On The Sensitivity Of CVaR Optimization Model To The Estimation Errors In The Underlying Mean Return

by

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A research report
presented to the University of Waterloo
in fulfillment of the
requirement for the degree of
Master of Math
in
Computational Mathematics

Waterloo, Ontario, Canada, 2009

Abstract

The estimation errors problem is very important for portfolio selection models. It has been shown that estimation errors in the asset returns can have a surprisingly big impact in the Mean Variance portfolio selection model (MV model). However, the issue has not yet been explored in the CVaR portfolio optimization model (CVaR model). The CVaR risk measure has attractive mathematical properties and is suitable for non-normal portfolio loss distribution. The CVaR model can be solved using a simple linear programming algorithm. Therefore, it is crucial to understand the sensitivity of the CVaR model. This research report investigates the estimation errors in the underlying mean returns in the CVaR model using a simulation example. This shows that the effects of estimation errors in the underlying mean returns can be large. The variation of the estimated portfolio's actual performance increases steadily as the relative error parameter increases. The magnitude of the variation decreases as the transaction cost increases and increase as the instruments' bounds increase. Moreover the estimated portfolio's performances are optimistically biased compared to the actual performance and thus can lead to a more aggressive investment strategy than it is advisable.

Keyword: CVaR, CVaR optimization model, estimation errors, estimated portfolio, actual portfolio, true portfolio, efficient frontier.

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1 Introduction

Parameter estimation errors for portfolio selection model have an impact on the resulting optimal portfolio and risk measures. It has been studied extensively in the framework of the Mean Variance Portfolio Selection Model (MV model). Optimal portfolios and efficient frontiers in MV model are very sensitive to the estimation errors, especially in the estimation errors in asset mean returns [3]. However, the MV model is very restrictive in the sense that it relies heavily on the assumption of a normal distribution in expected asset returns and uses variance as a risk measure which is misleading for nonnormal distributions.

The impacts of estimation errors in the CVaR portfolio optimization model has not been addressed. CVaR portfolio optimization model can have very broad applications in risk management and portfolio optimization. CVaR risk measure has very attractive mathematical properties and is suitable for general loss distributions. It is appropriate for a portfolio of derivatives. But the issue of estimation errors in the parameters still exists. In order to make use of this model in practice, it is crucial to investigate the sensitivity of the CVaR optimization model.

There are different sources of estimation errors. In this research report, we focus on the estimation errors in underlying mean returns as a first step to analyze the sensitivity of CVaR optimization models to estimation errors. The sensitivity of CVaR optimization model is investigated computationally. This study shows that the CVaR risk measure, the portfolio mean return and the CVaR optimal portfolio are quite sensitive to the estimation errors in the underlying mean returns. The efficient frontiers are quite sensitive as well.

In section 2, a description of the CVaR optimization model is summarized and a short comparison with Mean-Variance portfolio selection model is given. In section 3, the sensitivity of the CVaR optimization model is investigated through simulation examples. Section 4 is a conclusion.

2 CVaR Risk Measure and CVaR Portfolio Optimization Model

This section provides the background knowledge for this research report. It starts with the formal definition of the CVaR risk measure. Then the mathematical formulation of the CVaR optimization model is presented. Finally we compare the CVaR model with Mean-Variance model and shortly discuss the various techniques proposed in the research to analyze the sensitivity of Mean Variance portfolio selection model.

2.1 Conditional Value-at-risk

Conditional Value-at-risk (CVaR) is built on the concept of Value-at-Risk (VaR). VaR answers the following question: what is the minimum loss value that the loss does not exceed over a given time horizon with a certain confidence level. CVaR considers about the conditional left tail of the loss distribution, the expected loss that exceeds VaR.

Formally, let $L(x, S_t)$ denote the portfolio loss variable for a given portfolio $x \in R^N$ and a vector of random variables $S_t \in R^d$ which represents the sources of uncertainties. The subscript t indicates that the loss is calculated over a given time horizon t. Assume S_t follows a density function $p(S_t)$. A cumulative loss distribution function for the loss of a given portfolio not exceeding a threshold α is then given by [5]

$$\Phi(x, \alpha) = \int_{L(x, S_t) \leq \alpha} p(S_t) dS_t$$

The VaR associated with a portfolio x, for a specified confidence level β and a time horizon $\bar{t}$ is

$$VaR_\beta(x) = \min \{\alpha \in R, \Phi(x, \alpha) \geq \beta\}$$

CVaR, the conditional expectation of the loss that is equal and greater than VaR_α, for loss distribution which has no jumps is

$$CVaR_\beta(x) = (1 - \beta)^{-1} \int_{L(x, S) \geq VaR_\beta} L(x, S) p(S) dS$$

CVaR for a general loss distribution is defined as [5],

$$CVaR_\beta(x) = \inf_\alpha \left(\alpha + (1 - \beta)^{-1} \mathbf{E} \left[(L(x, S) - \alpha)^+ \right] \right)$$

where

$$(L(x, S) - \alpha)^+ = \max (L(x, S) - \alpha, 0)$$

CVaR is a consistent risk measure since it is sub-additive.[5].

2.2 CVaR Portfolio Optimization Model

CVaR portfolio optimization model minimizes CVaR risk measure for a portfolio. That is

$$\min_{x \in X} CVaR_\beta(x)$$

where $x \in R^N$, denotes the portfolio holding vectors.

Define the augmented function as in [5]

$$F_\beta(x, \alpha) \equiv \alpha + (1 - \beta)^{-1} \mathbf{E} \left[(L(x, S) - \alpha)^+ \right]$$

Rockafellar & Uryasev [8] showed that $F_\beta(x, \alpha)$ is convex and continuously differentiable with respect to α and $CVaR_\beta(x)$ is convex with respect to x. They also showed that VaR minimizes $F_\beta(x, \alpha)$. Moreover, minimizing the CVaR over $x \in X$, where X is a subset of R^N, is equivalent to minimizing $F_\beta(x, \alpha)$ over $(x, \alpha) \in (X \times R)$,

$$\min_{x \in X} CVaR_\beta(x) = \min_{(x, \alpha) \in (X \times R)} F_\beta(x, \alpha)$$

For a given portfolio of N instruments, the portfolio holdings are represented by an N-dimension vector $x \equiv [x_1, .., x_N]^T$. The instrument value vector is $V_t \equiv \{V_1(S_t, t), ..., V_N(S_t, t)\}$. The portfolio loss over a time horizon $\bar{t}$ is:

$$L(x, S) = -x^T (V_{\bar{t}} - V_0) = -(\delta V)^T x$$

Adding the budget,return constraints and bounds for instrument holdings, the CVaR optimization problem is formed as

$$\min_{(x, \alpha)} \left(\alpha + (1 - \beta)^{-1} E \left[(-(\delta V)^T x - \alpha)^+ \right] \right) \tag{1}$$

subject to:

$$\left(V^0 \right)^T x = 1 \tag{2}$$
$$\left(\delta \bar{V} \right)^T x = \bar{r}, \tag{3}$$
$$l \leq x \leq u \tag{4}$$

The CVaR optimization problem was proposed by Rockafellar and Uryasev [8]. Uryasev further explored the algorithms and applications [9] of this problem. Alexander, Coleman and Li [1] add transaction cost to the standard CVaR optimization model.

2.3   A Linear Programming Approach

There is usually no analytical formula for the loss distribution of a portfolio of derivatives. The CVaR optimization problem is solved by a Monte-Carlo simulation approach. It has been shown that, it can be optimized using linear programming (LP) and nonsmooth optimization algorithm [5].

Assume that there are M scenarios for the realization of underlying stock prices. Therefore there are M derivative prices for each instrument. $F_\beta(x, \alpha)$ can be approximated as following,

$$F_\beta(x, \alpha) \simeq \alpha + \frac{1}{M(1-\beta)} \sum_{i=1}^{M} (L(x, S_t)_i - \alpha)^+$$

where i is the ith scenario. The objective function becomes:

$$\min_{(x, \alpha)} \left(\alpha + \frac{1}{M(1-\beta)} \sum_{i=1}^{M} [-(\delta V)_i^T x - \alpha]^+ \right)$$

Let $y_i = [-(\delta V)_i^T x - \alpha]^+$. The CVaR optimization problem becomes

$$\min_{(x, y, \alpha)} \alpha + \frac{1}{M(1-\beta)} \sum_{i=1}^{M} y_i \tag{5}$$

subject to:

$$y_i \geq -(\delta V)_i^T x - \alpha, i = 1, ..., M \tag{6}$$
$$y_i \geq 0, i = 1, ..., M \tag{7}$$
$$\left(V^0 \right)^T x = 1 \tag{8}$$
$$\left(\delta \bar{V} \right)^T x = \bar{r}, \tag{9}$$
$$l \leq x \leq u \tag{10}$$

2.4 A Comparison with Mean Variance Portfolio Selection Model

The CVaR portfolio optimization model is very different from the classic Mean-Variance portfolio selection model. CVaR measures the risk of extreme loss while variance measures expected deviation from the mean loss. It therefore applies for a general loss distributions and is much less restritive than the MV model.

MV Model relies heavily on normal distribution assumption of expected returns for the assets. This model, developed by Markowtiz, uses variance as the risk meansure for the portfolio. It assumes that risk averse investors only care about expected return and volatility of the portfolio. The other charateristics of the distribution does not matter. This assumption is appropriate as long as the distribution of portfolio returns is normal. With a covariance matrix and returns of the assets as inputs, we can either maximize the expected return subject to certain variance constraint or minimize portfolio variance

subject to portfolio return higher than certain level[3]. However, The characteristics of the loss distributions for derivative portfolios are usually asymetric with large kurtosis. There is extensive evidence of asymetric loss distribution with fat tail.

The CVaR optimization model and MV model both have the problem of estimation errors in input parameters. The inputs for MV model are asset expected returns and a covariance matrix for assets. Estimation of these paremeters are usually inaccurate. The true parameter values are unknown. There has been mass studies in the literature that discuss the estimation error problem for the MV model. Stein(1995) shows that traditional sample statistics are not appropriate for multivariate problems. Barry(1974) and Michaud (1989) describe the problem in detail. It has been shown that MV portfolios are very sensitive the changes in the parameters, especially the mean returns. Chopra and Ziemba (1993)[3] examinethe relative impacts of estimation errors in means, variances and covariances in the mean-variance portfolio optimization. They conclude that errors in means are more important than those in variances and covariances. Jorion (1992) [4] and Broadie (1993) [2]use Monte Carlo simulations to estimate the magnitude of the problem. Similar estimation errors problem exists for the CVaR optimization problem. However, it has not been explored yet.

3 Investigating Estimation Errors Using Simulation

In this section, we investigate the effects of estimation errors in underlying mean returns on CVaR optimization problem. In this section, *estimation errors* is used instead of *estimation errors in underlying mean returns* in this chapter. This chapter starts with a simple description of the algorithm and the simulation example.[1] Then it illustrates the senstivity of CVaR and portfolio-mean-return profiles (denote as (CVaR,$\bar{r}$)) and the estimated optimal portfolios graphically. In section 3.3, we measure the estimation errors quantitatively. In the last section, the sensitivity of efficient frontiers is analyzed.

3.1 The Simulation Example

Consider an investor who holds a portfolio of N instruments. For each of the four correlated underlying stocks, there are 12 standard calls, 12 standard puts, 12 digital calls and 12 digital puts and the strike and expiry are all possible combinations of 3 strikes [0.8, 1.025, 1.25]S_0 and 4 expiries [2,4,6,8]$\bar{t}$ where $\bar{t}$ is 10 days (assuming 250 trading days in a year). Therefore there are $N = 196$ investment instruments including 192 options plus the

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[1]A detailed description of the simulation example is provided in the appendix A.1.

4 underlying assets. The required portfolio return $\bar{r}$ is twice the risk freee rate over the time horizon $[0,\bar{t}]$ with the annual risk free rate $r = 5\%$. The initial value of the portfolio is \$1. The lower bound of the instrument holdings is -0.3 and the upper bound of the instrument holdings is 0.4.

Let μ and $\tilde{\mu}$ denote the four dimensional underlying mean return vector without estimation errors and with estimation errors respectively. Assume that the estimated errors have independent normal distributions. Specifically,

$$\tilde{\mu}_i = \mu + \delta\mu_i\phi_i \quad i = 1, 2, 3, 4$$

Where i is the ith underlying.ϕ is a four elements random vector whose elements are independent with standard normal random variables. We use $randn$ in MATLAB to generate ϕ. The parameter $\delta \in [0, 1]$, indicates the size of the relative errors. We use the linear programming solver in the software package MOSEK version 5.0.0.127 to compute the optimal portfolios and CVaR & VaR.

α is equal to 0.1, 0.3 and 0.5 in our simulation. To see the effects of α, we conduct 50 simulations for each α. A 50×4 random number matrix, denoted as C is used to represent the simulation errors. This C matrix is used repeatedly. Let M denote the number of simulations and $M = 30000$. The 4 underlying stock prices are generated by Monte-Carlo simulation. Let n denote the number of steps in Monte-Carlo simulation. We use $randn$ in MATLAB to generate a $M \times 4n$ random number matrix, denoted as Q. This matrix is used repeatedly under different underlying mean returns vector as well. In this way, we control the simulation errors for better illustration of the effects of estimated errors in mean returns.

Besides different α, we also conduct experiments under different confidence levels β and different weighted cost parameter ω^2. The purpose of this experimental group setting is to see how the effects of estimation errors may change with β and ω. Let $\beta = 0.95$ and $\omega = 0$ be the benchmark experimental group. First we leave β unchanged and set $\omega = 0.005$ and 0.5. Then we leave ω unchanged and set $\beta = 0.85$ and 0.99. Thus we have 5 experimental goups in total and for each of the experimental groups, we need to investigate the effects of each of the 3 α values by 50 simulations.

The Crank-Nicholson method with Rannacher Smoothing (CN Rannacher method)is used to approximate the derivatives' values at $\bar{t}$. This method has quadratic truncation errors $O(\Delta t^2, \Delta S^2)$. The derivative prices for simulated underlying stock prices are calculated by linear interpolation from derivative values on the fixed grid of stock price.

The *true* portfolio return, CVaR and VaR are given in table 1. The *true* CVaR & VaR for the benchmark group are given in the row 1 and row 4 as a reference. The first 3 rows

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[2]Weighted cost is of the form: $c_i = \omega \times \overline{CVaR}^0$ [1]for $1 \le i \le 196$. $\overline{CVaR}^0$ denotes the optimal CVaR with no transaction cost.

show that the CVaR & VaR increase as ω increases and the last 3 rows show that the CVaR & VaR increase as β increases. It seems that β has very small impact on CVaR & VaR.

Table 1: *True* VaR & CVaR with $\bar{r}$=0.004

β	ω	CVaR	VaR
0.95	0	0.01252	0.01244
0.95	0.005	0.01298	0.01292
0.95	0.5	0.02600	0.02586
0.95	0	0.01252	0.01244
0.85	0	0.01235	0.01205
0.99	0	0.01259	0.01257

3.2   The Sensitivity of CVaR and CVaR Optimal Portfolios

The effects of estimation errors on CVaR optimal portfolios are reflected in (CVaR,$\bar{r}$) profile and the optimal portfolio holdings. This section presents the computational experiments from the above two aspects graphically.

Figure 1(a) plots the (CVaR,$\bar{r}$) profiles for $\alpha = 0.1$, $\omega = 0$ and $\beta = 0.95$. There are three kinds of (CVaR,$\bar{r}$) profiles: *actual*, *esitmated* and *true* (CVaR,$\bar{r}$) profiles in the figure. The *true* and *estimated* (CVaR,$\bar{r}$) profile are the (CVaR,$\bar{r}$) pairs calculated under the true underlying mean return parameters μ and under the underlying mean returns with estimation errors $\tilde{\mu}$ respectively. The *estimated* optimal portfolio is solved under estimated parameters $\tilde{\mu}$ at the same time as well. The concept of *actual* (CVaR,$\bar{r}$) profile is a little more complicated. It is *true* (CVaR,$\bar{r}$) profile for *estimated* optimal porfolio. *Estimated* optimal portfolio together with μ to generate a simulated porfolio loss distribution and the (CVaR, $\bar{r}$) is calculated with this loss distrbution.

Table 2: Algorithm: calculation of the actual CVaR and actual portfolio mean return

Actual portfolio mean return

1. Compute the estimated portfolio $\tilde{x}$ by solve the CVaR portfolio optimization problem with $\tilde{\mu}$;

2. Calculated the actual portfolio mean returns by $\tilde{r} = -\delta\bar{V}^T\tilde{x}$, where $\delta\bar{V}$ is a $N \times 1$ vector of simulated average changes under μ.

Actual CVaR

1. Calculated the correponding actual portfolio losses by $\tilde{L} = -\delta V^T\tilde{x}$, where δV is a $N \times M$ matrix for changes in n instruments values over $\bar{t}$ for M simulation.

2. Get the CVaR & VaR of the simulated losses distribution by the procedure described in Rockafellar & Uryasev (2002).

To summarize, the *true* (CVaR,$\bar{r}$) is unosbervable to investors since the true parameter values are unknown. The *estimated* (CVaR,$\bar{r}$) profile is the profile that investors calculate based on their knowledge of parameter values. Therefore *estimated* (CVaR,$\bar{r}$) profile is used by investors to evaluate a portfolio performance and the estimated optimal portfolio is used to guide investors' investment strategy. The *actual* (CVaR,$\bar{r}$) profile is unobservable to the investors as well. It is the true perfomance of the *estimated* optimal portfolio. It is useful here as a tool to see how large the impacts of estimation errors are. Table 2 describes the algorithm used to calculate *actual* (CVaR,$\bar{r}$).

In Figure 1(a), using the true underlying mean returns, the *true* (CVaR,$\bar{r}$) is plotted. Next using the underlying mean returns with estimation errors, the *estimated* (CVaR,$\bar{r}$) profile for 50 simulations are plotted. Finally, the *actual* (CVaR,$\bar{r}$) profiles for 50 simulations are plotted. Since the CVaR optimization problem is solved with the same target portfolio mean return constraint, the *true* and *estimated* (CVaR,$\bar{r}$) profiles are located on the line $\bar{r} = 0.004$. Figure 1(c) and 1(e) plot three kinds of (CVaR,$\bar{r}$) profiles for $\alpha = 0.3$ and $\alpha = 0.5$ with the same ω and β. The *actual* (CVaR,$\bar{r}$) profiles show larger and larger variation as α increases and almost all of them lie below target portfolio return $\bar{r} = 0.004$. Notice that the *estimated* (CVaR,$\bar{r}$) profiles mostly have smaller CVaR value and all of them achieve target returns. This suggests that the the *estimated* (CVaR,$\bar{r}$) profiles are mostly optimistically biased compared to its *actual* performance.

How sensitive are the *actual* (CVaR,$\bar{r}$) to the estimation errors relative to the *true* (CVaR,$\bar{r}$)? Let's define the following relative difference measures for CVaR and $\bar{r}$ for all

the simulations,

$$RelDif(CVaR) = \frac{CVaR_i^{actual} - CVaR_i^{true}}{CVaR_s^{true}};$$

$$RelDif(\bar{r}) = \frac{\bar{r}_i^{actual} - \bar{r}_i^{true}}{\bar{r}_i^{true}}.$$

Figure 1(b),1(d) and 1(f) show the relative differences. The $ReDif(CVaR)$ is in the range of [-30%, 15%] for $\alpha = 0.1$, [-60%, 30%] for $\alpha = 0.3$, [-80%, 60%] for $\alpha = 0.5$ and the $ReDif(\bar{r})$ is in the range of [-22%, -2%] for $\alpha = 0.1$, [-30%, 5%] for $\alpha = 0.3$, [-35%, 5%] for $\alpha = 0.5$. The range of $ReDif(CVaR)$ increases quite fast. The range for RelDif($\bar{r}$) are mostly negative, again indicating lower mean returns for estimated optimal portfolios. The results for the other three experimental groups are shown in Figure 2, 3, 4 and 5. They have the similar patterns as in Figure 1.

Figure 1, 2 and 3 together suggest that the transaction cost parameter ω have a big impact on the variation in the *actual* (CVaR,$\bar{r}$) profiles. In contrast with Figure 1, without transaction cost (i.e. $\omega = 0$), Figure 2 shows that the variation in the *actual* (CVaR,$\bar{r}$) profiles reduces with a very small transaction parameter, $\omega = 0.005$. When ω increases to 0.5, the variation is largely reduced. Some of (CVaR,$\bar{r}$) profiles, which used to be very close to each other, overlap with each other and most of them line up straight. Although *actual* (CVaR,$\bar{r}$) profiles are very different from the *true* (CVaR,$\bar{r}$) profiles, their variations are largely reduced.

There is no clear pattern from Figure 4 and 5 to show the effects of β on the variation. Figure 6 shows a comparison of instrument holdings for $\alpha = 0$ (i.e. no estimation errors), $\alpha = 0.1$, $\alpha = 0.3$ and $\alpha = 0.5$. For each of the subgraph, there are 4 groups (i.e. four instruments) and each group has 4 bars, their length representing instrument holdings for 4 different α. It shows that the instrument holding varies drastically as α changes.
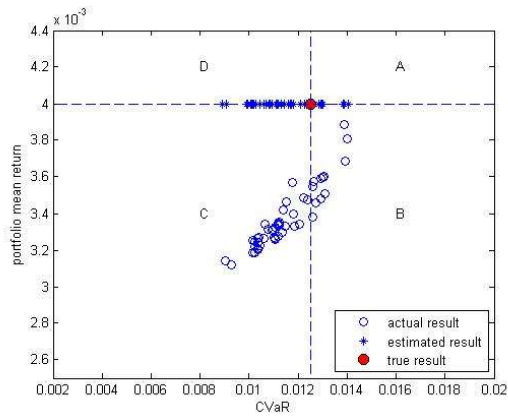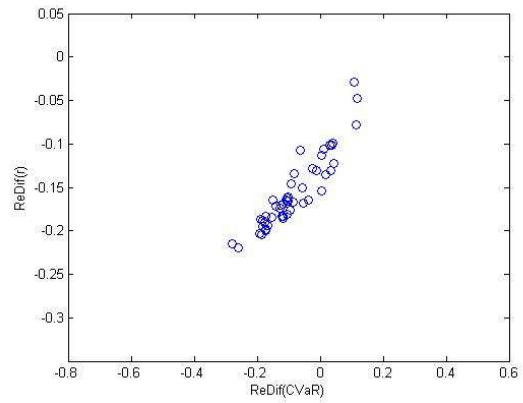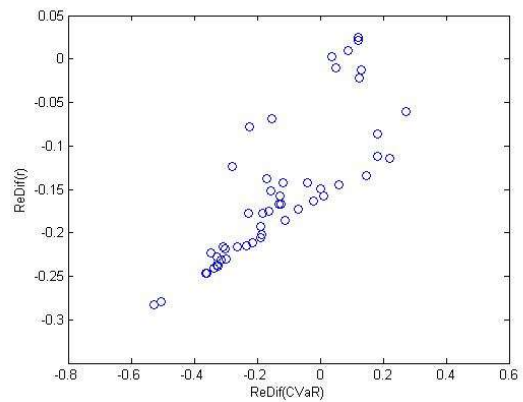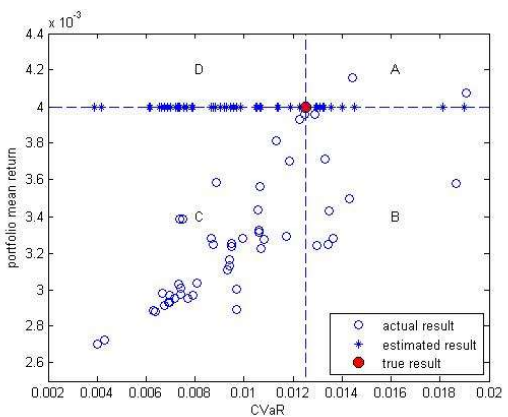
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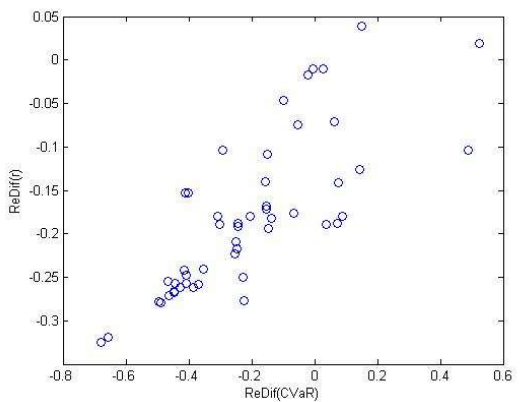
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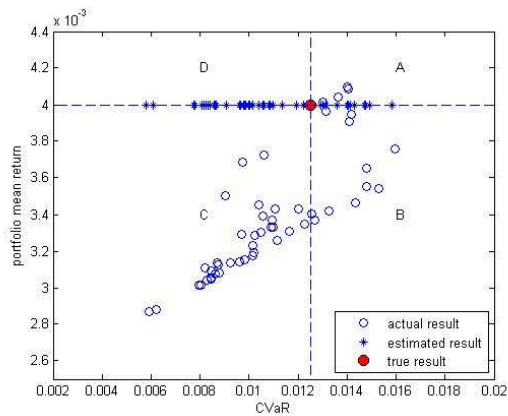
(c) (CVaR,$\bar{r}$), $\alpha = 0.3$

(d) (ReDif(CVaR),ReDif($\bar{r}$)), $\alpha = 0.3$

(e) (CVaR,$\bar{r}$), $\alpha = 0.5$
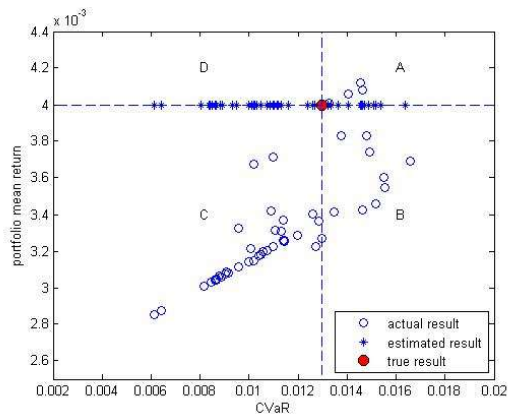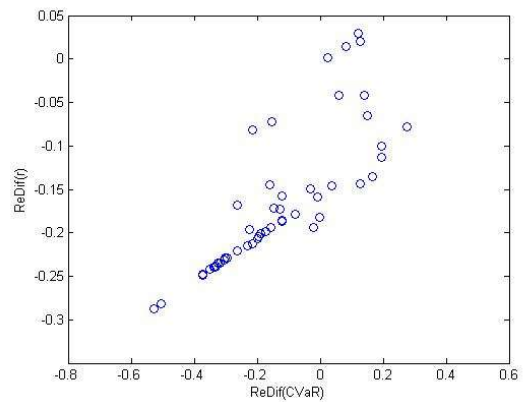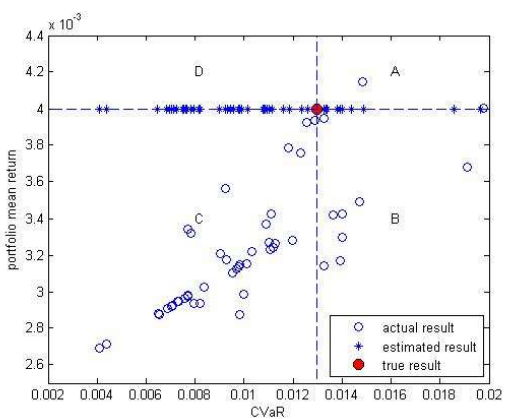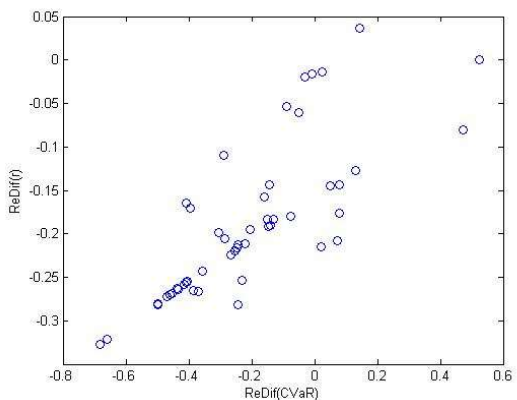
(f) (ReDif(CVaR),ReDif($\bar{r}$)), $\alpha = 0.5$

Figure 1: (CVaR,$\bar{r}$) profiles. $\beta = 0.95$, $\omega = 0$.
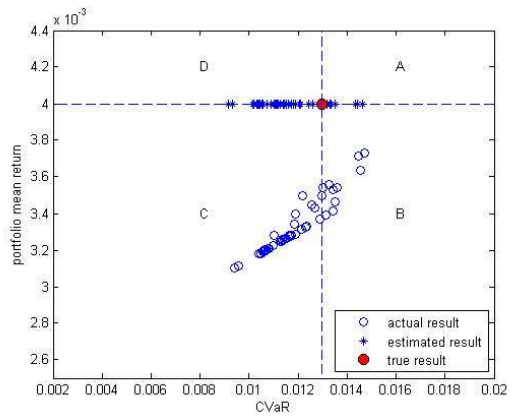
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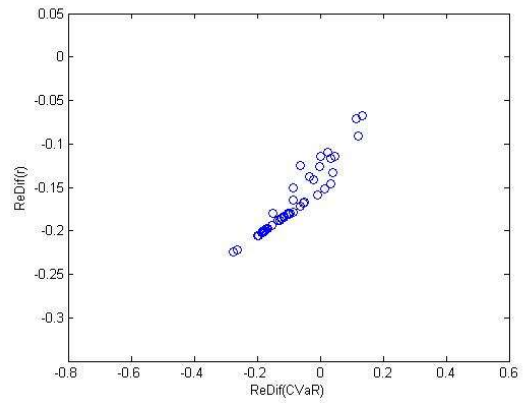
(b) (ReDif(CVaR),ReDif($\bar{r}$)), $\alpha = 0.1$

(c) (CVaR,$\bar{r}$), $\alpha = 0.3$
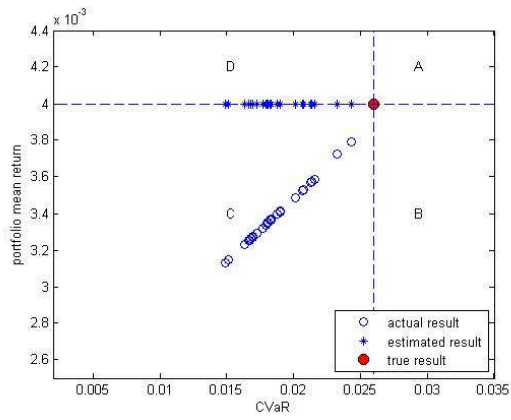
(d) (ReDif(CVaR),ReDif($\bar{r}$)), $\alpha = 0.3$
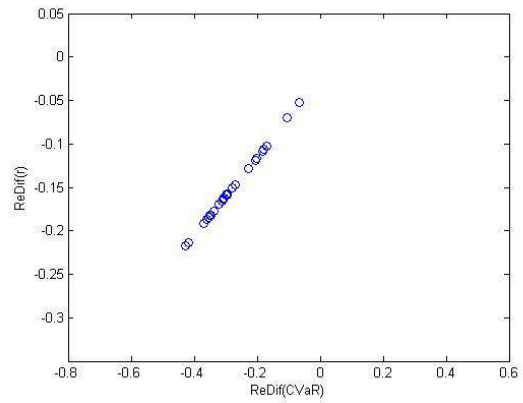
(e) (CVaR,$\bar{r}$), $\alpha = 0.5$

(f) (ReDif(CVaR),ReDif($\bar{r}$)), $\alpha = 0.5$

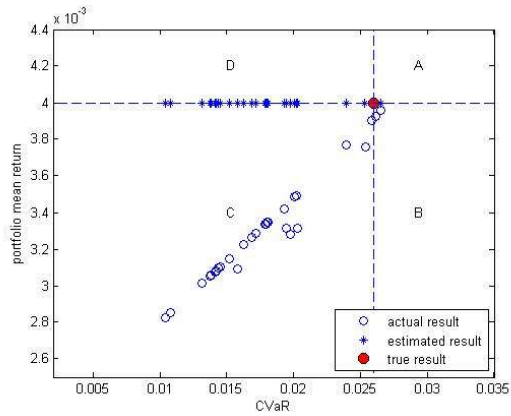Figure 2: (CVaR,$\bar{r}$) profiles. $\beta = 0.95$, $\omega = 0.005$.

(a) (CVaR,$\bar{r}$), $\alpha = 0.1$

(b) (ReDif(CVaR),ReDif($\bar{r}$)), $\alpha = 0.1$

(c) (CVaR,$\bar{r}$), $\alpha = 0.3$

(d) (ReDif(CVaR),ReDif($\bar{r}$)), $\alpha = 0.3$
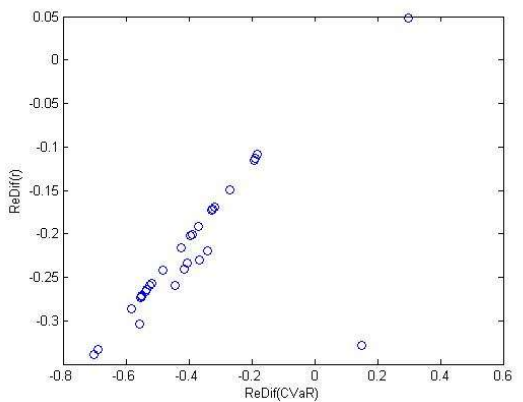
(e) (CVaR,$\bar{r}$), $\alpha = 0.5$

(f) (ReDif(CVaR),ReDif($\bar{r}$)), $\alpha = 0.5$

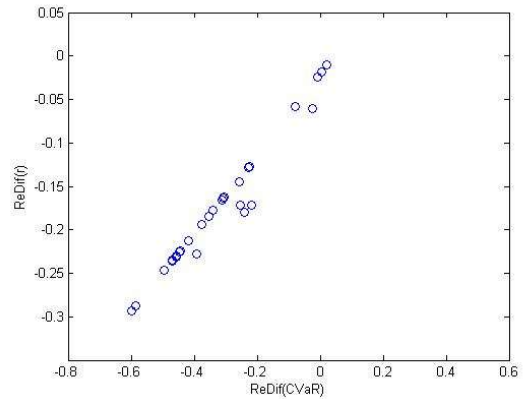Figure 3: (CVaR,$\bar{r}$) profiles. $\beta = 0.95$,$\omega = 0.5$.
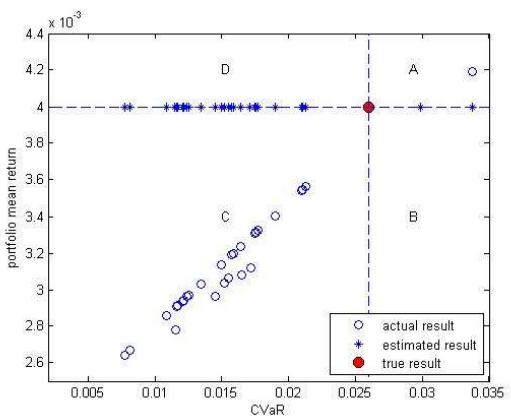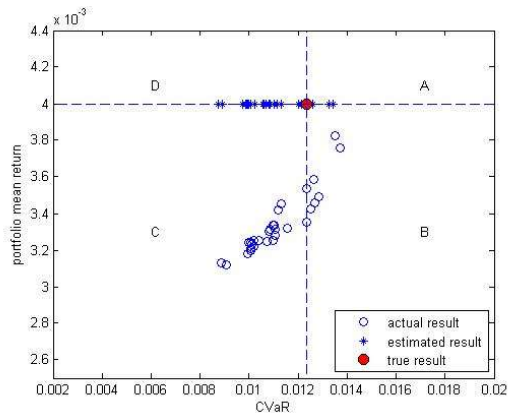
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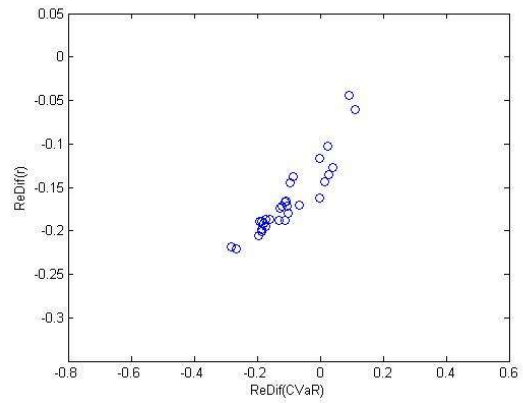
(b) (ReDif(CVaR),ReDif($\bar{r}$)), $\alpha = 0.1$

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(e) (CVaR,$\bar{r}$), $\alpha = 0.5$

(f) (ReDif(CVaR),ReDif($\bar{r}$)), $\alpha = 0.5$

Figure 4: (CVaR,$\bar{r}$) profiles. $\beta = 0.85$, $\omega = 0$.

(a) (CVaR,$\bar{r}$), $\alpha = 0.1$

(b) (ReDif(CVaR),ReDif($\bar{r}$)), $\alpha = 0.1$
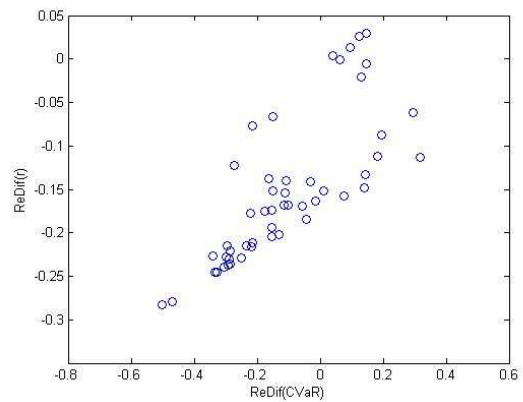
(c) (CVaR,$\bar{r}$), $\alpha = 0.3$

(d) (ReDif(CVaR),ReDif($\bar{r}$)), $\alpha = 0.3$
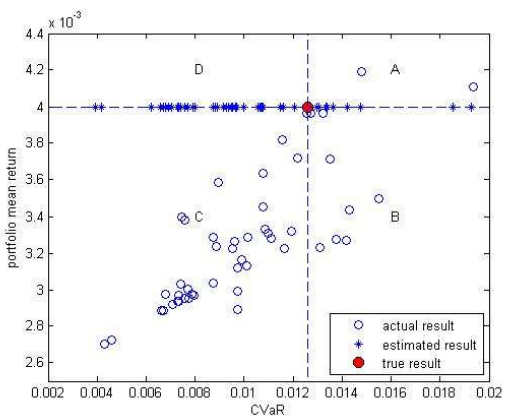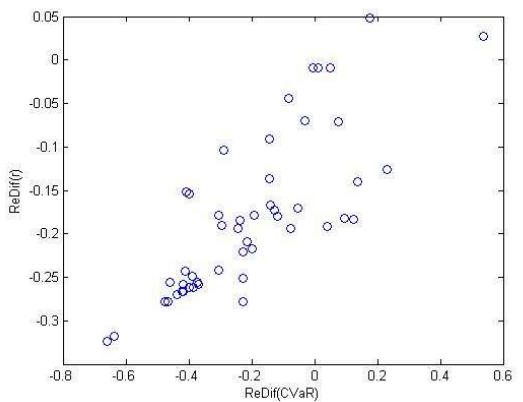
(e) (CVaR,$\bar{r}$), $\alpha = 0.5$
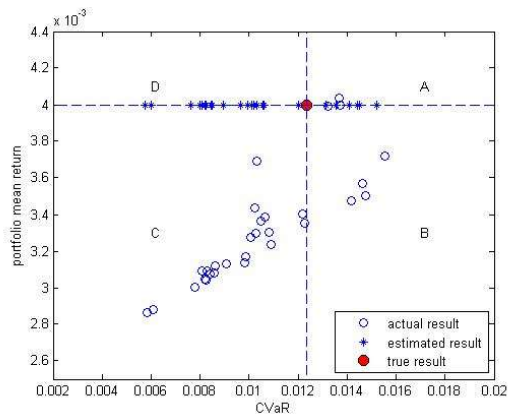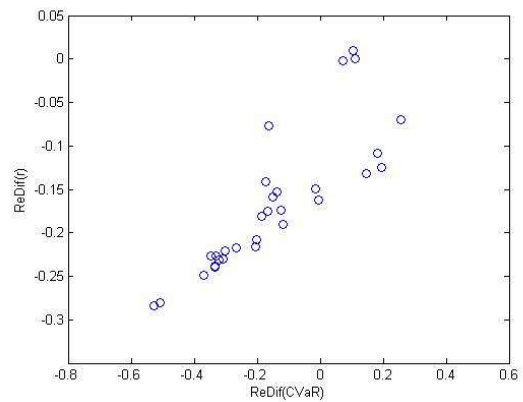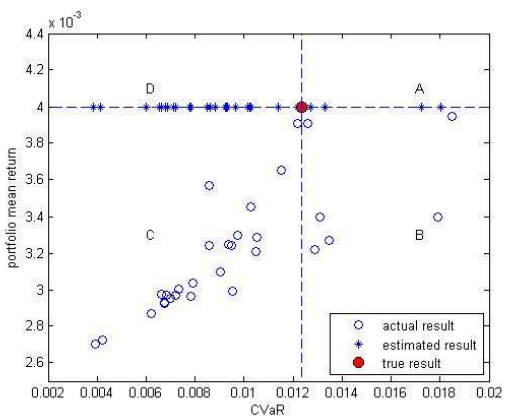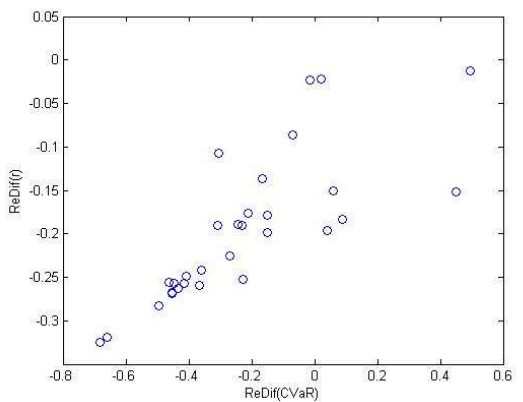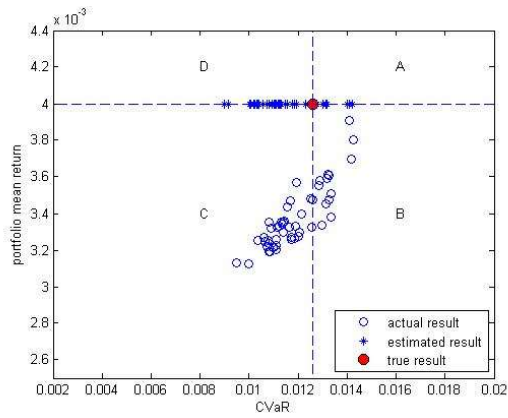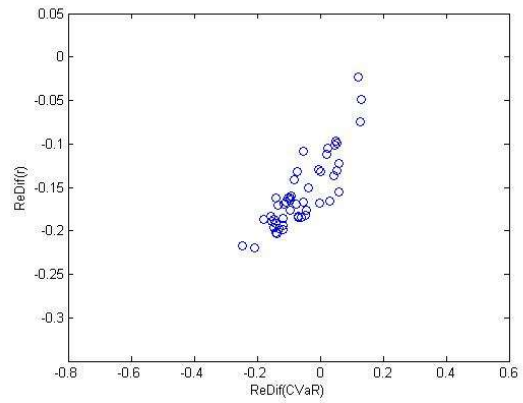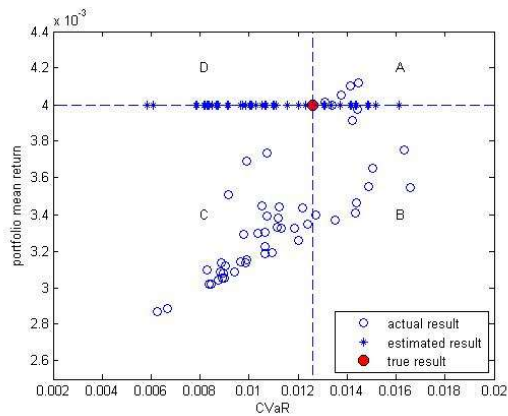
(f) (ReDif(CVaR),ReDif($\bar{r}$)), $\alpha = 0.5$

Figure 5: (CVaR,$\bar{r}$) profiles. $\beta = 0.99$, $\omega = 0$.

(a) (ReDif(CVaR),ReDif($\bar{r}$)), $\alpha = 0.3$

(b) (CVaR,$\bar{r}$), $\alpha = 0.5$
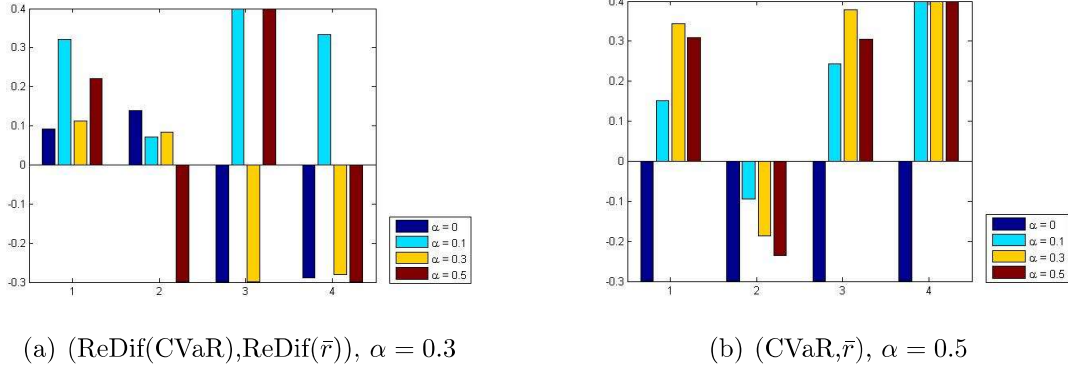
Figure 6: A comparison of instrument holdings$\omega = 0$, $\beta = 0.95$

3.3 Quantitative Measures of Estimation Errors

In order to see how sensitive CVaR portfolio optimization model is to the estimation errors, it is useful to have a quantitative measure of the error casued by using parameters with estimation errors. The distance between a *actual* and *true* (CVaR,$\bar{r}$) profile is one way to measure the error [2]. However absolute distance value can be misleading, a relative distance measure is introduced next. Let $(CVaR^{true}(\alpha), \bar{r}^{true}(\alpha))$ denote the true profile for a given α. Let $(CVaR_s^{actual}(\alpha), \bar{r}_s^{actual}(\alpha))$ represent the sth simulated actual profile (where range from 1 to S). Then the relative-root-mean-squared (RRMS) CVaR-error, denoted $f_{CVaR}(\alpha)$ and the RRMS $\bar{r}$-error, denoted $f_{\bar{r}}(\alpha)$ are given by

$$f_{CVaR}(\alpha) = \sqrt{\frac{1}{K} \sum_{k=1}^{K} \left(\frac{CVaR_k^{actual} - CVaR_k^{true}}{CVaR_k^{true}} \right)^2};$$

$$f_{\bar{r}}(\alpha) = \sqrt{\frac{1}{K} \sum_{k=1}^{K} \left(\frac{\bar{r}_k^{actual} - \bar{r}_k^{true}}{\bar{r}_k^{true}} \right)^2}.$$

Table 3: RRMS-error measures

β	ω	α	f_{CVaR}	$\frac{\Delta f_{CVaR}}{f_{CVaR}}$	$f_{\bar{r}}$	$\frac{\Delta f_{\bar{r}}}{f_{\bar{r}}}$
0.95	0	0.1	0.1267		0.1620	
		0.3	0.2314	0.8262	0.1738	0.0727
		0.5	0.3254	0.4058	0.2011	0.1574
0.95	0.005	0.1	0.1282		0.1721	
		0.3	0.2314	0.8054	0.1824	0.0594
		0.5	0.3250	0.4044	0.2085	0.1432
0.95	0.5	0.1	0.2996		0.1593	
		0.3	0.3584	0.1964	0.1884	0.1829
		0.5	0.4427	0.2354	0.2351	0.2477
0.85	0	0.1	0.1409		0.1694	
		0.3	0.2547	0.8080	0.1855	0.0951
		0.5	0.3544	0.3915	0.2133	0.1500
0.99	0	0.1	0.1061		0.1618	
		0.3	0.2205	1.0784	0.1735	0.0721
		0.5	0.3190	0.4466	0.2007	0.1569

Table 3 column 4 and 6 show how the RRMS increases steadily as α increases for each of the four experimental groups. For instance, $f_{CVaR}(\alpha)$ increases from 0.1267 for $\alpha = 0.1$ to 0.33254 for $\alpha = 0.5$ and $f_{\bar{r}}(\alpha)$ increases from 0.1620 to 0.2011, for $\beta = 0.95$ and $\omega = 0$ group.

There are some other interesting observations as well. First, for $\alpha = 0.1$, the f_{CVaR} is generally smaller than $f_{\bar{r}}$ for four experimental groups. For example, in the first group, that is $\beta = 0.95$ and $\omega = 0$, $f_{CVaR} = 0.1267$ while $f_{\bar{r}} = 0.1620$ for $\alpha = 0.1$. However, f_{CVaR} is larger than $f_{\bar{r}}$ for larger α. This indicates f_{CVaR} increases faster than $f_{\bar{r}}$ as α increase. $f_{CVaR} = 0.2314$, $f_{\bar{r}} = 0.1738$ for $\alpha = 0.3$ and $f_{CVaR} = 0.3254$, $f_{\bar{r}} = 0.2011$ for $\alpha = 0.5$. There is a similar trend in the other three experimental groups. The relative changes in f_{CVaR} and $f_{\bar{r}}$ are shown in column 5 and 7. The formula are given by

$$\frac{\Delta f_{CVaR}}{f_{CVaR}} = \frac{f_{CVaR,2} - f_{CVaR,1}}{f_{CVaR,1}}; \frac{\Delta f_{\bar{r}}}{f_{\bar{r}}} = \frac{f_{\bar{r},2} - f_{\bar{r},1}}{f_{\bar{r},1}};$$

where 2 represents the very next higher α than 1. These number verifies that the changes in f_{CVaR} is larger that that in $f_{\bar{r}}$ as α increase. It brings to the conclusion that CVaR is more sensitive to the estimation errors in current CVaR optimization model.

The second observation is that f_{CVaR} is less sensitive for large α than for small α. The

changes in $f_{\bar{r}}$ is larger for large α than for small α. More specifically, in the first experiment group, $\frac{\Delta f_{CVaR}}{f_{CVaR}}$ for $\alpha = 0.3$ is 0.8492 but this number is 0.4023, less than half, for $\alpha = 0.5$. At the same time, $\frac{\Delta f_{\bar{r}}}{f_{\bar{r}}}$ is 0.1037 for $\alpha = 0.3$ and 0.1473 for $\alpha = 0.5$.

To further verify the above two obeservations, table 3.4 shows how f_{CVaR}, $\frac{\Delta f_{CVaR}}{f_{CVaR}}$, $f_{\bar{r}}$ and $\frac{\Delta f_{\bar{r}}}{f_{\bar{r}}}$ change with α more carefully. RRMS error meaures increase as relative error α increases. f_{CVaR} is in general more sensitive to changes in α. $\frac{\Delta f_{CVaR}}{f_{CVaR}}$ is always larger than $\frac{\Delta f_{\bar{r}}}{f_{\bar{r}}}$. However, f_{CVaR}'s sensitivity is decreasing as α is increasing.

Table 4: $\beta = 0.95$, $\omega = 0$

α	f_{CVaR}	$\frac{\Delta f_{CVaR}}{f_{CVaR}}$	$f_{\bar{r}}$	$\frac{\Delta f_{\bar{r}}}{f_{\bar{r}}}$
0.04	0.1024		0.1659	
0.08	0.1244	0.2139	0.1663	0.0025
0.10	0.1369	0.1010	0.1665	0.0013
0.16	0.1751	0.2789	0.1687	0.0130
0.20	0.1997	0.1408	0.1716	0.0174
0.24	0.2214	0.1083	0.1760	0.0254
0.28	0.2428	0.0969	0.1810	0.0289
0.30	0.2532	0.0426	0.1838	0.0150
0.32	0.2635	0.0408	0.1863	0.0136
0.36	0.2843	0.0789	0.1916	0.0286
0.40	0.3050	0.0728	0.1968	0.0274
0.44	0.3253	0.0667	0.2023	0.0278
0.48	0.3452	0.0611	0.2080	0.0282
0.50	0.3550	0.0285	0.2108	0.0135

Table 4 also shows that results in experimental group $\beta = 0.95$, $\omega = 0.5$ have large CVaR estimation errors f_{CVaR} while $\bar{r}$ estimation errors $f_{\bar{r}}$ are almost the same as the group without transaction cost i.e. $\beta = 0.95$, $\omega = 0$. Moreover, the changes in f_{CVaR}, $\frac{\Delta f_{CVaR}}{f_{CVaR}}$ is much smaller than group $\beta = 0.95$, $\omega = 0$. This together with previous observation suggests that transaction cost reduces the variations in *actual* (CVaR,$\bar{r}$) profiles but increases the estimation errors.

3.4 The Sensitivity of Efficient Frontiers

This section illustrates the effects of estimation errors on the efficient frontiers. The definition of efficient frontier is similar as in the Modern Porfolio Theory. Points on the efficient frontier represents portfolios (explicitly excluding the risk-free alternative) for which there is lowest risk for a given level of return. Conversely, for a given amount of risk, the portfolio lying on the efficient frontier represents the combination offering the best possible return. Similar concepts are used here as in section 3.2 to compute the following three efficient frontiers for CVaR portfolio optimization models:

- *The true efficient frontier* is the curve for $(CVaR,\bar{r})$ pairs computed from different portfolio target mean returns constraints under the true underlying mean return parameters μ.

- *The estimated efficient frontier* is the curve for $(CVaR,\bar{r})$ pairs computed from different portfolio target mean returns constraints under the estimated underlying mean return parameters, $\hat{\mu} = \mu + \Delta\mu$.

- *The actual efficient frontier* is the curve for actual $(CVaR,\bar{r})$ pairs that are computed with the estimated portfolio holdings and the true underlying mean return parameters. The computation procedure is the same as one described in table 2. The actual efficient frontier depicts the true performance of the estimated frontier.

In Figure 9(a), one simulation of estimation efficient frontier and its corresponding actual efficient frontier are graphed with the true efficient frontier. The points on estimated efficient frontier representing very small expected portfolio returns, $\bar{r} < 0.08$, are very close to those on the actual efficient frontier. They are also very close the points of true efficient frontier. The distances between three kinds of efficient frontiers are larger for larger expected portfolio returns. There seems to be a subtle tendency that this difference between actual efficient frontier and estimated frontier (or between actual efficient frontier and true efficient frontier)is becoming larger. Notice that although short selling is allowed here, the bound for instrument holdings is $x \in$ [-0.3, 0.4]. To see if there is such an increasing difference between estimated efficient frontier and actual efficient frontier as the expected portfolio return increases, Figure 9(b) graphs a comparsion for three kinds of efficient frontiers for $x \in$ [-3, 4]. It shows that as larger instrument holdings are allowed, the distance between estimated efficient frontier and actual efficient frontier are increasing.
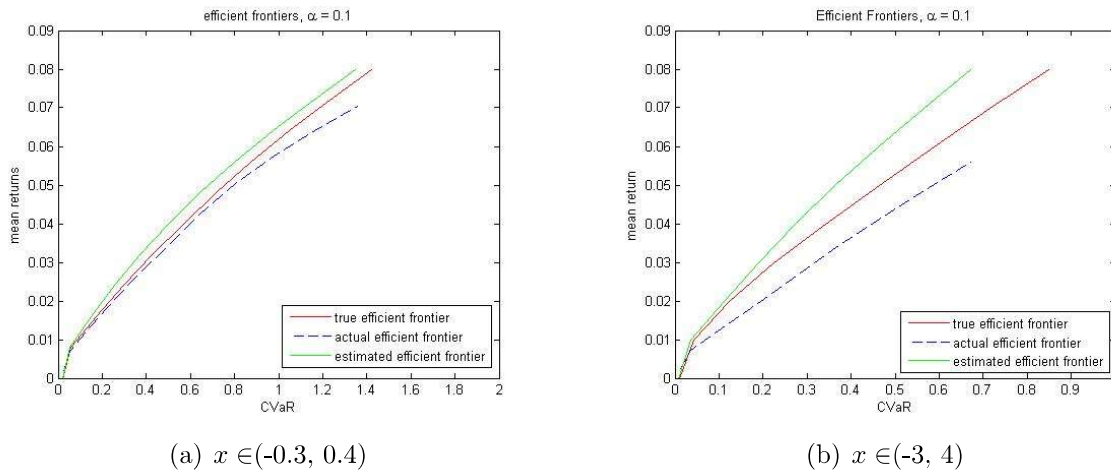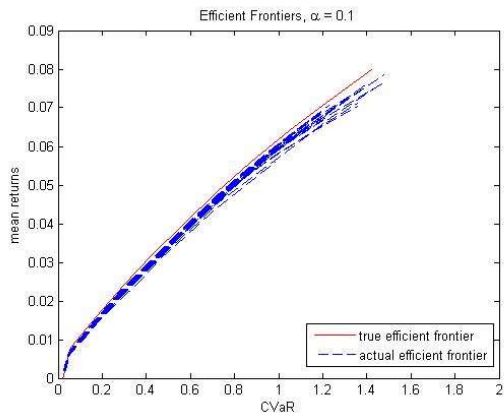
(a) $x \in$(-0.3, 0.4)   (b) $x \in$(-3, 4)

Figure 7: Efficient frontiers for $\alpha = 0.1$ $\omega = 0$ $\beta = 0.95$.

This observation suggests that the CVaR optimization problem with estimation errors can be solved more accurately when the target portfolio return is low. When the target portfolio is high, the estimation errors have a big impact on the resulting optimal portfolios.

The figure 7 are two typical examples for all the simulations. They also show that the estimated efficient frontier generally over-performance and the actual efficient frontier ususlly under-performance compared to true efficient frontier. Figure 8 graph all the simulated *actual* efficient frontiers and the *true* efficient frontier. Figure 9 graph all the simulated *estimated* efficient frontiers and the *true* efficient frontier. They verify this observation. It is very dangerous for investors to use estimated optimal portfolio to guide their investment strategies because it leads to a more aggressive investment strategies than the true optimal portfolio does. The actual performance of the estimated optimal portfolio is much lower than the investors' expectations.

The CVaR optimization problem with large admissible on holdings is more sensitive to the the estimation errors. It attains better performance in the sense that it reached the same target returns with smaller CVaR values. Notice that the CVaR-axis for $x \in$ [-3,4] is [0, 1] while it is [0, 2] for $x \in$ [-0.3,0.4]. This is explained by the large freedom to choose the instrument holdings. If there is no constrains on instrument holdings at all, estimation errors problem can be very servere.
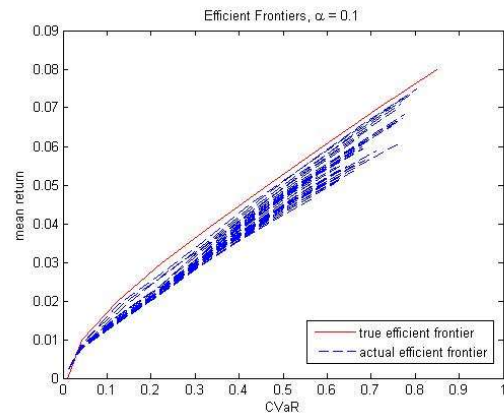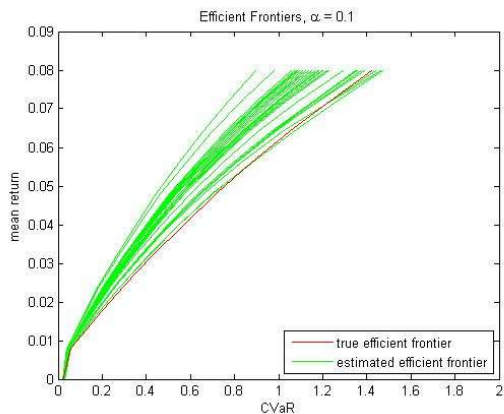
(a) $x \in$(-0.3, 0.4)

(b) $x \in$(-3, 4)

Figure 8: *Actual* and *true* efficient frontier for $\alpha = 0.1$,$\omega = 0$,$\beta = 0.95$

(a) $x \in$(-0.3, 0.4)

(b) $x \in$(-3, 4)

Figure 9: *Estimated* and *true* efficient frontier for $\alpha = 0.1$,$\omega = 0$,$\beta = 0.95$

Figure 10 and 11 graph efficient frontiers for $\alpha = 0.3$. Notice that the instrument holdings lie between -0.3 and 0.4. The observation that the differences between estimated efficient frontier and actual efficient frontier are larger as the target returns increase is even more obvious for $\alpha = 0.3$. The variations of estimated efficient frontiers and actual frontiers are much larger as well.
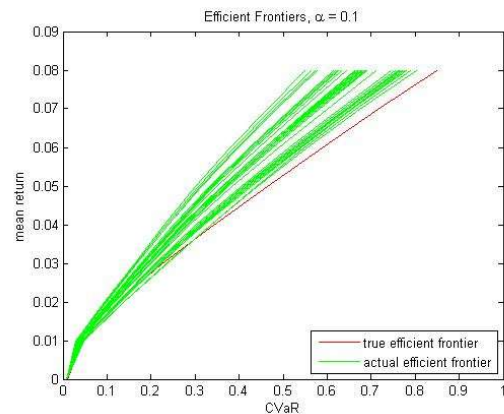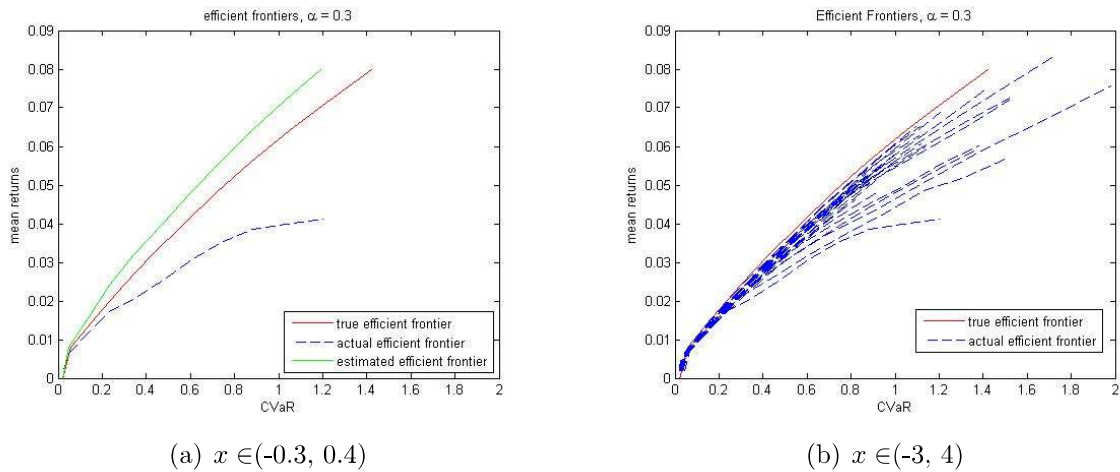
(a) $x \in$(-0.3, 0.4)

(b) $x \in$(-3, 4)

Figure 10: *Actual* and *true* efficient frontier for $\alpha = 0.3, \omega = 0, \beta = 0.95$
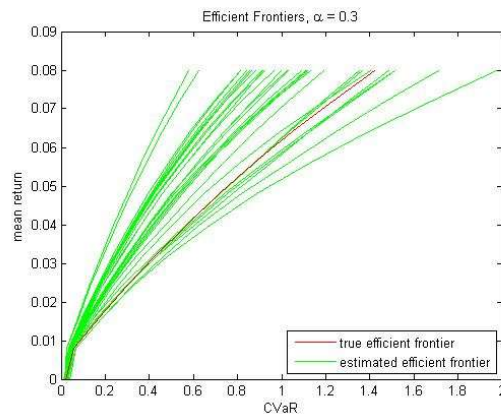
(a) $x \in$(-0.3, 0.4)

Figure 11: *Estimated* and *true* efficient frontier for $\alpha = 0.3, \omega = 0, \beta = 0.95$

Actual efficient frontiers are usually below *estimated* efficient frontiers. This again indicates that *estimated* results are optimistically biased. It is very dangerous to use an estimated optimal portfolio to guidance investors' investment strategy. The corresponding estimated CVaR and portfolio mean return $\bar{r}$ either underestimate the risk inherent to achieve certain return target or overestimate its ability to achieve a return target with a certain risk constraint.

4 Conclusion

Esitmation of the underlying mean returns in the CVaR optimization model inevitably has errors. It is very important to analyze how sensitive the CVaR risk measure, portfolio mean returns, CVaR optimal portfolio and efficient frontiers are to these estimation errors. In this research report, we consider the independent estimation errors in underlying mean returns and study their effects on the CVaR optimization problem.

The report first shows that the optimal CVaR risk measure and portfolio mean returns profile are very sensitive to the relative estimation errors for different confidence levels and transaction cost parameters graphically. A comparison of corresponding CVaR optimal portfolio is also presented. The estimated optimal portfolio is very different from the true optimal portfolio without estimation errors. Some evidence is presented showing that, in this context, the CVaR risk measure is more sensitive to the estimation errors than is the portfolio mean returns measure. The optimization problem solved under higher confidence levels is generally less sensitive to the estimation errors. Finally, the sensitivity of efficient frontiers to estimation error is analyzed. It is shown that with larger instrument holding bounds, the efficient frontiers are more sensitive to estimation errors. Also, the part of efficient frontier representing higher mean portfolio returns and higher CVaR risk measures have more severe estimation errors than the lower mean portfolio returns and lower CVaR risk measures.

In addition to the above efforts to show how sensitive the CVaR optimization model to the estimation errors, this research report considers a performance evaluation of estimated CVaR optimal portfolios. In general, estimation errors can be very dangerous for investors because estimated CVaR optimal portfolios usually have much higher estimated performance than actual performance. The computational results indicate that the estimation errors in underlying mean returns can result in big estimation errors in CVaR optimal portfolios.

APPENDICES

A Description of the Computational Example

This section provides a detail description of the computational example. In this example, we construct a portfolio of derivatives. Assume that there are four correlated assets following the CEV model assumptions

$$\frac{dS^i}{S^i} = \mu^i dt + \frac{c^i}{S^i} dZ^i, i = 1, 2, 3, 4. \tag{11}$$

where the correlation matrix of dZ^1, dZ^2, dZ^3, dZ^4 is Qdt with

$$Q = \begin{bmatrix} 1.0000 & 0.3769 & 0.1003 & 0.4596 \\ 0.3769 & 1.0000 & 0.3959 & 0.6372 \\ 0.1003 & 0.3959 & 1.0000 & 0.3138 \\ 0.4596 & 0.6372 & 0.3138 & 1.0000 \end{bmatrix} \tag{12}$$

$$c^1 = 53.7587, c^2 = 17.0294, c^3 = 4.4497, c^4 = 28.1069 \tag{13}$$

The annual expected returns are

$$\mu^1 = 0.1091, \mu^2 = 0.0619, \mu^3 = 0.0279, \mu^4 = 0.0649 \tag{14}$$

and the initial prices are

$$S_0^1 = 100, S_0^2 = 50, S_0^3 = 30, S_0^4 = 100 \tag{15}$$

For each asset, we consider 12 standard calls, 12 standard puts, 12 digital calls and 12 digital puts. Here the strike and expiry are all possible combinations of 3 strikes $[0.8, 1.025, 1.25]S_0$ and 4 expiries $[2,4,6,8]\bar{t}$ where $\bar{t}$ is 10 days (assuming 250 trading days in a year). Therefore there are 196 investment instruments including 192 options plus the 4 underlying assets. The required portfolio return $\bar{r}$ is twice the risk freee rate over the time horizon $[0,\bar{t}]$ with the annual risk free rate $r = 5\%$.[9]

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