r)\alpha\frac{\Delta\tau}{2}] + \alpha\frac{\Delta\tau}{2}b(C_k)^n W^{(j+1)k} + \alpha\frac{\Delta\tau}{2}a(C_k)^n W^{(j-1)k},
\end{aligned} \tag{4.20}$$

and for equation (4.19), we have:

$$(C_k)^{n+1}W^{jk}[1+(a+b+r)\frac{1-\alpha}{2-\alpha}\Delta\tau]-\frac{1-\alpha}{2-\alpha}\Delta\tau b(C_k)^{n+1}W^{(j+1)k}-\frac{1-\alpha}{2-\alpha}\Delta\tau a(C_k)^{n+1}W^{(j-1)k}$$
$$=\frac{1}{\alpha(2-\alpha)}C_k^*W^{jk}-\frac{(1-\alpha)^2}{\alpha(2-\alpha)}(C_k)^nW^{jk}. \tag{4.21}$$

Dividing both equations by $(C_k)^nW^{jk}$, equation (4.20) becomes:

$$\begin{aligned}\frac{C_k^*}{(C_k)^n}[1+(a+b+r)\alpha\frac{\Delta\tau}{2}]-\alpha\frac{\Delta\tau}{2}b\frac{C_k^*}{(C_k)^n}W^k-\alpha\frac{\Delta\tau}{2}a\frac{C_k^*}{(C_k)^n}W^{-k}\\ =[1-(a+b+r)\alpha\frac{\Delta\tau}{2}]+\alpha\frac{\Delta\tau}{2}bW^k+\alpha\frac{\Delta\tau}{2}aW^{-k},\end{aligned} \tag{4.22}$$

and equation (4.21) becomes:

$$\begin{aligned}C_k[1+(a+b+r)\frac{1-\alpha}{2-\alpha}\Delta\tau]-\frac{1-\alpha}{2-\alpha}\Delta\tau bC_kW^k-\frac{1-\alpha}{2-\alpha}\Delta\tau aC_kW^{-k}\\ =\frac{1}{\alpha(2-\alpha)}\frac{C_k^*}{(C_k)^n}-\frac{(1-\alpha)^2}{\alpha(2-\alpha)}.\end{aligned} \tag{4.23}$$

By factoring out $\frac{C_k^*}{(C_k)^n}$ and C_k from the two equations above respectively, equation (4.22) becomes:

$$\frac{C_k^*}{(C_k)^n}=\frac{[1-(a+b+r)\frac{\alpha\Delta\tau}{2}]+\frac{\alpha\Delta\tau}{2}bW^k+\frac{\alpha\Delta\tau}{2}aW^{-k}}{[1+(a+b+r)\frac{\alpha\Delta\tau}{2}]-\frac{\alpha\Delta\tau}{2}bW^k-\frac{\alpha\Delta\tau}{2}aW^{-k}}, \tag{4.24}$$

and equation (4.23) becomes:

$$C_k=\frac{\frac{1}{\alpha(2-\alpha)}\frac{C_k^*}{(C_k)^n}-\frac{(1-\alpha)^2}{\alpha(2-\alpha)}}{[1+(a+b+r)\frac{1-\alpha}{2-\alpha}\Delta\tau]-\frac{1-\alpha}{2-\alpha}\Delta\tau bW^k-\frac{1-\alpha}{2-\alpha}\Delta\tau aW^{-k}}. \tag{4.25}$$

Since

$$a=\frac{1}{2}\sigma^2\frac{1}{\Delta x^2}-(r-\frac{1}{2}\sigma^2)\frac{1}{2\Delta x}, b=\frac{1}{2}\sigma^2\frac{1}{\Delta x^2}+(r-\frac{1}{2}\sigma^2)\frac{1}{2\Delta x}, \tag{4.26}$$

we can simplify the expressions by noting:

$a+b+r=\frac{\sigma^2}{\Delta x^2}+r$, and $bW^k+aW^{-k}=\frac{\sigma^2}{\Delta x^2}\cos(\frac{2\pi}{N}k)+i\frac{1}{\Delta x}(r-\frac{1}{2}\sigma^2)\sin(\frac{2\pi}{N}k)$.

Thus we can write equation (4.24) as:

$$\frac{C_k^*}{(C_k)^n}=\frac{[1-(\frac{\sigma^2}{\Delta x^2}+r)\alpha\frac{\Delta\tau}{2}]+\frac{1}{2}[\frac{\alpha\Delta\tau\sigma^2}{\Delta x^2}\cos(\frac{2\pi}{N})k+i\frac{\alpha\Delta\tau}{\Delta x}(r-\frac{1}{2}\sigma^2)\sin(\frac{2\pi}{N}k)]}{[1+(\frac{\sigma^2}{\Delta x^2}+r)\alpha\frac{\Delta\tau}{2}]-\frac{1}{2}[\frac{\alpha\Delta\tau\sigma^2}{\Delta x^2}\cos(\frac{2\pi}{N}k)+i\frac{\alpha\Delta\tau}{\Delta x}(r-\frac{1}{2}\sigma^2)\sin(\frac{2\pi}{N}k)]}, \tag{4.27}$$

and equation (4.25) as:

$$C_k=\frac{\frac{1}{\alpha(2-\alpha)}\frac{C_k^*}{(C_k)^n}-\frac{(1-\alpha)^2}{\alpha(2-\alpha)}}{[1+(\frac{\sigma^2}{\Delta x^2}+r)\frac{1-\alpha}{2-\alpha}\Delta\tau]-\frac{1-\alpha}{2-\alpha}[\frac{\Delta\tau\sigma^2}{\Delta x^2}\cos(\frac{2\pi}{N}k)+i\frac{\Delta\tau}{\Delta x}(r-\frac{1}{2}\sigma^2)\sin(\frac{2\pi}{N}k)]}. \tag{4.28}$$

Substituting (4.27) into (4.28), we can obtain the expression of C_k:

$$C_k=\frac{\frac{[1-\frac{1+(1-\alpha)^2}{\alpha(2-\alpha)}(\frac{\sigma^2}{\Delta x^2}+r)\alpha\frac{\Delta\tau}{2}]+\frac{1+(1-\alpha)^2}{\alpha(2-\alpha)}\frac{1}{2}[\frac{\alpha\Delta\tau\sigma^2}{\Delta x^2}\cos(\frac{2\pi}{N}k)+i\frac{\alpha\Delta\tau}{\Delta x}(r-\frac{1}{2}\sigma^2)\sin(\frac{2\pi}{N}k)]}{[1+(\frac{\sigma^2}{\Delta x^2}+r)\alpha\frac{\Delta\tau}{2}]-\frac{1}{2}[\frac{\alpha\Delta\tau\sigma^2}{\Delta x^2}\cos(\frac{2\pi}{N}k)+i\frac{\alpha\Delta\tau}{\Delta x}(r-\frac{1}{2}\sigma^2)\sin(\frac{2\pi}{N}k)]}}{[1+(\frac{\sigma^2}{\Delta x^2}+r)\frac{1-\alpha}{2-\alpha}\Delta\tau]-\frac{1-\alpha}{2-\alpha}[\frac{\Delta\tau\sigma^2}{\Delta x^2}\cos(\frac{2\pi}{N}k)+i\frac{\Delta\tau}{\Delta x}(r-\frac{1}{2}\sigma^2)\sin(\frac{2\pi}{N}k)]}. \tag{4.29}$$

The $\|\|_2$ norm of E^n is:

$$\|E^n\|_2=\sum_{j=\frac{-N}{2}+1}^{\frac{N}{2}}E_j^n(E_j^n)^* \tag{4.30}$$

where $(E_j^n)^*$ is the complex conjugate of E_j^n . Since

$$E_j^n = \frac{1}{X_N} \sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} (C_k)^n W^{jk}, \tag{4.31}$$

we have

$$\|E^n\|_2 = \frac{N}{X_N^2} \sum_{k=\frac{-N}{2}+1}^{\frac{N}{2}} (|C_k|^2)^n. \tag{4.32}$$

Thus, in order for $\|E^n\|_2$ to be bounded, each Fourier coefficient $|C_k|^{2n}$ needs to be bounded, when n approaches infinity. Consequently, we need:

$$|C_k| \leq 1. \tag{4.33}$$

In our case, we can find the value of $|C_k|$ from equation (4.29):

$$|C_k|^2 = \frac{\frac{\{1-\frac{1+(1-\alpha)^2}{\alpha(2-\alpha)}[\frac{\alpha\Delta\tau}{2}r+\frac{\alpha\Delta\tau}{2}\frac{\sigma^2}{\Delta x^2}(1-\cos(\frac{2\pi}{N}k))]\}^2+\{\frac{1+(1-\alpha)^2}{\alpha(2-\alpha)}\frac{1}{2}\frac{\alpha\Delta\tau}{\Delta x}(r-\frac{1}{2}\sigma^2)\sin(\frac{2\pi}{N}k)\}^2}{\{1+[\frac{\alpha\Delta\tau}{2}r+\frac{\alpha\Delta\tau}{2}\frac{\sigma^2}{\Delta x^2}(1-\cos(\frac{2\pi}{N}k))]\}^2+\{\frac{1}{2}\frac{\alpha\Delta\tau}{\Delta x}(r-\frac{1}{2}\sigma^2)\sin(\frac{2\pi}{N}k)\}^2}}{\{1+\frac{2(1-\alpha)}{\alpha(2-\alpha)}[\frac{\alpha\Delta\tau}{2}r+\frac{\alpha\Delta\tau}{2}\frac{\sigma^2}{\Delta x^2}(1-\cos(\frac{2\pi}{N}k))]\}^2+\{\frac{2(1-\alpha)}{\alpha(2-\alpha)}\frac{1}{2}\frac{\alpha\Delta\tau}{\Delta x}(r-\frac{1}{2}\sigma^2)\sin(\frac{2\pi}{N}k)\}^2}. \tag{4.34}$$

If we write:

$$M = \frac{1+(1-\alpha)^2}{\alpha(2-\alpha)}, N = \frac{2(1-\alpha)}{\alpha(2-\alpha)}, \tag{4.35}$$

then, since $0 < \alpha < 1$, we have $M > 0$,$N > 0$. Also

$$P = \frac{\alpha\Delta\tau}{2}r + \frac{\alpha\Delta\tau}{2}\frac{\sigma^2}{\Delta x^2}(1-\cos(\frac{2\pi}{N}k)), \tag{4.36}$$

$$Q = \frac{1}{2}\frac{\alpha\Delta\tau}{\Delta x}(r-\frac{1}{2}\sigma^2)\sin(\frac{2\pi}{N}k), \tag{4.37}$$

where $P > 0$. Then $|C_k|^2$ can be written as:

$$|C_k|^2 = \frac{\frac{(1-MP)^2+(MQ)^2}{(1+P)^2+Q^2}}{(1+NP)^2+(NQ)^2}. \tag{4.38}$$

We can further simplify the equation above:

$$|C_k|^2 = \frac{1+M^2P^2+M^2Q^2-2MP}{1+(N^2+1)P^2+(N^2+1)Q^2+R} \tag{4.39}$$

where

$$R = 2(N+1)P+2(N^2+N)P^3+4NP^2+2N^2Q^2P+N^2(P^4+Q^4)+2N^2Q^2P^2+2NPQ^2. \tag{4.40}$$

Notice that $R > 0$. A little manipulation shows that

$$N^2+1 = M^2 \tag{4.41}$$

and so we have:

$$|C_k|^2 = \frac{(1+M^2P^2+M^2Q^2)-2MP}{(1+M^2P^2+M^2Q^2)+R}. \tag{4.42}$$

Since $MP > 0$, we have $|C_k|^2 < 1$ and so $|C_k| < 1$.

As the initial perturbation is bounded as the number of steps increases, we prove that the TR-BDF2 time stepping method is unconditionally stable.

Chapter 5

Numerical Tests

In this chapter, we solve the Black-Scholes equation to price European options and American options. Both the TR-BDF2 time stepping method and the Crank-Nicolson time stepping method are used, so we can compare the two in terms of the stability of Greeks (V_S, V_{SS}) and rate of convergence.

For the American option pricing problem, we formally state the problem as a linear complementarity problem:

$$\min[V_\tau - (\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV), V - P] \geq 0. \tag{5.1}$$

where $P(S, \tau)$ is the payoff condition which as:

$$P(S, \tau = 0) = \max(K - S, 0) \quad \text{for a put} \tag{5.2}$$
$$P(S, \tau = 0) = \max(S - K, 0) \quad \text{for a call} \tag{5.3}$$

with K the strike price.

The boundary conditions at $S \to 0$ is:

$$\min[V_\tau - rV, V - P] \geq 0. \tag{5.4}$$

And boundary conditions for large S is:

$$V \simeq 0, S \to \infty; \quad \text{for a put} \tag{5.5}$$
$$V \simeq S, S \to \infty; \quad \text{for a call} \tag{5.6}$$

We will use the penalty method [7] to solve this linear complementarity problem.

5.1 European option case

In this section, we will use both the TR-BDF2 and the Crank-Nicolson time stepping methods to solve the Black-Scholes equation and thus price European options. In addition, the properties of Greeks stability and rate of convergence of both methods are compared.

5.1.1 Numerical results

With the data assumed in Table 5.1, a convergence study for the option price was carried out. A convergence table, which has a series of non-uniform grids, is used. In addition, variable timestep sizes are used and the initial timestep size on the coarsest grid is $\Delta\tau = T/25$. In the convergence study, on each grid refinement, new grid nodes are added halfway between the original ones. The new grid has twice as many nodes as the previous grid, the timestep control parameter *dnorm* is halved, and the initial timestep size is divided by four.

Variables	Values
σ	0.8
r	0.10
Time to expiry (T)	0.25 years
Strike Price	\$ 100
Initial asset price S^0	\$ 100

Table 5.1: Data for European Put Option

Nodes	Timesteps	Value	Change	Ratio
TR-BDF2				
62	58	14.43089		
123	141	14.44664	0.01575	
245	312	14.45059	0.00395	4.0
489	654	14.45158	0.00099	4.0
977	1339	14.45182	0.00025	4.0
Crank-Nicolson (Rannacher smoothing)				
62	56	14.42767		
123	142	14.44643	0.01876	
245	315	14.45058	0.00414	4.5
489	658	14.45158	0.00010	4.1
977	1343	14.45182	0.00025	4.0

Table 5.2: Value of a European Put. Exact solution:14.45191. Change is the difference in the solution from the coarser grid. Ratio is the ratio of changes on successive grids.

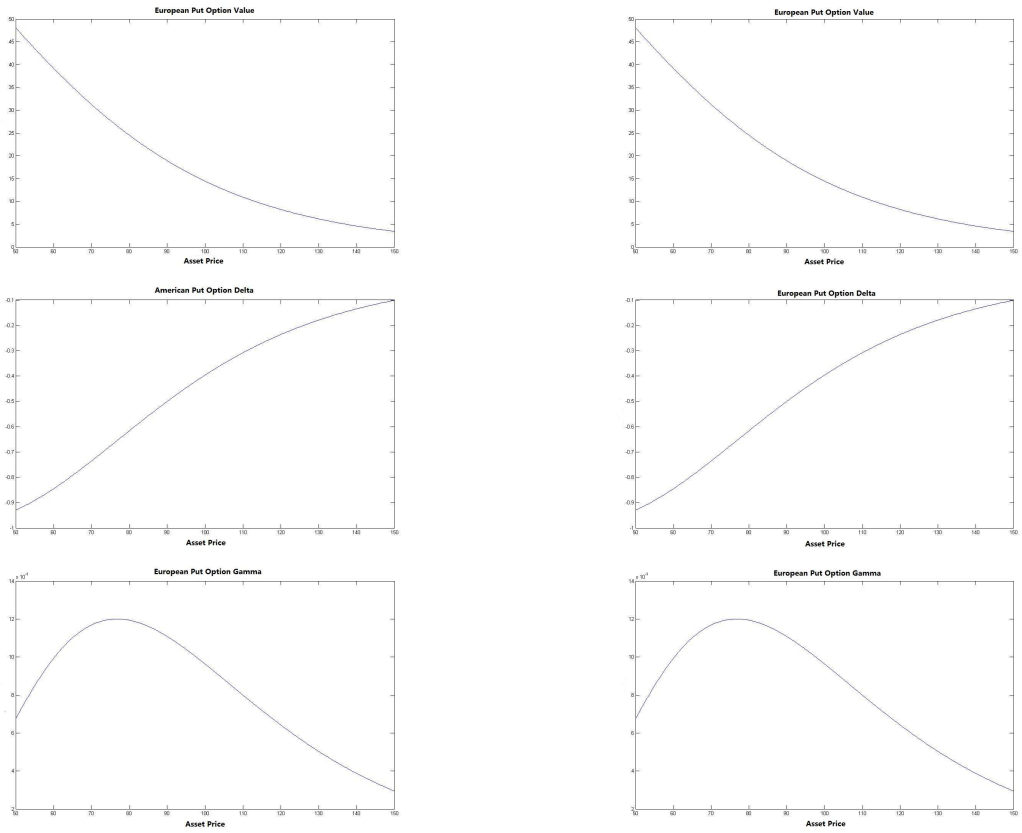

Figure 5.1: Value, delta(V_S), and gamma(V_{SS}) of a European Put, σ=0.8, T=0.25, r=0.1, K=100. Left: TR-BDF2 time stepping method, right: Crank-Nicolson time stepping method with Rannacher smoothing. Top: option value (V), middle: delta (V_S), bottom: gamma(V_{SS}).

5.1.2 Analysis

As can be seen from Table 5.2, both time stepping methods converge. In addition, second order convergence was obtained by both methods. It is known that, the Crank-Nicolson time stepping method is only A-stable [12] and can introduce spurious oscillations in the Greeks [9]. As can be observed in the Figure 5.1, there are no oscillations in delta and gamma for the Crank-Nicolson time stepping method using Rannacher smoothing. On the other hand, no oscillations are observed in delta and gamma for the TR-BDF2 method as well.

5.2 American option case

In this section, we will use both the TR-BDF2 and the Crank-Nicolson time stepping methods to solve American option pricing problem with a penalty method. In addition, the stability of Greeks and rate of convergence of both methods are compared.

A convergence study of pricing different American option examples was carried out in a similar way as the European case. The convergence table, which has a series of non-uniform grids, is used. The tolerance value tol is chosen to be 10^{-6}. In addition, variable timestep sizes are used and the initial timestep size on the coarsest grid is $\Delta\tau = T/25$. In the convergence study, on each grid refinement, the new grid has twice as many nodes as the previous grid, the timestep control parameter $dnorm$ is halved, and the initial timestep size is divided by four. The convergence table, and figures of the Greeks are given in next section.

5.2.1 Numerical results

Variables	Values
σ	0.2
r	0.10
Time to expiry (T)	0.25 years
Strike Price	\$ 100
Initial asset price S^0	\$ 100

Table 5.3: Data for American Put Option

Nodes	Timesteps	Value	Change	Ratio
TR-BDF2				
62	25	3.06321		
123	55	3.06838	0.00517	
245	114	3.06967	0.00129	4.0
489	231	3.07000	0.00032	4.0
977	464	3.07008	0.00008	4.0
Crank-Nicolson (Rannacher smoothing)				
62	26	3.06049		
123	56	3.06794	0.00746	
245	115	3.06960	0.00165	4.5
489	232	3.06999	0.00039	4.3
977	465	3.07008	0.00009	4.2

Table 5.4: Value of an American Put. Change is the difference in the solution from the coarser grid. Ratio is the ratio of changes on successive grids.

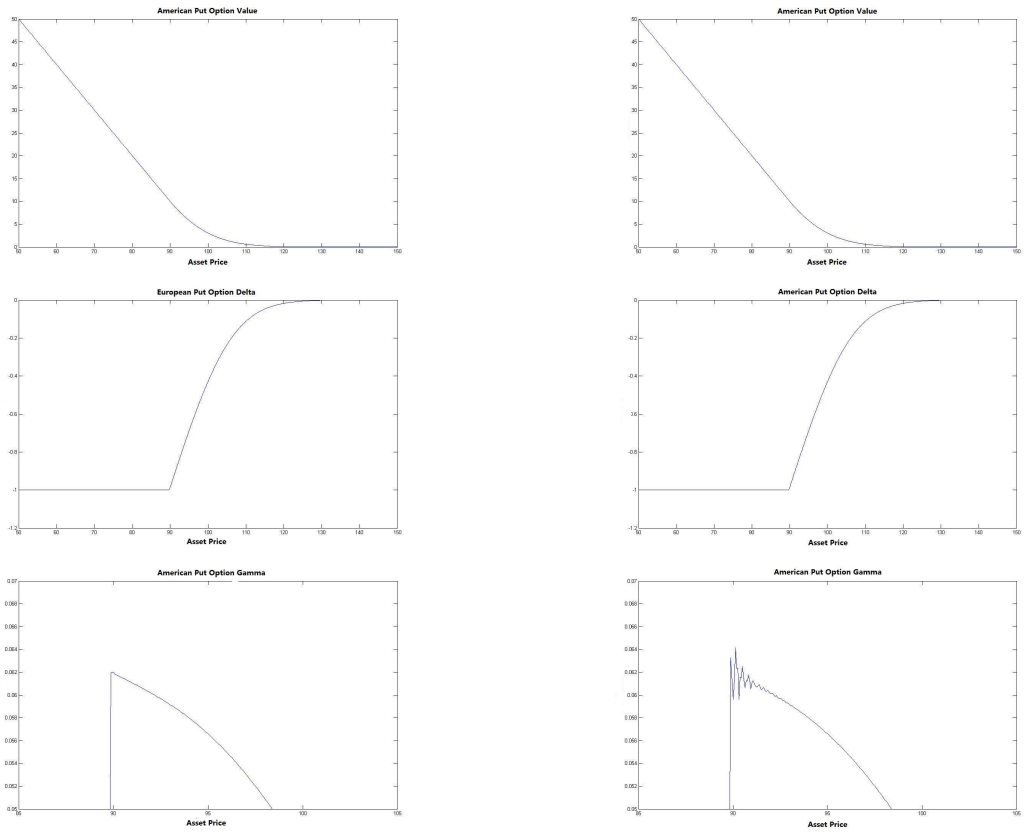

Figure 5.2: Value, delta(V_S), and gamma(V_{SS}) of an American Put, σ=0.2, T=0.25, r=0.1, K=100. Left: TR-BDF2 time stepping method, right: Crank-Nicolson time stepping method with Rannacher smoothing. Top: option value (V), middle: delta (V_S), bottom: gamma(V_{SS}).

Variables	Values
σ	0.3
r	0.15
Time to expiry (T)	0.25 years
Strike Price	\$ 100
Initial asset price S^0	\$ 100

Table 5.5: Data for American Put Option

Nodes	Timesteps	Value	Change	Ratio
TR-BDF2				
62	32	4.57778		
123	73	4.58455	0.00677	
245	154	4.58627	0.00172	3.9
489	315	4.58670	0.00043	4.0
977	638	4.58681	0.00011	4.0
Crank-Nicolson (Rannacher smoothing)				
62	33	4.57365		
123	74	4.58391	0.01026	
245	155	4.58616	0.00226	4.5
489	317	4.58668	0.00052	4.3
977	640	4.58681	0.00012	4.3

Table 5.6: Value of an American Put. Change is the difference in the solution from the coarser grid. Ratio is the ratio of changes on successive grids.

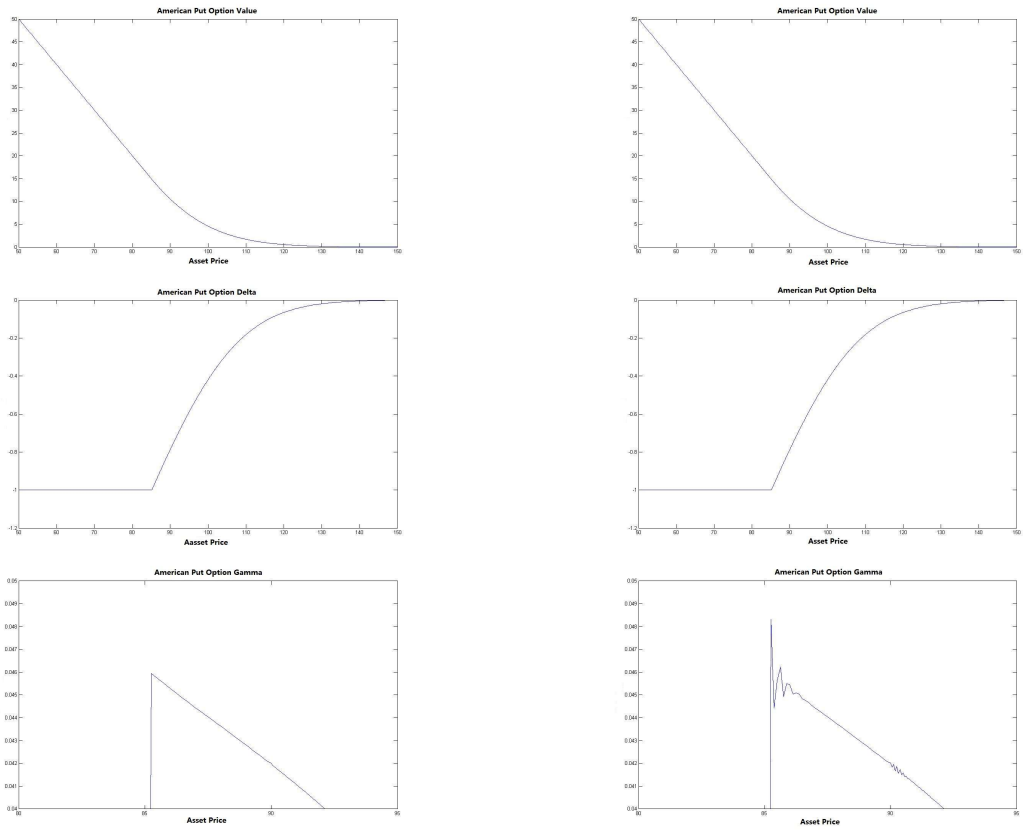

Figure 5.3: Value, delta(V_S), and gamma(V_{SS}) of an American Put, σ=0.3, T=0.25, r=0.15, K=100. Left: TR-BDF2 time stepping method, right: Crank-Nicolson time stepping method with Rannacher smoothing. Top: option value (V), middle: delta (V_S), bottom: gamma(V_{SS}).

Variables	Values
σ	0.4
r	0.03
Time to expiry (T)	5.00 years
Strike Price	\$ 100
Initial asset price S^0	\$ 100

Table 5.7: Data for American Put Option

Nodes	Timesteps	Value	Change	Ratio
TR-BDF2				
62	73	27.70744		
123	192	27.74057	0.03314	
245	442	27.74948	0.00891	3.7
489	953	27.75195	0.00247	3.6
977	1980	27.75256	0.00061	4.1
Crank-Nicolson (Rannacher smoothing)				
62	67	27.68477		
123	186	27.73715	0.05238	
245	443	27.74892	0.01177	4.5
489	961	27.75186	0.00294	4.0
977	1989	27.75255	0.00069	4.3

Table 5.8: Value of an American Put. Change is the difference in the solution from the coarser grid. Ratio is the ratio of changes on successive grids.

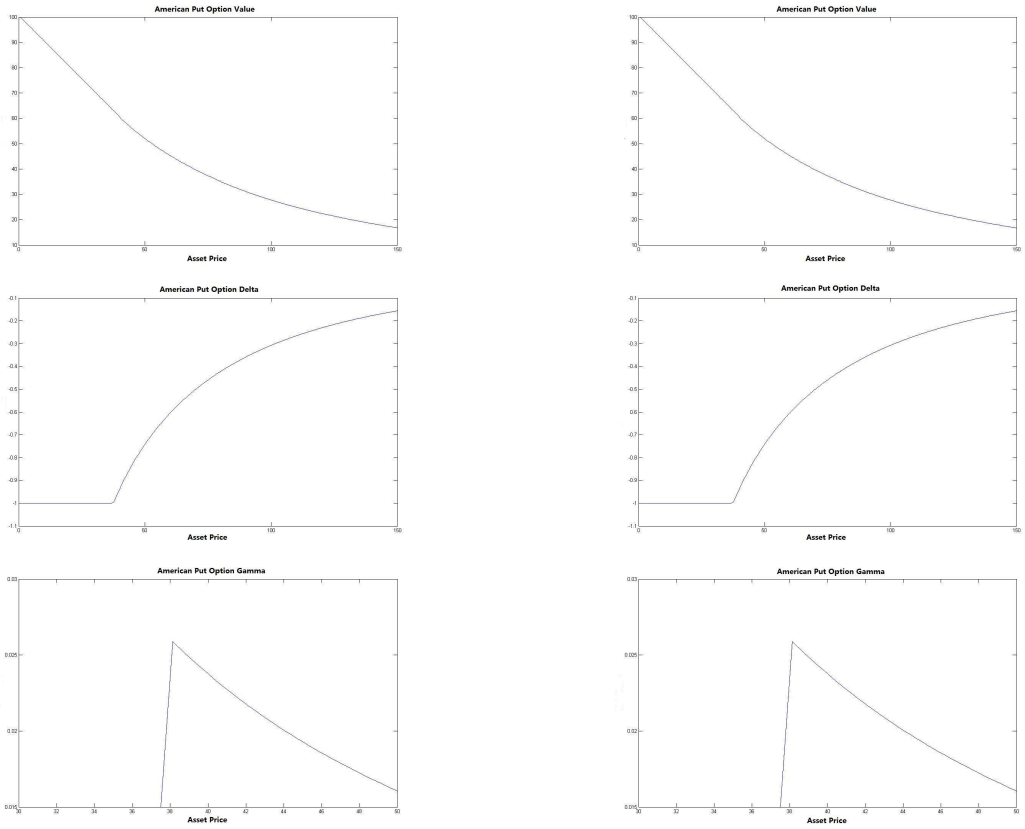

Figure 5.4: Value, delta(V_S), and gamma(V_{SS}) of an American Put, σ=0.4, T=5.00, r=0.03, K=100. Left: TR-BDF2 time stepping method, right: Crank-Nicolson time stepping method with Rannacher smoothing. Top: option value (V), middle: delta (V_S), bottom: gamma(V_{SS}).

Variables	Values
σ	0.3
r	0.04
Time to expiry (T)	0.50 years
Strike Price	\$ 100
Initial asset price S^0	\$ 100

Table 5.9: Data for American Put Option

Nodes	Timesteps	Value	Change	Ratio
TR-BDF2				
62	42	7.56866		
123	99	7.58049	0.01183	
245	213	7.58348	0.00298	4.0
489	440	7.58422	0.00075	4.0
977	894	7.58440	0.00018	4.0
Crank-Nicolson (Rannacher smoothing)				
62	43	7.56265		
123	101	7.57959	0.01694	
245	215	7.58333	0.00375	4.5
489	442	7.58420	0.00086	4.3
977	896	7.58440	0.00020	4.2

Table 5.10: Value of an American Put. Change is the difference in the solution from the coarser grid. Ratio is the ratio of changes on successive grids.

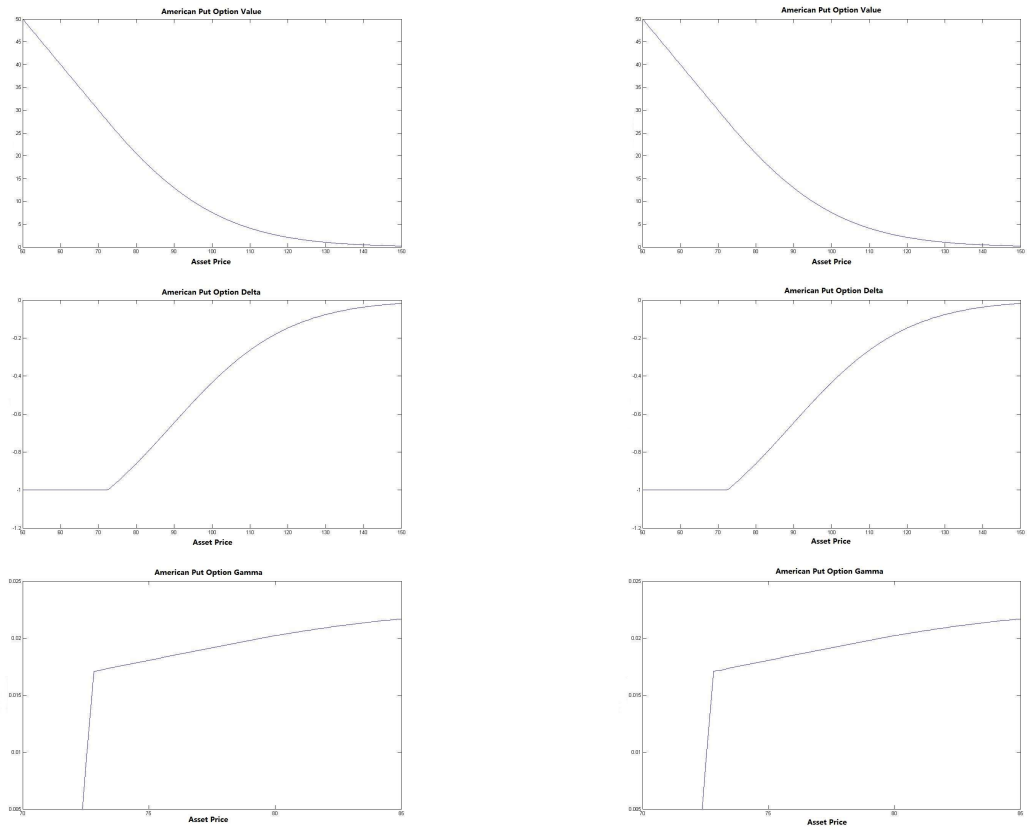

Figure 5.5: Value, delta(V_S), and gamma(V_{SS}) of an American Put, σ=0.3, T=0.50, r=0.04, K=100. Left: TR-BDF2 time stepping method, right: Crank-Nicolson time stepping method with Rannacher smoothing. Top: option value (V), middle: delta (V_S), bottom: gamma(V_{SS}).

Variables	Values
σ	0.2
r	0.05
Time to expiry (T)	1.00 years
Strike Price	\$ 100
Initial asset price S^0	\$ 100

Table 5.11: Data for American Put Option

Nodes	Timesteps	Value	Change	Ratio
TR-BDF2				
62	37	6.07958		
123	87	6.08762	0.00804	
245	187	6.08969	0.00207	3.9
489	387	6.09020	0.00051	4.0
977	785	6.09033	0.00013	4.0
Crank-Nicolson (Rannacher smoothing)				
62	38	6.07395		
123	89	6.08677	0.01282	
245	189	6.08955	0.00277	4.6
489	389	6.09018	0.00063	4.4
977	787	6.09032	0.00015	4.3

Table 5.12: Value of an American Put. Change is the difference in the solution from the coarser grid. Ratio is the ratio of changes on successive grids.

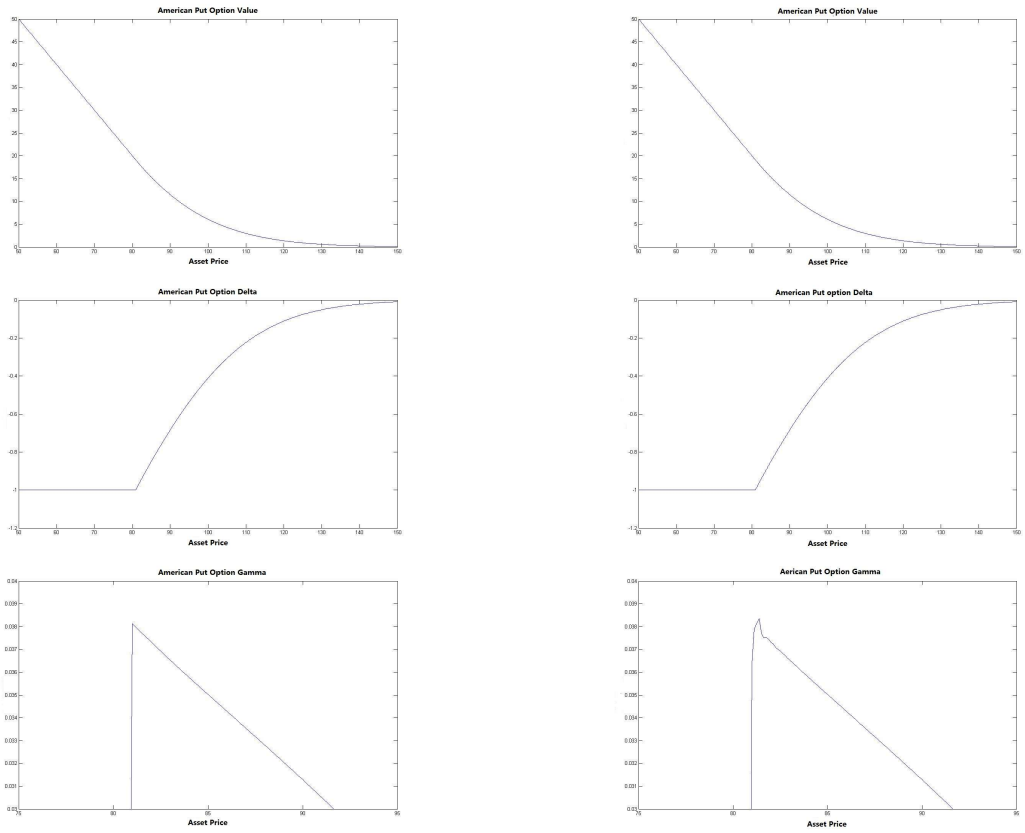

Figure 5.6: Value, delta(V_S), and gamma(V_{SS}) of an American Put, σ=0.2, T=1.00, r=0.05, K=100. Left: TR-BDF2 time stepping method, right: Crank-Nicolson time stepping method with Rannacher smoothing. Top: option value (V), middle: delta (V_S), bottom: gamma(V_{SS}).

Variables	Values
σ	0.1
r	0.02
Time to expiry (T)	1.00 years
Strike Price	\$ 100
Initial asset price S^0	\$ 100

Table 5.13: Data for American Put Option

Nodes	Timesteps	Value	Change	Ratio
TR-BDF2				
62	26	3.21806		
123	57	3.22310	0.00503	
245	118	3.22445	0.00135	3.7
489	239	3.22479	0.00034	4.0
977	482	3.22487	0.00008	4.0
Crank-Nicolson (Rannacher smoothing)				
62	27	3.21521		
123	58	3.22267	0.00746	
245	119	3.22438	0.00171	4.4
489	240	3.22478	0.00040	4.3
977	483	3.22487	0.00009	4.2

Table 5.14: Value of an American Put. Change is the difference in the solution from the coarser grid. Ratio is the ratio of changes on successive grids.

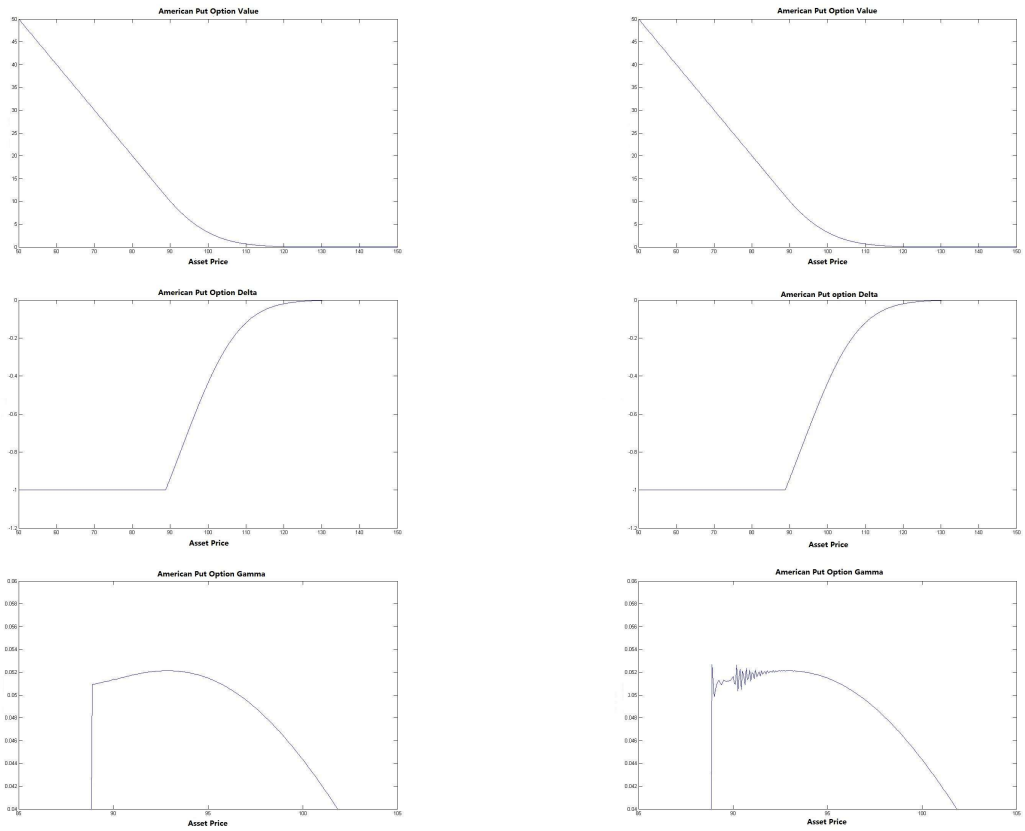

Figure 5.7: Value, delta(V_S), and gamma(V_{SS}) of an American Put, σ=0.1, T=1.00, r=0.02, K=100. Left: TR-BDF2 time stepping method, right: Crank-Nicolson time stepping method with Rannacher smoothing. Top: option value (V), middle: delta (V_S), bottom: gamma(V_{SS}).

5.2.2 Analysis and conclusion

As can be observed from the examples in last section, results obtained by both time stepping methods converged to the same value and both methods achieved second order convergence.

Figure. 5.2-5.7 compare option values, delta (V_S), gamma (V_{SS}) for both time stepping methods. It can be observed that though the value and delta are similar for both methods, Rannacher smoothing cannot reduce all the oscillations in gamma. So there are oscillations observed in the gamma corresponding to the early exercise boundary for the Crank-Nicoson time stepping method. This is due to the stability properties of the Crank-Nicolson time stepping method [4].

However, for the TR-BDF2 time stepping method which has better stability properties, there are still no oscillations observed in Greeks. So, with comparable speed of convergence, the TR-BDF2 time stepping method offers better results for Greeks, which are of practical importance as commonly used hedging parameters.

Chapter 6

Summary

In this paper, the Trapezoidal Rule with the second order Backward Difference Formula (TR-BDF2) time stepping method was studied for option pricing. The Crank-Nicolson time stepping method, as an alternative method, was compared against this method in the study.

We first derived the solution to the option pricing problem under the Black-Scholes model using the TR-BDF2 time stepping method which is a second order fully implicit Runge Kutta method. Then Von Neumann stability analysis of the TR-BDF2 time stepping method was carried out. It was known that the Crank-Nicolson time stepping method is algebraically unconditionally stable, and in our analysis, it was proved that the TR-BDF2 time stepping method is also unconditionally stable.

When pricing options, the Crank-Nicolson time stepping method can introduce spurious oscillations, particularly in Greeks. We did numerical tests and showed that, though Rannacher smoothing can fix this problems for European options priced using the Crank-Nicolson time stepping method, it does not work well for American options. Obvious oscillations corresponding to early exercise boundaries can still be observed in gamma in the American option case.

On the other hand, the TR-BDF2 time stepping method, with better stability properties, does not introduce any oscillations in Greeks and has comparable speed of convergence as the Crank-Nicolson time stepping method with Rannacher smoothing.

APPENDICES

Appendix A

Definition of A-stable and L-stable

Consider a linear model problem:

$$u^{'}(t) = \lambda u(t) \tag{A.1}$$

where λ is a complex number.

We say a numerical method is A-stable if its stability region contains the whole left half plane. If we define $z = \lambda \Delta t$, A-stable can be written as: $\{z \in \mathbb{C} : Re(z) \leq 0\}$. For example, forward Euler scheme leads to a discretization of $u_{j+1} = (1 + \lambda \Delta t)u_j$ for the problem stated above and its stability region is $|1 + z| < 1$. By definition, it is not A-stable because the stability region is a disc of radius centred at the point -1.

L-stable is stronger than A-stable, we define a numerical method is L-stable if it is A-stable and $\frac{u_{j+1}}{u_j} \to 0$ as $|z| \to \infty$.

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