Optimal Trade Execution under Constant Volatility and Realized GARCH Volatility Model

by

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I hereby declare that I am the sole author of this report. This is a true copy of the report, including any required final revisions, as accepted by my examiners.

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Abstract

Current literature on optimal trade execution generally assumes a constant volatility model over the execution time horizon. Using Monte Carlo simulations, we compare execution cost evaluation between a recent Realized GARCH volatility model and the constant volatility Geometric Brownian Motion, and the results show that under RGARCH model trading cost is less sensitive to volatility change. In addition we consider volatility-based market impact functions. We then compute optimal strategies by minimizing mean and Conditional Value-at-Risk (CVaR) of execution costs. The monte carlo simulation with smoothing method we proposed can solve the nonlinear optimal execution problem easily. The efficient frontiers of optimal execution cost show that RGARCH model is always more preferable to GBM, especially considering either expected costs alone or risk alone.

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To my parents

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1 Introduction

The execution costs of stock and portfolio liquidation have been recognized as a large determinant in investment performance (see, for example, Freyre-Sanders, Guobuzaite, and Byrne (2004)). A general goal in recent literature is to find the optimal execution strategies under a particular optimal execution formulation, such as the minimum mean cost approach of Bertsimas and Lo (1998), the mean and variance approach of Almgren and Chriss (2000), and the mean-quadratic-variation approach of Forsyth, Kennedy, Tse, and Windcliff (2012). However, most of these studies assume constant volatility over the execution time horizon.

Stochastic volatility models have been intensively discussed and adopted to a wide variety of applications, such as option pricing (see Duan (1995)) and portfolio risk management (see Gron, Jorgensen, and Polson (2011)). Past studies, including Almgren, Thum, Hauptmann and Li (2005), and Beltran, Durré, Giot (2009), also provide evidence that increased volatility is a dominant cause of higher trading costs in stock transactions. Almgren (2012) discussed the mean-variance optimal trade strategies based on stochastic volatility and liquidity, but they used linear impact models to simplify discussion.

In this paper, we investigate the potential use of a recently proposed Realized GARCH volatility model by Hansen, Huang and Shek (2011), for execution costs modeling and analysis. We use Monte Carlo simulation to compare the execution costs to that of the constant volatility Geometric Brownian Motion (GBM). To evaluate trading cost, we also need to choose the market impact models that account for price shifts in respond to stock trades. As a benchmark for comparison, we first use the linear permanent and temporary impact functions under GBM and Realized GARCH. Then we contrast that to the GBM and Realized GARCH model under the volatility-based temporary impact functions. Using Monte Carlo Simulations under same initial volatility, we can obtain some comparison results for the naïve trading strategy, i.e. trading equal amount in each period over the trading horizon. Our analysis of the trading costs can also be applied to any other latent volatility models.

Our next investigation is to compute optimal execution strategy by minimizing mean and CVaR of the execution costs under GBM and Realized GARCH. To keep the discussion simple, we ignore permanent impact in the optimization formulation as it is relatively small compared to temporary impact. This yields a piecewise differentiable nonlinear convex optimization problem. To resolve the non-differentiable points in the objective function, we propose a change-of-variable as well as a smoothing technique, and the latter gives significant reduction in computational complexity. This approach can also be applied to minimize mean and other types of risk measure, such as variance and down-side risk.

2 Stock Price and Trading Impact Models

To effectively estimate the trading costs of a particular financial security, first we need to model normal price dynamics of the securities to be traded. In addition, we need to model the price shifts in respond to large stock trading activities.

2.1 Price Dynamics

In this section we introduce models of price dynamics of a financial security.

Suppose at time 0 we hold X units of a single security, that we wish to completely liquidate this asset before time T. We divide $[0, T]$ into N discrete intervals of equal time length

$$\Delta t = T/N$$

To simplify notation, we define $t = 1 \dots N$ as each trading stage, and n_t as the number of units we want to sell between $t-1$ and t.

We assume the initial price of the security is S_0, thus the initial market value of our position is XS_0, and the security price S_t follows a geometric random walk

$$S_t = S_{t-1}\left(1 + r_t - \Delta t g\left(\frac{n_t}{\Delta t}\right)\right),\ t = 1 \dots N \tag{1}$$

where r_t represents the return of the asset, and the permanent market impact $g(v)$ is a function of the average rate of trading $v = \frac{n_t}{\Delta t}$ during the interval $t-1$ and t.

2.1.1 Geometric Brownian Motion (GBM)

As our goal is to consider the trading costs under a stochastic volatility model, we want to compare its trading cost to traditional constant volatility model. Therefore we use the well-known constant volatility Geometric Brownian Motion as a benchmark for comparison

$$r_t^{GBM} = \sqrt{h} z_t \tag{2}$$

where $\sqrt{h}$ is the constant volatility parameter, $z_t \sim iid\,(0,1)$. Here the return r_t^{GBM} replaces r_t in the price dynamics model (1) (same with r_t^{RG}).

2.1.2 The Realized GARCH (1, 2) Model

Latent volatility models have been successful in modeling financial returns since the early seminal work published by Engle (1982). The recent Realized GARCH model of Hansen, Huang and Shek (2011) provides a framework that jointly models return and realized measure of volatility. Due to its generality in nesting existing ARCH and GARCH models, and its improvement from standard GARCH models, it has generated wide interests since its publication. We have chosen this model as the focus of our discussion in estimating trading costs under a stochastic volatility.

The structure of the Log-Linear specification of the Realized GARCH(p, q) model can be described as follows

$$\begin{aligned} r_t^{RG} &= \sqrt{h_t} z_t \qquad (3) \\ \log h_t &= \omega + \sum_{i=1}^{p} \beta_i \log h_{t-i} + \sum_{j=1}^{q} \gamma_j \log x_{t-j} \\ \log x_t &= \xi + \phi \log h_t + \tau\left(z_t\right) + u_t \end{aligned}$$

Except for change of notation, the parameters are the same as those in Hansen, Huang and Shek (2011). Here the return r_t^{RG} replaces r_t in the price dynamics model (1), $z_t \sim iid\,(0,1)$, $h_t = \mathrm{var}\left(r_t|\mathcal{F}_{t-1}\right)$ is the stochastic volatility parameter, with $\mathcal{F}_t = \sigma(r_t, x_t, r_{t-1}, x_{t-1}\ldots)$, $u_t = iid\,(0, \sigma_u^2)$, and $\tau\left(z_t\right)$ is the leverage function, for which we choose the quadratic form $\tau\left(z_t\right) = \tau_1 z_t + \tau_2\left(z_t^2 - 1\right)$ because it ensures that $\mathbf{E}\left(\tau\left(z_t\right)\right) = 0$, for any distribution z_t, as long as $\mathbf{E}\left(z_t\right) = 0$, and $\mathrm{var}\left(z_t\right) = 1$.

We choose the Log-Linear specification of the Realized GARCH model, due to its appealing features: it ensures positive variance, and it does not require non-zero returns, as they are occasionally observed in practice.

Following the test results and recommendations of Hansen, Huang and Shek (2011), We are particularly interested in the Realized GARCH(1, 2) model (RGARCH), i.e.

$$\log h_t = \omega + \beta \log h_{t-1} + \gamma_1 \log x_{t-1} + \gamma_2 \log x_{t-2} \tag{4}$$

To estimate the parameters in the RGARCH model, we follow the Quasi-Maximum Likelihood Analysis of Hansen, Huang and Shek (2011) with the data set sample

$$\{r_t | t = s_0 \ldots s_T\}$$

$$\{x_t | t = s_0 \ldots s_T\}$$

where t_B marks the beginning of sample, and t_E marks the end of sample.

Note: the choice of stochastic volatility model is not limited to the RGARCH model. Our discussion is also applicable to any other latent volatility models.

2.2 Market Impact Functions

The next step of our model is to determine the price impact on trading a particular security. Following the discussion of Almgren and Chriss (2000), we assume two types of price impact. Temporary market impact is caused by temporary change in price caused by our trading as it drains liquidity from the market; it only affects the price of orders we are trading at the current trading stage. Permanent impact characterizes permanent change in the security price due to our trading, which remains at least until the end of the trading horizon.

2.2.1 Permanent Market Impact Function

We have assumed the permanent market impact in the price dynamics (1). Huberman and Stanzl (2004) showed that permanent market impact must be linear such that no quasi-arbitrage is possible

$$g\left(\frac{n_t}{\Delta t}\right) = \ddot{\mu} \frac{n_t}{\Delta t}$$

where $\ddot{\mu}$ is a constant multiplier.

2.2.2 Temporary Market Impact

We define the execution price S_t^* as follows

$$S_t^* = S_{t-1}\left(1 - f\left(\frac{n_t}{\Delta t}\right)\right) \tag{5}$$

Note: the temporary market impact function $f\left(\frac{n_t}{\Delta t}\right)$ only affects execution price in the current trading stage. Here we provide a review of several different specifications of temporary impact function f.

Similar to permanent impact, Almgren and Chriss (2000) use the linear temporary impact function

$$\ddot{f}\left(\frac{n_t}{\Delta t}\right) = \ddot{\epsilon}\,\text{sgn}\left(n_t\right) + \frac{\ddot{\eta}}{\Delta t} n_t$$

where $\ddot{\epsilon}\,\text{sgn}\left(n_t\right)$ is the fixed cost of selling, such as half of the bid-ask spread plus certain transaction fees, and $\ddot{\eta}$ is a constant parameter.

Non-linear impact functions were also proposed, such as the power law function of Almgren, Thum, Hauptmann, and Li (2005)

$$\hat{f}\left(\frac{n_t}{\Delta t}\right) = \text{sgn}\left(n_t\right)\widehat{\eta}\left|\frac{n_t}{\Delta t}\right|^{\alpha}$$

Grinold and Kahn (1994) and Toth et. (2011) investigated the volatility-based impact function, and provided evidence of good approximation of the model

$$\tilde{f}\left(\frac{n_t}{\Delta t}\right) = \text{sgn}\left(n_t\right)\left(\tilde{\epsilon} + \frac{\widetilde{\eta}}{\Delta t}\sigma_t\sqrt{\frac{|n_t|}{EDV}}\right) \tag{6}$$

2.3 Capture and Cost of Trading Strategies

Once we have price dynamics and trading impact, we can estimate the trading costs under a particular trading strategy. We define a trading strategy as

$$\{n_1 \dots n_N\}$$

$$\text{subject to } \begin{cases} \sum_{t=1}^{N} n_t = X \\ n_t \geq 0, t = 1 \dots N \end{cases}$$

The capture of a trading strategy

$$\sum_{t=1}^{N} n_t S_t^*$$

And the implementation shortfall is:

$$XS_0 - \sum_{t=1}^{N} n_t S_t^*$$

Although in practice trading costs also involve human labor, systems and other costs, we only consider trading costs as the implementation shortfall, which is the direct financial loss and the most easily quantifiable costs of security liquidation. In the rest of the paper we compare the trading costs under the GBM and the RGARCH model.

3 Monte Carlo Simulation

To evaluate the execution costs of a particular security, we report the mean, variance, Value-at-Risk with confidence level 95% (VaR 95%) and Conditional Value-at-Risk (CVaR 95%). As the analytical expression of these measures under the RGARCH model is very complicated, we use M Monte Carlo simulations instead to approximate them. For example, the mean of execution costs can be approximated as

$$\mathbf{E}\left(XS_0 - \sum_{t=1}^{N} n_t S_t^*\right) \approx \frac{1}{M}\sum_{j=1}^{M}\left(XS_0 - \sum_{t=1}^{N} n_t S_t^{*(j)}\right)$$

where M is the number of independent simulations, and j indicates the jth scenario.

3.1 Initialization Issues

For a reasonable comparison, we need same constant parameters for both GBM and RGARCH. Then we also need to set the initial volatility for both models (2) (3) to be equal, i.e. $h = h_0$. In addition, we want to infer the initial volatility h_0 from reasonable observations of realized measures of volatility x_{-2} and x_{-1} before the simulation starts

$$\log h_0 = \omega + \beta \log h_0 + \gamma_1 \log x_{-1} + \gamma_2 \log x_{-2}$$

$$\Rightarrow h_0 = \exp\left(\frac{\omega + \gamma_1 \log x_{-1} \ + \gamma_2 \log x_{-2}}{1-\beta}\right) \tag{7}$$

To make our comparison realistic, we choose the realized measures of volatility from the data sample

$$\begin{array}{ll} \min_{x^*_{-2}, x^*_{-1}} & (x^*_{-2} - x_{-2})^2 + (x^*_{-1} - x_{-1})^2 \\ s.t. & \{x^*_{-2}, x^*_{-1}\} \in \{\{x_t, x_{t+1}\} | t = s_0 \ldots s_T - 1\} \end{array}$$

where x^*_{-2} and x^*_{-1} are the real values we use in simulations to replace x_{-2} and x_{-1} in (7).

To make our comparison comprehensive, we choose several reasonable scenarios for x_{-2} and x_{-1}. Specifically we consider average initial volatility estimated from the RGARCH model, high initial volatility, and low initial volatility.

3.1.1 Average Initial Volatility

We define expected realized measure $\overline{x}$ as the average of all x_t over the estimation period. The average volatility $\overline{h}$ is the value of volatility we obtain by setting

$$x_{-2} = x_{-1} = \overline{x}$$

Then we can infer $\overline{h}$ from (7), same below.

3.1.2 High Initial Volatility

To determine the effect of increased initial volatility on trading costs, we define the high with low initial volatility h^{high} as of when high realized measures of volatility are observed

$$\log x_{-2} \ = \log x_{-1} \ = \log \overline{x} \ + \sigma_u$$

3.1.3 Low Initial Volatility

Similarly, we set the low initial volatility h^{low} as of when

$$\log x_{-2} \ = \log x_{-1} \ = \log \overline{x} \ - \sigma_u$$

3.2 Empirical Analysis

In this section we present results using returns and realized measures of volatility for 18 stock indices and their respected exchange-traded funds (ETFs).

3.2.1 Data Description

We use the daily open-to-close stock index data from Jan 1, 2000 to Dec 31 2001 to estimate the RGARCH model parameters. Following the methods described in 3.1, we also compute the initial volatility from this sample.

Since we cannot trade the stock indices directly, we select one index-tracking ETF for each stock index in our sample to compute the average daily open price S_0, average daily volume EDV and average bid-ask spread. To keep the study up-to-date, we use the open-to-close index ETF data from Jan 1, 2012 to Jun 30, 2012.[1]

3.2.2 Choice of Parameters

For a reasonable comparison of different execution cost models, we choose the following parameters

$$M = 10,000,000$$

$$X = 1/5 \ of \ average \ daily \ trading \ volume \ (shares)$$

$$T = 5 \ (days)$$

$$N = 5$$

In addition, we choose the naïve strategy as an example of cost comparison, i.e. trading equal amount of volume over 5 stages

$$n_t = \overline{n} = X/N, \ t = 1 \ldots N \tag{8}$$

We follow the same approach from Almgren and Chriss (2000) to obtain parameters for the linear impact functions, except we need to scale the parameters by initial price to account for the geometric model

$$\ddot{\mu} = \frac{bid \ ask \ spread}{10\% \ of \ EDV * S_0} \ (\$/share^2)$$

[1]This sample is obtained from ©Bloomberg

$$\ddot{\epsilon} = \frac{half\ of\ bid\ ask\ spread}{S_0}$$

$$\ddot{\eta} = \frac{bid\ ask\ spread}{1\%\ of\ EDV * S_0}\ (\$/share)/(share/day)$$

For the volatility-based volatility impact function $\tilde{f}$ to be comparable to the linear impact based function $\ddot{f}$, we set the parameters such that their costs are equal if we are trading the average volume $n_t = \overline{n}$ under average volatility $\sigma_t = \sqrt{\overline{h}}$

$$\ddot{f}\left(\frac{\overline{n}}{\Delta t}\right) = \tilde{f}\left(\frac{\overline{n}}{\Delta t}\right)$$

Thus we can obtain the parameters for $\tilde{h}\left(\frac{n_t}{\Delta t}\right)$

$$\tilde{\epsilon} = \frac{half\ of\ bid\ ask\ spread}{S_0}$$

$$\widetilde{\eta} = \frac{20 * bid\ ask\ spread}{S_0 * average\ volatility}\ (\$/share)/(share/day)$$

3.3 Simulation Results

In this section, we present our test result of the Monte Carlo Simulations. To save space, we report some selected comparison results of the trading costs under GBM and RGARCH: Table 2 shows trading costs based on linear impact models, with average initial volatility. Table 3 shows trading costs under linear impact models, with high initial volatility. Table 4 shows trading costs under volatility-based impact models, with high initial volatility. Table 5 shows the initial volatiilty, observed realized measures of volatility and the expected volatility at the last stage of the RGARCH model.

Index Code	ω	β	γ_1	γ_2	ξ	ϕ	τ_1	τ_2	σ_u
SPX	0.11	0.70	0.43	-0.18	-0.37	1.01	-0.10	0.10	0.45
FTSE	0.06	0.73	0.27	-0.06	-0.28	1.13	-0.07	0.11	0.47
N225	0.09	0.66	0.35	-0.10	-0.21	1.05	-0.02	0.14	0.45
GDAXI	0.03	0.76	0.39	-0.17	-0.07	0.99	-0.12	0.11	0.42
RUT	0.37	0.69	0.36	-0.04	-1.00	0.77	-0.07	0.15	0.61
DJI	0.13	0.70	0.41	-0.12	-0.41	0.93	-0.09	0.07	0.43
IXIC	0.14	0.79	0.50	-0.31	-0.53	1.00	-0.08	0.10	0.48
FCHI	0.08	0.71	0.29	-0.04	-0.23	1.00	-0.05	0.13	0.42
HSI	0.07	0.79	0.20	-0.05	-0.37	1.21	-0.08	0.14	0.38
KS11	0.11	0.72	0.43	-0.22	-0.37	1.17	-0.01	0.12	0.44
AEX	0.01	0.72	0.46	-0.17	-0.06	0.90	-0.12	0.07	0.44
SSMI	0.15	0.67	0.42	-0.10	-0.51	0.96	-0.08	0.11	0.35
IBEX	0.08	0.78	0.34	-0.17	-0.36	1.09	-0.08	0.12	0.38
MXX	0.25	0.80	0.49	-0.28	-1.00	0.73	-0.04	0.12	0.58
BVSP	0.28	0.69	0.34	-0.15	-1.00	1.22	-0.03	0.14	0.47
STOXX50E	0.11	0.76	0.29	-0.10	-0.49	1.12	-0.12	0.13	0.50
FTSTI	0.15	0.72	0.37	-0.14	-0.59	1.04	-0.01	0.12	0.44
FTSEMIB	0.09	0.72	0.40	-0.15	-0.31	0.98	-0.09	0.11	0.49

Table 1: Estimated parameters for the log-linear Realized GARCH(1,2) model

		Constant Volatility				**Realized RGARCH**			
Stock Index	Index ETF	Mean	Std	VaR (95%)	CVaR (95%)	Mean	Std	VaR (95%)	CVaR (95%)
SPX	SPY	5.9	301	497	619	5.9	300	500	636
FTSE	ISF	137	1,025	1,814	2,232	135	1,022	1,811	2,253
N225	NKY	8,002	19,008	39,059	46,774	7,976	19,011	38,972	47,358
GDAXI	DAXEX	18.7	141	248	305	18.5	141	250	313
RUT	IWM	6.0	204	339	420	6.2	204	340	432
DJI	DIA	6.8	255	422	525	6.9	254	426	538
IXIC	QQQ	6.3	244	402	498	6.4	244	405	513
FCHI	CAC FP	9.7	73	129	158	9.8	73	129	160
HSI	2800 HK	23.5	43	93	110	23.4	43	93	111
KS11	EWY	7.2	165	275	341	7.3	165	275	347
AEX	IAEX	12.4	58	107	130	12.5	58	108	134
SSMI	CSSMI	16.0	97	175	215	16.0	97	176	220
IBEX	BBVAI	12.9	17	41	47	12.9	17	41	48
MXX	NAFTRAC	10.1	105	180	222	10.4	105	181	228
BVSP	BOVA11	20.1	182	317	389	20.0	182	316	394
STOXX50E	SX5EEX	10.9	57	103	126	10.9	56	104	129
FTSTI	STTF SP	8.4	5	17	19	8.4	5	17	19
FTSEMIB	LEVMIB	5.4	13	27	33	5.4	13	28	34

Table 2: Trading costs (cents/share) with average initial volatility under linear impact models

		Constant Volatility				Realized GARCH			
Stock Index	Index ETF	Mean	Std	VaR (95%)	CVaR (95%)	Mean	Std	VaR (95%)	CVaR (95%)
SPX	SPY	6.2	347	571	710	5.9	343	568	724
FTSE	ISF	135	1,286	2,231	2,752	136	1,276	2,224	2,771
N225	NKY	7,955	21,933	43,736	52,600	7,961	21,647	43,218	52,700
GDAXI	DAXEX	18.5	173	300	369	18.5	172	301	378
RUT	IWM	5.7	283	465	577	6.3	278	458	581
DJI	DIA	6.6	320	527	656	6.6	317	528	666
IXIC	QQQ	5.7	331	539	666	6.0	329	540	684
FCHI	CAC FP	9.6	90	156	193	9.6	89	155	194
HSI	2800 HK	23.5	51	106	127	23.4	51	106	128
KS11	EWY	7.3	198	328	406	7.4	196	325	410
AEX	IAEX	12.5	75	134	164	12.5	74	135	168
SSMI	CSSMI	16.0	119	211	259	15.9	118	210	263
IBEX	BBVAI	13.0	21	47	56	12.9	21	47	57
MXX	NAFTRAC	10.3	141	239	296	10.3	138	234	295
BVSP	BOVA11	19.8	209	359	442	20.1	205	354	442
STOXX50E	SX5EEX	11.0	67	120	147	10.9	66	120	149
FTSTI	STTF SP	8.4	6	18	21	8.3	6	18	21
FTSEMIB	LEVMIB	5.4	17	33	40	5.4	17	33	40

Table 3: Trading costs (cents/share) with high initial volatility under linear impact models

		Constant Volatility				Realized GARCH			
Stock Index	Index ETF	Mean	Std	VaR (95%)	CVaR (95%)	Mean	Std	VaR (95%)	CVaR (95%)
SPX	SPY	6.6	347	572	711	6.1	342	563	703
FTSE	ISF	162	1,284	2,255	2,779	159	1,273	2,233	2,752
N225	NKY	8,776	21,948	44,574	53,372	8,524	21,642	43,858	52,732
GDAXI	DAXEX	21.0	173	303	372	20.6	172	300	370
RUT	IWM	7.8	283	467	579	7.3	277	457	569
DJI	DIA	8.9	320	530	659	7.9	317	525	654
IXIC	QQQ	7.2	331	540	668	7.2	330	538	669
FCHI	CAC FP	11.2	90	158	194	10.8	89	156	192
HSI	2800 HK	26.5	51	109	130	26.2	50	108	129
KS11	EWY	8.4	198	329	407	8.2	197	327	406
AEX	IAEX	15.0	75	137	167	14.6	74	135	166
SSMI	CSSMI	18.1	119	213	261	17.9	119	211	260
IBEX	BBVAI	15.2	21	49	58	14.8	21	49	58
MXX	NAFTRAC	12.6	141	242	298	12.2	138	236	292
BVSP	BOVA11	21.6	209	361	444	20.4	206	355	438
STOXX50E	SX5EEX	12.3	67	121	148	12.0	66	120	147
FTSTI	STTF SP	9.4	6	19	22	9.1	6	19	22
FTSEMIB	LEVMIB	6.4	17	34	40	6.2	17	33	40

Table 4: Trading costs (cents/share) with high initial volatility under volatility-based impact models

		Average Initial Volatility				**High Initial Volatility**				**Low Initial Volatility**			
Stock Index	σ_u	x^*_{-2}	x^*_{-1}	h_0	$\overline{h_5}$	x^*_{-2}	x^*_{-1}	h_0	$\overline{h_5}$	x^*_{-2}	x^*_{-1}	h_0	$\overline{h_5}$
SPX	0.45	1.52	1.62	2.26	2.16	2.32	2.35	2.99	2.63	0.95	0.91	1.30	1.42
FTSE	0.47	1.25	1.23	1.47	1.45	1.91	2.12	2.32	2.12	0.75	0.76	1.02	1.06
N225	0.45	1.74	1.71	1.92	1.87	2.57	2.53	2.55	2.22	1.15	1.14	1.43	1.56
GDAXI	0.42	2.58	2.51	2.53	2.48	3.72	3.78	3.83	3.57	1.68	1.78	1.96	2.02
RUT	0.61	0.90	0.92	3.02	2.96	1.69	1.74	5.79	4.64	0.49	0.51	1.66	1.96
DJI	0.43	1.18	1.20	1.81	1.78	1.73	1.88	2.85	2.56	0.77	0.77	1.20	1.28
IXIC	0.48	4.30	4.24	6.74	6.69	7.31	7.61	12.35	11.37	2.68	2.87	5.37	5.88
FCHI	0.42	1.87	1.90	2.26	2.18	2.57	3.06	3.46	3.05	1.23	1.29	1.64	1.70
HSI	0.38	1.74	1.71	2.04	2.00	2.28	2.63	2.89	2.71	1.19	1.19	1.57	1.59
KS11	0.44	3.43	3.44	3.82	3.81	5.63	5.60	5.51	5.08	2.24	2.20	2.68	2.86
AEX	0.44	1.46	1.47	1.56	1.52	2.12	2.31	2.60	2.38	0.91	0.95	1.02	1.07
SSMI	0.35	0.74	0.71	1.13	1.12	1.01	1.06	1.71	1.60	0.52	0.51	0.82	0.84
IBEX	0.38	1.73	1.73	2.22	2.18	2.48	2.74	3.47	3.21	1.18	1.22	1.73	1.79
MXX	0.58	0.98	0.97	3.43	3.38	1.74	1.74	6.27	5.06	0.52	0.56	2.18	2.57
BVSP	0.47	2.43	2.31	4.19	4.07	3.91	3.63	5.50	4.78	1.49	1.52	3.33	3.57
STOXX50E	0.50	1.83	1.84	2.55	2.51	3.34	2.98	3.57	3.33	1.16	1.17	1.78	1.85
FTSTI	0.44	0.90	0.86	1.44	1.43	1.40	1.36	2.13	1.93	0.60	0.59	1.07	1.14
FTSEMIB	0.49	1.76	1.77	2.35	2.32	2.97	2.93	3.69	3.36	1.06	1.05	1.45	1.55

Table 5: Volatility parameters of the RGARCH model under different initial volatility

The results demonstrate that in both models increased initial volatility leads to substantial increase in risk, and the expected costs also increase under volatility-based impact model. In the same way expected costs and risk decrease with volatility for both models (not reported).

In addition, if we start trading with high initial volatility, the expected volatility under RGARCH model generally declines over time, therefore the trading costs under RGARCH and volatility-based impact models are lower than that under GBM. In the same way, the costs under RGARCH are higher than GBM with low initial volatility (not reported), which makes RGARCH model less sensitive to volatility change than GBM.

4 Optimal Execution

Our next step is to formulate the optimal execution problem under the RGARCH model. First we need to compute the optimal execution strategies obtained under both GBM and RGARCH models. Then instead of comparing both strategies under their respective models, we are interested in comparing the performance of the strategies under the RGARCH model, since it apparently provides much better fit to the market.

4.1 Optimal Execution Problem

Following the assumptions of Moazeni, Coleman and Li (2010), the decision maker wants to minimize the mean and CVaR of the implementation shortfall, which leads to the stochastic programming problem:

$$\min_{\substack{n_1 \dots\ n_N \\ n_t : \mathcal{F}_t - \text{measurable}}} \quad \mathbf{E}\left(XS_0 - \sum_{t=1}^{N} n_t S_t^*\right) + \lambda\Phi\left(XS_0 - \sum_{t=1}^{N} n_t S_t^*\right) \tag{9}$$

$$\text{subject to} \quad \begin{cases} \sum_{t=1}^{N} n_t = X \\ n_t \geq 0, t = 1 \dots N \end{cases}$$

Φ is a risk measure of the cost, and λ is a constant risk tolerance parameter. In particular, Φ is often chosen as Variance, VaR, or CVaR.

4.2 Minimizing Mean and CVaR

In this paper, we focus our discussion of risk measure on CVaR, following the definition of Rockafellar and Uryasev (2000)

$$\mathrm{CVaR}_p(Y) = \min_a \left(a + \frac{1}{1-p}\mathbf{E}\left([Y-a]^+\right)\right)$$

So the objective function becomes

$$\min_{\substack{n_1 \dots\ n_N, a \\ n_t : \mathcal{F}_t - \text{measurable}}} \quad \mathbf{E}\left(XS_0 - \sum_{t=1}^{N} n_t S_t^* \right) + \lambda \mathrm{CVaR}_p \left(XS_0 - \sum_{t=1}^{N} n_t S_t^* \right) \tag{10}$$

which is equivalent to

$$\min_{\substack{n_1 \dots\ n_N, a \\ n_t : \mathcal{F}_t - \text{measurable}}} \quad \mathbf{E}\left(XS_0 - \sum_{t=1}^{N} n_t S_t^* \right) + \lambda a + \frac{\lambda}{1-p}\mathbf{E}\left(\left[XS_0 - \sum_{t=1}^{N} n_t S_t^* - a \right]^+ \right)$$

As many studies of optimal execution have been conducted on linear impact models (see, for example, Almgren (2012)), we focus on the volatility-based temporary impact function (6)

$$S_t^* = S_{t-1}\left(1 - \tilde{\epsilon} - \frac{\widetilde{\eta}}{\Delta t}\sigma_t \sqrt{\frac{n_t}{EDV}} \right)$$

Note that Moazeni, Coleman and Li (2010) developed a computational technique to solve this stochastic dynamic programming problem, but based only on constant volatility GBM. Our goal here is to obtain a static strategy by solving for the constant values of $n_1 \dots\ n_N$ (under $\mathcal{F}_0$)

$$\min_{n_1 \dots\ n_N, a} \mathbf{E}\left(XS_0 - \sum_{t=1}^{N} n_t S_t^* \right) + \lambda a + \frac{\lambda}{1-p}\mathbf{E}\left(\left[XS_0 - \sum_{t=1}^{N} n_t S_t^* - a \right]^+ \right) \tag{11}$$

4.3 Monte Carlo Simulation Method

Since the objective function (11) does not have a simple analytic expression, we again use the Monte Carlo Simulation approach to approximate it

$$\min_{n_1 \dots n_N, a} \frac{1}{M} \sum_{j=1}^{M} \left(XS_0 - \sum_{t=1}^{N} n_t S_t^{*(j)} \right) + \lambda a + \frac{\lambda}{M(1-p)} \sum_{j=1}^{M} \left(XS_0 - \sum_{t=1}^{N} n_t S_t^{*(j)} - a \right)^+ \quad (12)$$

If we consider the execution price S_t^* as it is defined, the objective function becomes a very complex nonlinear polynomial. One way to formulate a convex optimization problem is to ignore the permanent impact function $g\left(\frac{n_t}{\Delta t}\right)$ as it is relatively small compared to temporary impact $f\left(\frac{n_t}{\Delta t}\right)$. This way we have the new price function $\breve{S}_t$

$$\breve{S}_t = \breve{S}_{t-1}(1 + r_t), \ \breve{S}_0 = S_0$$

As far as optimization is concerned, $\breve{S}_t, \ t = 1 \dots N$ are constants, and the execution price $\breve{S}_t^*$ becomes

$$\breve{S}_t^* = \breve{S}_{t-1}\left(1 - f\left(\frac{n_t}{\Delta t}\right)\right)$$

The objective function becomes

$$\min_{n_1 \dots n_N, a} \frac{1}{M} \sum_{j=1}^{M} \left(XS_0 - \sum_{t=1}^{N} n_t \breve{S}_t^{*(j)} \right) + \lambda a + \frac{\lambda}{M(1-p)} \sum_{j=1}^{M} \left(XS_0 - \sum_{t=1}^{N} n_t \breve{S}_t^{*(j)} - a \right)^+ \quad (13)$$

4.3.1 Change of Variable Method

To better formulate the problem, we can use a change of variables

$$y_j = \left(XS_0 - \sum_{t=1}^{N} n_t \breve{S}_{t-1}^{(j)} \left(1 - \tilde{\epsilon} - \frac{\widetilde{\eta}}{\Delta t} \sigma_t \sqrt{\frac{n_t}{EDV}}\right) - a \right)^+, \ j = 1 \dots M$$

The problem then becomes

$$\min_{\substack{n_1 \dots n_N, \\ a, y_1 \dots y_M}} \quad XS_0 - \frac{1}{M} \sum_{j=1}^{M} \sum_{t=1}^{N} \left(n_t \breve{S}_{t-1}^{(j)} \left(1 - \tilde{\epsilon} - \frac{\widetilde{\eta}}{\Delta t} \sigma_t \sqrt{\frac{n_t}{EDV}}\right)\right) + \lambda a + \frac{\lambda}{M(1-p)} \sum_{j=1}^{M} y_j \quad (14)$$

$$\text{subject to} \begin{cases} \sum_{t=1}^{N} n_t = X \\ n_t \geq 0, & t = 1 \dots N \\ y_j \geq 0, & j = 1 \dots M \\ y_j \geq XS_0 - \sum_{t=1}^{N} n_t \breve{S}_{t-1}^{(j)} \left(1 - \tilde{\epsilon} - \frac{\tilde{\eta}}{\Delta t} \sigma_t \sqrt{\frac{n_t}{EDV}}\right) - a, & j = 1 \dots M \end{cases}$$

This is a convex optimization problem as it has a nonlinear convex objective function and the constraints form a nonlinear convex set. However, solving this problem can be computationally very expensive due to the number of scenarios M is typically very large.

4.3.2 Smoothing

An alternative way to avoid the large number of constraints is to adopt the smoothing technique proposed by Alexander et al. (2006). See also Moazeni, Coleman and Li (2010). To simplify notation, we define

$$z_j = XS_0 - \sum_{t=1}^{N} n_t \breve{S}_{t-1}^{(j)} \left(1 - \tilde{\epsilon} - \frac{\tilde{\eta}}{\Delta t} \sigma_t \sqrt{\frac{n_t}{EDV}}\right) - a, \; j = 1 \dots M$$

We can approximate the piecewise convex function $[z_j]^+$ with a continuously piecewise differentiable function $\rho_\kappa(z_j)$ with a small resolution parameter κ

$$\rho_\kappa(z_j) = \begin{cases} z_j, & \text{if } z_j > \kappa \\ \frac{{z_j}^2}{4\kappa} + \frac{z_j}{2} + \frac{\kappa}{4}, & \text{if } -\kappa \leq z_j \leq \kappa \\ 0, & \text{if } z_j < -\kappa \end{cases}$$

The problem becomes

$$\min_{n_1 \dots n_N, a} \frac{1}{M} \sum_{j=1}^{M} (z_j + a) + \lambda a + \frac{\lambda}{M(1-p)} \sum_{j=1}^{M} \rho_\kappa(z_j) \tag{15}$$

$$\text{subject to} \begin{cases} \sum_{t=1}^{N} n_t = X \\ n_t \geq 0, t = 1 \dots N \end{cases}$$

Now we have a convex optimization problem with nonlinear convex objective function and linear constraints. To simplify notation, we define L to be the objective function

$$L = \frac{1}{M}\sum_{j=1}^{M}(z_j + a) + \lambda a + \frac{\lambda}{M(1-p)}\sum_{j=1}^{M}\rho_\kappa(z_j)$$

Proposition 1 The gradient of the objective function $\nabla L = \left(\frac{\partial L}{\partial n_1} \dots \frac{\partial L}{\partial n_N}, \frac{\partial L}{\partial a}\right)$ is given by

$$\frac{\partial L}{\partial n_t} = \frac{1}{M}\sum_{j=1}^{M}\frac{\partial z_j}{\partial n_t} + \frac{\lambda}{M(1-p)}\sum_{j=1}^{M}\frac{\partial z_j}{\partial n_t}\rho'_\kappa(z_j), \ t = 1\dots N$$

$$\frac{\partial L}{\partial a} = \lambda - \frac{\lambda}{M(1-p)}\sum_{j=1}^{M}\rho'_\kappa(z_j)$$

where

$$\rho'_\kappa(z_j) = \frac{\partial \rho_\kappa(z_j)}{\partial z_j} = \begin{cases} 1, & \text{if } \ z_j > \kappa \\ \frac{z_j}{2\kappa} + \frac{1}{2}, & \text{if } \ -\kappa \le z_j \le \kappa \\ 0, & \text{if } \ z_j < -\kappa \end{cases}$$

$$\frac{\partial z_j}{\partial n_t} = -\breve{S}_{t-1}^{(j)}\left(1 - \tilde{\epsilon} - \frac{3}{2}\frac{\widetilde{\eta}}{\Delta t}\sigma_t\sqrt{\frac{n_t}{EDV}}\right), \ t = 1\dots N$$

Proposition 2 The Hessian of the objective function is

$$\nabla^2(L) = \begin{bmatrix} \frac{\partial^2 L}{\partial n_1^2} & \frac{\partial^2 L}{\partial n_1 \partial n_2} & \cdots & \frac{\partial^2 L}{\partial n_1 \partial n_N} & \frac{\partial^2 L}{\partial n_1 \partial a} \\ \frac{\partial^2 L}{\partial n_2 \partial n_1} & \frac{\partial^2 L}{\partial n_2^2} & \cdots & \frac{\partial^2 L}{\partial n_2 \partial n_N} & \frac{\partial^2 L}{\partial n_2 \partial a} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^2 L}{\partial n_N \partial n_1} & \frac{\partial^2 L}{\partial n_N \partial n_2} & \cdots & \frac{\partial^2 L}{\partial n_N^2} & \frac{\partial^2 L}{\partial n_N \partial a} \\ \frac{\partial^2 L}{\partial a \partial n_1} & \frac{\partial^2 L}{\partial a \partial n_2} & \cdots & \frac{\partial^2 L}{\partial a \partial n_N} & \frac{\partial^2 L}{\partial a^2} \end{bmatrix}$$

where

$$\frac{\partial^2 L}{\partial n_t^2} = \frac{1}{M}\sum_{j=1}^{M}\frac{\partial^2 z_j}{\partial n_t^2} + \frac{\lambda}{M(1-p)}\sum_{j=1}^{M}\left(\frac{\partial^2 z_j}{\partial n_t^2}\rho'_\kappa(z_j) + \left(\frac{\partial z_j}{\partial n_t}\right)^2\rho''_\kappa(z_j)\right), \ t = 1\dots N$$

$$\frac{\partial^2 L}{\partial n_s \partial n_t} = \frac{\partial^2 L}{\partial n_t \partial n_s} = \frac{\lambda}{M(1-p)} \sum_{j=1}^{M} \frac{\partial z_j}{\partial n_s} \frac{\partial z_j}{\partial n_t} \rho''_\kappa(z_j), \; s = 1 \ldots N, \; t = 1 \ldots N, \; s \neq t$$

$$\frac{\partial^2 L}{\partial a \partial n_t} = \frac{\partial^2 L}{\partial n_t \partial a} = -\frac{\lambda}{M(1-p)} \sum_{j=1}^{M} \frac{\partial z_j}{\partial n_t} \rho''_\kappa(z_j), \; t = 1 \ldots N$$

$$\frac{\partial^2 L}{\partial a^2} = \frac{\lambda}{M(1-p)} \sum_{j=1}^{M} \rho''_\kappa(z_j)$$

where $\rho'_\kappa(z_j)$ and $\frac{\partial z_j}{\partial n_t}$ are given in Proposition 1, and

$$\rho''_\kappa(z_j) = \frac{\partial^2 \rho_\kappa(z_j)}{\partial z_j^2} = \begin{cases} 0, & \text{if } \; z_j > \kappa \\ \frac{1}{2\kappa}, & \text{if } \; -\kappa \leq z_j \leq \kappa \\ 0, & \text{if } \; z_j < -\kappa \end{cases}$$

$$\frac{\partial^2 z_j}{\partial n_t^2} = \breve{S}_{t-1}^{(j)} \frac{3}{4} \frac{\widetilde{\eta}}{\Delta t} \sigma_t \sqrt{\frac{1}{EDV n_t}}, \; t = 1 \ldots N$$

4.4 Optimization Results

We now present the results obtained by implementing the smoothing method described in chapter 5.4.[2] For all the results presented, we use the same data set and parameters given in chapter 3.2, all but the naïve trading strategy (8), and to reduce computing time we run $M = 1,000,000$ simulations for the optimization.

Table 6 shows trading costs and optimal strategies with high initial volatility given risk tolerance level 0.1. Table 7 shows those results with low initial volatility. Using the RGARCH model for price dynamics, the figures show efficient frontiers obtained by GBM and RGARCH optimal strategies for selected ETFs.

[2]We solved the optimization problem with the interior point algorithm in the Matlab fmincon solver. For details of the algorithm see Byrd, Hribar, and Nocedal (1999), and Waltz, Morales, Nocedal, and Orban (2006).

	Constant Volatility								Realized GARCH							
	Optimal Costs (cents/share)			Optimal Strategy (1,000 shares)					Optimal Costs (cents/share)			Optimal Strategy (1,000 shares)				
Index ETF	Mean (95%)	VaR (95%)	CVaR	n_1	n_2	n_3	n_4	n_5	Mean (95%)	VaR (95%)	CVaR	n_1	n_2	n_3	n_4	n_5
SPY	8.6	403	505	24,446	4,472	1,322	611	373	8.8	402	503	25,049	4,385	1,038	575	176
ISF	145.3	1,861	2,312	565	297	184	129	108	144.8	1,864	2,317	562	296	181	131	113
NKY	7053.0	39,882	48,766	18	14	11	10	9	7017.3	40,133	49,098	18	14	11	10	10
DAXEX	19.4	251	314	120	62	38	27	23	19.6	249	311	124	62	37	26	22
IWM	9.4	336	421	7,614	1,816	566	283	189	9.4	336	421	7,648	1,699	604	315	202
DIA	10.8	379	474	951	209	72	32	26	10.9	378	473	963	206	66	33	22
QQQ	9.5	388	485	7,609	1,495	533	219	142	9.7	386	483	7,811	1,426	459	190	114
CAC FP	10.3	131	162	59	31	19	14	12	10.3	131	163	58	31	19	14	12
2800 HK	21.6	100	121	859	700	600	540	514	21.5	100	121	847	693	602	547	526
EWY	9.6	244	305	297	90	37	19	14	9.8	243	304	304	85	37	19	13
IAEX	12.9	120	148	16	10	7	6	5	12.9	120	148	16	10	7	6	5
CSSMI	16.2	180	224	21	12	8	6	5	16.3	179	224	21	12	8	6	5
BBVAI	12.3	46	55	8	7	6	6	6	12.2	46	55	8	7	6	6	6
NAFTRAC	12.2	193	241	10,706	4,916	2,739	1,779	1,459	12.1	193	242	10,650	4,800	2,719	1,855	1,574
BOVA11	19.9	290	362	195	95	53	36	30	19.8	291	363	193	93	54	37	31
SX5EEX	10.6	105	130	69	43	29	23	20	10.6	104	130	70	42	29	23	20
STTF SP	7.4	17	20	11	10	9	9	9	7.4	18	20	10	10	9	9	9
LEVMIB	5.2	31	38	260	199	164	142	132	5.1	31	38	258	197	162	144	137

Table 6: Optimal trading costs under RGARCH with high initial volatility, risk confidence level 0.95, risk tolerance 0.1

	Constant Volatility								Realized GARCH							
	Optimal Costs (cents/share)			Optimal Strategy (1,000 shares)					Optimal Costs (cents/share)			Optimal Strategy (1,000 shares)				
Index ETF	Mean (95%)	VaR (95%)	CVaR	n_1	n_2	n_3	n_4	n_5	Mean (95%)	VaR (95%)	CVaR	n_1	n_2	n_3	n_4	n_5
SPY	6.0	266	333	24,768	4,500	1,244	488	223	6.1	266	332	25,222	4,118	1,246	428	209
ISF	104.8	1,246	1,547	574	297	179	127	106	105.6	1,240	1,539	585	298	174	124	102
NKY	5738.1	30,837	37,696	18	14	11	10	9	5760.7	30,656	37,454	19	14	11	10	9
DAXEX	14.5	182	227	120	63	38	27	22	14.8	180	224	126	63	36	25	20
IWM	5.6	181	227	7,648	1,724	650	265	181	5.8	180	226	7,859	1,667	528	240	174
DIA	7.5	246	308	959	209	72	30	20	7.7	245	306	990	193	61	30	16
QQQ	6.7	256	321	7,711	1,460	463	212	151	6.8	255	319	7,942	1,422	378	148	108
CAC FP	7.5	91	113	59	31	19	13	11	7.6	91	112	60	31	19	13	11
2800 HK	16.7	75	90	860	700	600	540	514	16.8	75	90	871	704	597	536	507
EWY	7.0	171	213	300	88	37	19	13	7.1	170	212	309	85	34	17	12
IAEX	8.8	77	95	16	10	7	6	5	8.9	76	94	16	10	7	5	5
CSSMI	11.9	126	157	21	12	8	6	5	12.0	125	156	22	12	8	5	5
BBVAI	9.3	33	40	8	7	6	6	6	9.3	33	40	8	7	6	6	6
NAFTRAC	8.2	117	146	10,845	4,918	2,675	1,744	1,416	8.4	116	144	11,430	4,795	2,534	1,595	1,244
BOVA11	16.7	229	286	197	95	53	36	29	16.8	228	284	200	93	52	35	28
SX5EEX	8.0	75	93	70	43	29	22	20	8.0	74	92	72	43	28	22	19
STTF SP	5.7	13	15	11	10	9	9	9	5.7	13	15	11	10	9	9	9
LEVMIB	3.5	20	24	263	201	162	141	131	3.6	20	24	271	202	161	137	126

Table 7: Optimal trading costs under RGARCH with low initial volatility, risk confidence level 0.95, risk tolerance 0.1

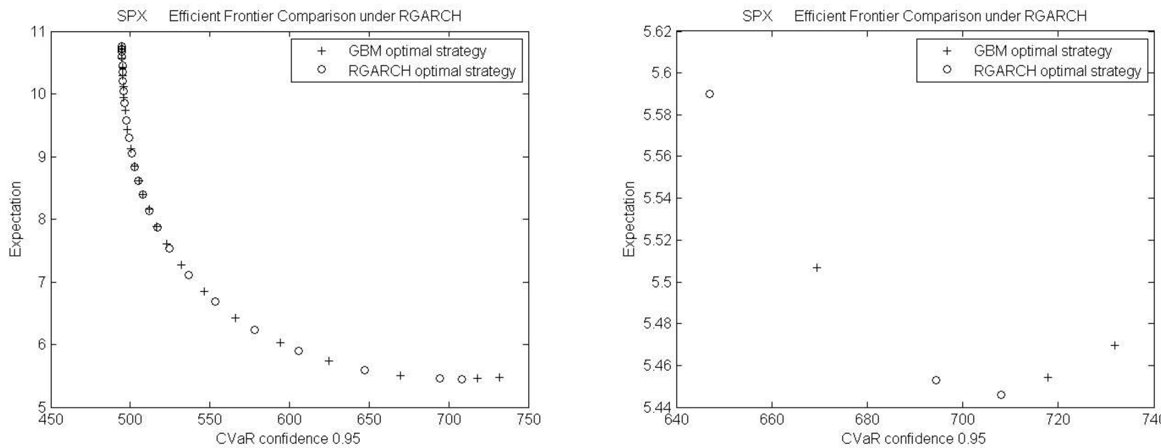

Figure 1: Efficient frontier comparison of stock index ETF SPY with high initial volatility. Left: entire efficient frontier. Right: zoomed in on right end.

As shown in the tables and figures, even if we assume the RGARCH model to be the accurate market model, the costs computed under GBM optimal strategies are generally close to the efficient frontier of RGARCH optimal strategies.

However, if we compare the same risk aversion level under both GBM and RGARCH optimal strategies, the trading costs obtained by the GBM strategies can still be far away from RGARCH strategies, for example getting less expected costs in the expenses of incurring more risk, or vice versa. Especially when we want to only minimize the expected cost or only minimize the risk, GBM optimal strategy can produce both larger expected costs and larger risk altogether, which can never be optimal compared to the RGARCH strategy.

5 Conclusion

In this paper, we have compared the trading costs under the recent RGARCH model to the constant volatility GBM by using Monte Carlo Simulations. We have demonstrated that increased volatility leads to increased trading costs. However, the difference in trading costs between GBM and RGARCH is small.

To obtain the optimal trading strategies under the volatility-based impact model, we have developed a change-of-variable and a smoothing method to formulate the minimization of mean and CVaR as a convex optimization problem, and the smoothing method proves to

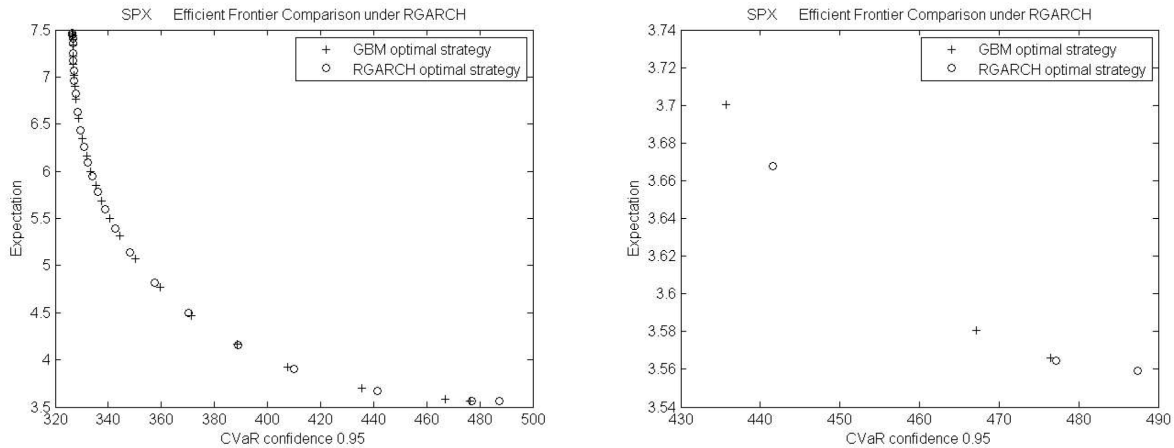

Figure 2: Efficient frontier comparison of stock index ETF SPY with low initial volatility. Left: entire efficient frontier. Right: zoomed in on right end.

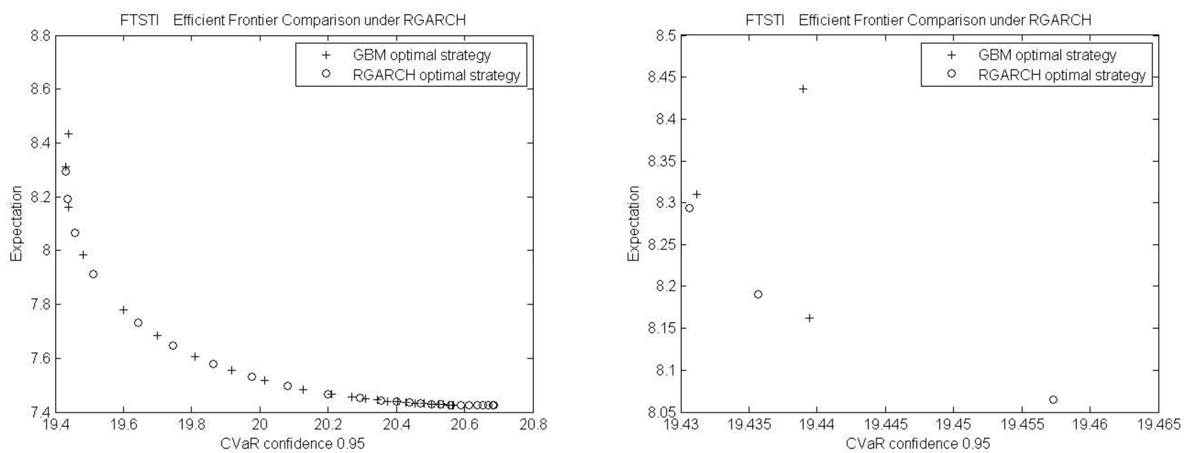

Figure 3: Efficient frontier comparison of stock ETF STTF SP with high initial volatility. Left: entire efficient frontier. Right: zoomed in on left end.

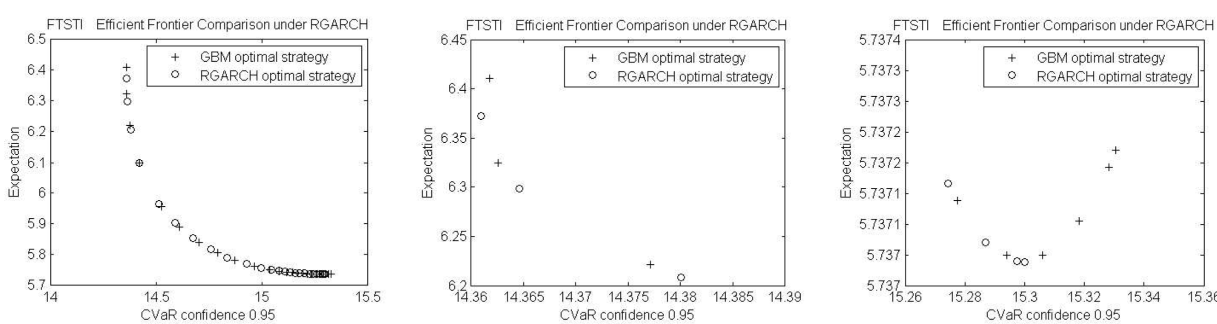

Figure 4: Efficient frontier comparison of stock ETF STTF SP with low initial volatility. Left: entire efficient frontier. Center: zoomed in on left end. Right: zoomed in on right end.

be very effective to this problem. The results have shown that although the difference is small between optimal strategies under the two models, RGARCH is always more preferable, especially when only minimizing expected costs or only minimizing risk without considering the other.

Possible extension of the analysis is either to consider the trading costs of a portfolio of stocks (ETFs), or to compare the costs under the constant volatility GBM to a different volatility model. Models for highly volatile markets, such as regime switching models or volatility models with jumps, may reveal larger discrepancies from the constant volatility model within a short liquidation time frame.

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