Error sensitive multivariate polynomial interpolation

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Abstract

In this paper, we make a strong connection between algebraic geometry and interpolation. In particular, we use algebraic geometry tools to develop a machinery for the analysis of Newton or nested multivariate interpolation schemes. The main practical result coming out of our analysis is that for robustness, one should replace the condition of minimal degree with a minimally complete condition that is introduced in this paper. We show how to construct minimally complete schemes and provide examples.

1 Introduction

Interpolation is the mechanism by which one solves the problem of finding an element of a given function space that agrees with a given set of observations. Formally,

Basic Problem: Given a set of distinct locations $\Theta := \left\{\theta_1, \ldots, \theta_n : \theta_i \in \mathbb{R}^d\right\}$ and a function space F, for any set of data $\{d_\theta : d_\theta \in \mathbb{R},\ \theta \in \Theta\}$, find $f \in F$ such that

$$f(\theta) = d_\theta \qquad \forall \theta \in \Theta.$$

The generic solution to this problem is known. One finds a subspace of $F_\Theta = \operatorname{span}\{f_1, \ldots, f_n : f_i \in F\}$, such that the **Vandermonde** matrix

$$(f_c(\theta_r) : r, c = 1, \ldots, n)$$

is invertible. Then,

$$f = \sum_{i=1}^{n} c_i f_i \qquad \text{where} \qquad \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} f_1(\theta_1) & \cdots & f_n(\theta_1) \\ \vdots & & \vdots \\ f_1(\theta_n) & \cdots & f_n(\theta_n) \end{pmatrix}^{-1} \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}.$$

Thus, the solution to the interpolation problem is reduced to the problem of finding the appropriate subspace F_Θ and inversion of a square matrix. When the Vandermonde matrix is invertible, we say that F_Θ is *correct* for the interpolation problem.

For many spaces of functions, the selection of a correct subspace is easy. For example, when $d = 1$ and $F = \Pi(\mathbb{R})$, the polynomials, the standard choice is

$$F_\Theta = \Pi_{n-1}(\mathbb{R}) := \{p \in \Pi(\mathbb{R}) : \deg p < n\}.$$

Most of the effort here is in selecting a basis for Π_{n-1} such that the Vandermonde is easily or accurately inverted. Some choices include:

Newton Basis: $\{\nu_i \in F : i = 1, \ldots, n\}$ such that

$$\nu_i(\theta_j) = 0 \quad \text{if } j < i, \qquad \nu_i(\theta_i) \neq 0.$$

The Newton basis yields a lower triangular Vandermonde.

Lagrange Basis: $\{\ell_\theta \in F : \theta \in \Theta\}$ such that

$$\ell_\theta(\vartheta) = \begin{cases} 0 & \text{if } \vartheta \neq \theta \\ 1 & \text{if } \vartheta = \theta \end{cases}$$

For the Lagrange basis, the Vandermonde is the identity matrix, which means

$$f = \sum_{\theta \in \Theta} f(\theta)\ell_\theta.$$

For $\Pi_{n-1}(\mathbb{R})$, the Newton basis has the form

$$\nu_i(t) = \prod_{j=1}^{i-1}(t - \theta_j),$$

and the Lagrange basis has the form

$$\ell_\theta(t) = \prod_{\substack{\vartheta \neq \theta \\ \vartheta \in \Theta}} \frac{(t - \vartheta)}{(\theta - \vartheta)}.$$

As suggested by the notation, the Newton basis depends on an ordering of the points in Θ, while the Lagrange basis does not. We will use both Newton and Lagrange bases in our analysis.

For multivariate polynomials, the selection of a correct interpolation subspace is more difficult. There are two fundamental difficulties in generating a correct interpolation subspace:

1. Dimension: The number of data points is insufficient to determine the appropriate subspace of $\Pi(\mathbb{R}^d)$.

$$\dim \Pi_k(\mathbb{R}^d) = \binom{k+d}{d} = \frac{(k+d)!}{k!\ d!}.$$

   For example, let $\#\Theta$ denote the number of elements of Θ. If $\#\Theta = 5$, and $d = 2$, then $\dim \Pi_1(\mathbb{R}^2) = 3$ and $\dim \Pi_2(\mathbb{R}^2) = 6$. Thus, in this case (and in general for $d > 1$) we have to do more work than just select k to choose a basis.

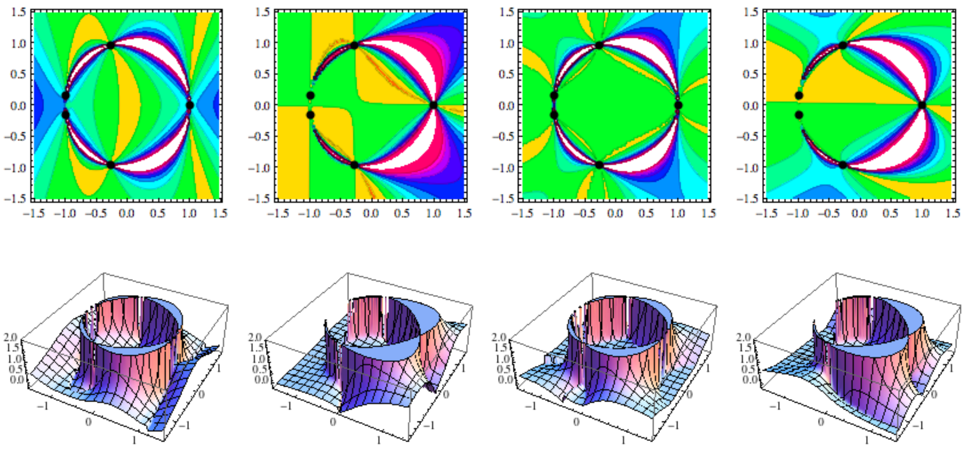

Figure 1: Experiment: Fix five points on the unit circle and allow the sixth point to vary on a 101x101 grid. For each point, compute how much the addition of the sixth point has improved the interpolant, i.e., the maximum error for interpolation of x^3, x^2y, xy^2, and y^2 at six points minus the respective maximum error for interpolation by 5 fixed points. In the contour plot, yellow and brown are improvements, all other colors are places where the interpolation with that point is worse.

2. Zero sets: In general, if there is a $f \in F_\Theta$ such that $f \neq 0$, and $f(\theta) = 0$ for all $\theta \in \Theta$, then the corresponding Vandermonde matrix $V(F_\Theta, \Theta)$ is rank deficient, and as such can not be inverted for general data. For this reason, we must avoid basis functions that vanish on Θ.

   Avoiding these zeros sets poses an additional challenge in multivariate polynomial interpolation, since even when there is agreement between the number of data points and the dimension of $\Pi_k(\mathbb{R}^d)$, there may be a nontrivial polynomial that vanishes on the data points. For example, let $\#\Theta = 6 = \dim \Pi_2(\mathbb{R}^2)$, but position all six points on the unit circle, i.e., $\forall \theta \in \Theta$, $\|\theta\| = 1$. Then, $p((x, y)) = x^2 + y^2 - 1$ is in $\Pi_2(\mathbb{R}^2)$, but $\forall \theta \in \Theta$, $p(\theta) = 0$ and so $\Pi_2(\mathbb{R}^2)$ cannot be correct for this Θ.

The zero set of a polynomial is a set of measure zero, so the probability of a random set of data locations being on the zero set of a polynomial is also 0. However, what happens when we are near that zero set? To explore this, consider the following experiment illustrated in Figure 1: fix five points on the unit circle, and let a sixth point vary on a 101×101 lattice in the square $[-1.5, 1.5] \times [-1.5, 1.5]$. Since no point in this lattice is on the unit circle, interpolation to the five points plus a point on the lattice by $\Pi_2(\mathbb{R}^2)$ is correct. To understand the impact of adding a point to the interpolant, we will numerically compute the maximum error within the convex hull of the five fixed points using both the five fixed points and all six points. At each of the lattice points, we have plotted the difference between interpolation with and without that point. In these plots, a negative number is an improvement and a positive number is reduction in quality of the interpolant by the introduction of the sixth point. With brown less than -0.25, yellow between -0.25 and zero, and the other contours at 0.25 intervals greater than 0, one can see that most points make the error worse, with some much worse.

Again, $\Pi_2(\mathbb{R}^2)$ is correct with the five fixed point and the addition of any single lattice point. The interpolation conditions are satisfied with double precision accuracy, but clearly as we get close to the zero set of $x^2 + y^2 - 1$ the resulting interpolant is undesirable. What is occurring in this example is that the correct space for interpolating the first five points is $\Pi_2(\mathbb{R}^2) \setminus \{x^2 + y^2 - 1\}$. This means that the sixth Lagrange polynomial must be

$$\ell_\theta((x,y)) = \frac{x^2 + y^2 - 1}{\theta_x^2 + \theta_y^2 - 1} \qquad \text{where } \theta \text{ is the sixth point.}$$

As θ gets closer to the unit circle, this Lagrange polynomial starts to dominate the interpolant. We will later show that choosing a Lagrange polynomial to interpolate data near its zero set will result in poor approximation, but for the moment note the following:

1. The use of points on the unit circle was only for ease of exposition. If we start with any six points for which $\Pi_2(\mathbb{R}^2)$ is correct, and allow one of the points, θ, to vary, the results would be the same. Namely, there is only one unscaled polynomial, ν, associated with θ. As θ varies only the scaling factor on ν will change. In particular, as θ gets close to the zero set of ν, ν will dominate the interpolant away from the zero set.

2. The method for constructing the interpolation was immaterial. The problem was the choice of the minimal correct interpolation space, $\Pi_2(\mathbb{R}^2)$, without regards to the distance from zero of the Lagrange polynomials.

3. This effect is not due to numerical instability. Even in the worst case, the interpolant constructed is accurate, in that it interpolates the data to within machine accuracy. However, the error in interpolation is abysmal. The highly osculating Lagrange polynomial completely dominates outside of a small region around the interpolation points.

4. Here, one needs to allow for more choice in the selection of the Lagrange polynomial to reduce error, which means that degree three polynomials must be considered. In general, it is not obvious how to choose this "last" polynomial to generate a reasonable interpolation scheme. The aim of this paper is to determine how to construct interpolation schemes that avoid problems of the type illustrated in this example. In particular, when choosing the next basis function for the interpolant, one should choose from a complete generating set for the ideal.

Our work relies on the relationship of zero sets and correct subspaces, so in Section 2 we review varieties and ideals. Further, our work builds on the Polynomial Least, which we review in detail in Section 3 as a means of introducing notation and important ideas. In Section 4 and Section 5, we provide a constructible connection between the polynomials that vanish on Θ, an ideal, and the collection of Lagrange polynomials that define the interpolation space. In Section 6, we prove that Example 1 is not an exceptional case, but is a structural issue with schemes that force interpolation from an insufficiently robust polynomial space. Schemes with fixed monomial order and minimal schemes fall into this category. Sections 7 and 8 show how to construct schemes that avoid this problem and give some example schemes. One of these schemes has the further property of being continuous with respect to Θ as a function of a single point θ. Finally, in Sections 9 and 10 we give some computational examples, draw conclusions and make suggestions for further work.

2 Varieties and Ideals

The Lagrange and Newton bases point to the fundamental connection between zero sets and correct subspaces. There is a rich and elegant theory on the connection between zero sets and subspaces for multivariate polynomials (see, e.g., [1]). For our purpose, we need to introduce a couple of concepts from this theory.

Definition 1. *A subset $V \subset \mathbb{R}^d$ is called a* **variety** *if there exists a finite collection of polynomials $P \subset \Pi(\mathbb{R}^d)$ such that $p(v) = 0$ for all $v \in V$ and all $p \in P$.*

Fact 1. *If V, W are varieties, then $V \cap W$ and $V \cup W$ are also varieties.*

Using these facts, one can see that point sets are varieties. For any point, $(\theta_1, \dots, \theta_d) \in \mathbb{R}^d$, one takes the collection of polynomials $p_i((x_1, \dots, x_d)) = x_i - \theta_i$. Clearly, $\theta = V(\{p_1, \dots, p_d\}) := \bigcap_{i=1}^d \{v : p_i(v) = 0\}$. Thus, a single point is a variety. So a finite point set, as a union of varieties, is a variety.

Closely related to varieties is the notion of an ideal.

Definition 2. *An* **ideal** *is a collection of polynomials, I, such that*

1. *$0 \in I$.*
2. *I is closed under addition, i.e., if $p, q \in I$, then $p + q \in I$.*
3. *I is closed under multiplication by Π, i.e., if $q \in I$, then for any $p \in \Pi$, $pq \in I$.*

Definition 3. *Let V be a variety, then $I(V) := \left\{p \in \Pi(\mathbb{R}^d) : p(v) = 0, \quad \forall v \in V\right\}$.*

Fact 2. *As the notation suggests, $I(V)$ is an ideal.*

In the previous section, we suggested that finding a correct subspace for multivariate polynomial interpolation may not be a sufficient criterion for the construction a "good" interpolant, however it is a necessary criterion. The problem of finding a correct subspace of $\Pi(\mathbb{R}^d)$ for interpolation at points Θ can be seen as finding polynomials

$$p_\theta \in I(\Theta \setminus \theta) \setminus I(\Theta).$$

This has led to a series of **Newton methods** for construction of correct subspaces.

Definition 4. *Given an ordered set of points $\Theta = \{\theta_1, \ldots, \theta_n\}$, a* **Newton Method** *determines, for each i, $I(\{\theta_1, \ldots, \theta_{i-1}\})$, and picks a $\nu_i \in I(\{\theta_1, \ldots, \theta_{i-1}\})$ such that $\nu_i(\theta_i) \neq 0$. The resulting basis, $\{\nu_1, \ldots, \nu_n\}$, would be called a* **Newton Basis** *resulting from this method.*

This definition of Newton Methods focuses exclusively on the invertability of the Vandermonde matrix (as a lower triangular matrix with non-zeros on the diagonal). That differs from methods suggested by Sauer [7] in that it does not imply **minimal degree**, a characterization of interpolation spaces first introduced in [3]. For example, with our definition, the set $\{1, x^2, x^2(x-1)\}$ would be a Newton Basis for the ordered points $\{0, 1, 2\}$. The definition does capture the **monotonicity property** from [3], where the connection between monotonicity and Newton methods was first noted.

Much of this paper was motivated by trying to understand examples where the Least construction in [3] did not give expected results. The fundamental computational approaches we take are drawn from and extensions of techniques in that paper. We will provide a mechanism that will generate a Newton basis (and, with a little extra work, a Lagrange basis), and also avoid the problems noted in Section 6. This Newton approach to generating correct interpolation spaces is consistent with most of the other mechanisms that exist.

3 Least Polynomial

In [3, 5], de Boor and Ron introduced a novel construction of correct polynomial interpolation spaces. This construction leveraged the "correctness" of the space of exponentials to generate a correct polynomial subspace of minimum degree. As our construction builds on theirs, we review here the Least as both a means of introducing our notation and to better compare our method to theirs.

Following de Boor and Ron, we define the **exponential** e_t **with frequency** t as

$$e_t(s) := \exp(s \cdot t) = \sum_{n=0}^{\infty} (s \cdot t)^n / n! = \sum_{\alpha \in \mathbb{Z}_+^d} s^\alpha t^\alpha / \alpha!.$$

For any analytic function f, let

$$T_\alpha(f) := D^\alpha f(0),$$

the α-th Taylor coefficient. Thus,

$$T_\alpha(e_t(s)) = t^\alpha.$$

This relationship directly connects the exponential function to the multinomials.

By the point separating property of the exponentials, we know that the set $\{e_\theta : \theta \in \Theta\}$ is linearly independent and define

$$\mathrm{Exp}_\Theta := \mathrm{span}\{e_\theta : \theta \in \Theta\}.$$

Since Exp_Θ has the same dimension as Θ, the Vandermonde matrix, $(e_\theta(\vartheta) : \vartheta, \theta \in \Theta)$, must be invertible. Therefore, Exp_Θ must be correct for interpolation on Θ.

The Least operation maps any function that is analytic about the origin to the lowest degree, non-zero terms in its Taylor expansion. Formally, let $f(s) = \sum_{\alpha\in\mathbb{Z}_+^d} T_\alpha(f)s^\alpha/\alpha!$, then

$$f_\downarrow(s) := \sum_{|\alpha|=n} T_\alpha(f)s^\alpha/\alpha!$$

with n the largest integer such that $T_\beta(f) = 0$ if $|\beta| < n$. Clearly, $f_\downarrow$ is a homogeneous polynomial in $\mathbb{R}^d$. As examples, $(y^3 + x^2 + 3xy + 4y^2 + x + 2y)_\downarrow = x + 2y$, $(e_\theta)_\downarrow(s) = 1$ and $(e_\theta - e_\vartheta)_\downarrow(s) = (\theta - \vartheta)\cdot s$.

de Boor and Ron show that the operation of taking the Least does not reduce the dimension of Exp_Θ, which means that the **Least Polynomial space** for Θ, defined by

$$\Pi_\Theta := (\mathrm{Exp}_\Theta)_\downarrow,$$

must be correct for interpolation on Θ and of minimum degree.

The computational algorithm for generating the Least involves an ingenious view of the exponential as a sequence of vectors, with each vector representing the multinomials of a given degree,

$$e_t = (1, v_1(t), v_2(t), \ldots) \qquad \text{where} \quad v_k(t) = (t^\alpha/\alpha! : |\alpha| = k).$$

The method for computing the Least Basis, called **Gauss Elimination by Segments** [4], involves setting up a matrix e_Θ and using a Gram-Schmidt orthogonalization process as the elimination step. I.e., to "eliminate" vector w with vector v, one computes

$$w - \frac{w\cdot v}{v\cdot v}v.$$

This is best understood by looking at an example.

Consider $\Theta = \{(1,0),(0,1),(-1,0),(0,-1)\}$. Then

$$e_\Theta = \begin{pmatrix} e_{(1,0)} \\ e_{(0,1)} \\ e_{(-1,0)} \\ e_{(0,-1)} \end{pmatrix} = \begin{pmatrix} 1 & (1,0) & (\frac{1}{2},0,0) & \ldots \\ 1 & (0,1) & (0,0,\frac{1}{2}) & \ldots \\ 1 & (-1,0) & (\frac{1}{2},0,0) & \ldots \\ 1 & (0,-1) & (0,0,\frac{1}{2}) & \ldots \end{pmatrix} \begin{pmatrix} 1 \\ \begin{pmatrix} t_x \\ t_y \end{pmatrix} \\ \begin{pmatrix} t_x^2 \\ t_xt_y \\ t_y^2 \end{pmatrix} \\ \ldots \end{pmatrix}$$

For ease of exposition, this example was constructed so that a correct interpolation space exists in Π_2, and we will only consider those terms. We will also omit the matrix of monomials in further equations. The first step is elimination with the scalar 1:

$$\begin{pmatrix} \mathbf{1} & (1,0) & (\frac{1}{2},0,0) \\ 1 & (0,1) & (0,0,\frac{1}{2}) \\ 1 & (-1,0) & (\frac{1}{2},0,0) \\ 1 & (0,-1) & (0,0,\frac{1}{2}) \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & (1,0) & (\frac{1}{2},0,0) \\ 0 & (-1,1) & (-\frac{1}{2},0,\frac{1}{2}) \\ 0 & (-2,0) & (0,0,0) \\ 0 & (-1,-1) & (-\frac{1}{2},0,\frac{1}{2}) \end{pmatrix}.$$

Next, we will do the elimination using the **bold** vectors by a Gram-Schmidt orthogonalization process.

$$\begin{pmatrix} 1 & (1,0) & (\frac{1}{2},0,0) \\ 0 & (-\mathbf{1},\mathbf{1}) & (-\frac{1}{2},0,\frac{1}{2}) \\ 0 & (-2,0) & (0,0,0) \\ 0 & (-1,-1) & (-\frac{1}{2},0,\frac{1}{2}) \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & (1,0) & (\frac{1}{2},0,0) \\ 0 & (-1,1) & (-\frac{1}{2},0,\frac{1}{2}) \\ 0 & (-\mathbf{1},-\mathbf{1}) & (\frac{1}{2},0,-\frac{1}{2}) \\ 0 & (-1,-1) & (-\frac{1}{2},0,\frac{1}{2}) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & (1,0) & (\frac{1}{2},0,0) \\ 0 & (-1,1) & (-\frac{1}{2},0,\frac{1}{2}) \\ 0 & (-1,-1) & (\frac{1}{2},0,-\frac{1}{2}) \\ 0 & 0 & (-\mathbf{1},\mathbf{0},\mathbf{1}) \end{pmatrix}$$

Associated with each v_k, there is the homogeneous polynomial $p(t) = \sum_{|\alpha|=k} v_k(\alpha)t^\alpha$. Thus, from the collection of bold terms used for elimination, we generate the Least basis

$$\left\{1, -t_x + t_y, -t_x - t_y, -t_x^2 + t_y^2\right\}.$$

By inspection, one can see that in this example,

$$I(\Theta) = I\left(\left\{t_x t_y, t_x^2 + t_y^2 - 1\right\}\right).$$

The vectors associated with the leading terms of the generators of $I(\Theta)$ are $(0,1,0)$ and $(1,0,1)$. Note that both of these vectors are orthogonal to the vector, $(-1,0,1)$, which is associated with the final homogeneous polynomial in the Least basis. This property is one of the defining characteristics of the Least space and suggests that there is a connection between the Least and the ideal.

4 A Framework for Analysis

One of the premises of this paper is that generation of correct subspaces, while necessary, is not sufficient for construction of a good interpolant. To that end, we need a mechanism to constructively generate possible choices for our Newton Basis, by constructively generating the ideal. Our mechanism is a modification of the tools for generating the Least Polynomial space. It will allow us to construct correct interpolation spaces that are sensitive to error and provide a tool for analysis.

Let $L(\Theta) = \left\{\ell_\theta \in \Pi(\mathbb{R}^d) : \theta \in \Theta\right\}$ be any collection of Lagrange polynomials on Θ. A common operation in the analysis of interpolation methods is projection. Here

$$P_L(f) = \sum_{\vartheta\in\Theta} f(\vartheta)\ell_\vartheta$$

is the projection of a function onto the space Π_L. To construct an interpolation scheme that is sensitive to error, we need be concerned with the error in this projection, namely,

$$f - P_L(f) = f - \sum_{\vartheta\in\Theta} f(\vartheta)\ell_\vartheta.$$

Connecting this error formula back to the tools for the Least Polynomial space, we define

$$g_L(t,s) := e_s(t) - \sum_{\vartheta\in\Theta} e_s(\vartheta)\ell_\vartheta(t), \tag{1}$$

which is the error of the projection of the exponential with frequency s onto the polynomial space $L(\Theta)$ as a function of t. Leveraging the dual representation of the exponential,

$$g_L(t,s) = e_t(s) - \sum_{\vartheta\in\Theta} e_\vartheta(s)\ell_\vartheta(t),$$

we will also use

$$g_{L,\alpha}(t) := T_\alpha(g_L(t,s)), \tag{2}$$

where T is understood to operate on the variable s.

The operator T_α allows us to consider g_L as a vector indexed by $\mathbb{Z}_+^d$. For example, if L_0 is the empty set (no points of interpolation) and we are in $\mathbb{R}^2$, then we have

$$g_{L_0}(t,s) = [1,\ t_x,\ t_y,\ t_x^2,\ t_xt_y,\ t_y^2,\ \dots\ ]\cdot[1,\ s_x,\ s_y,\ \frac{1}{2}s_x^2,\ s_xs_y,\ \frac{1}{2}s_y^2,\ \dots\ ],$$

where the $g_{L_0,\alpha} = t^\alpha$. However, the vector $[1,\ s_x,\ s_y,\ \frac{1}{2}s_x^2,\ s_xs_y,\ \frac{1}{2}s_y^2,\ \dots\ ]$ does not change with L, so we usually drop it from consideration during computations and think of

$$g_L(t,s) = [g_{L,\alpha}(t) : \alpha\in\mathbb{Z}_+^d].$$

Theorem 1. *$T_\alpha(g_L(t,s)) = g_{L,\alpha}(t)$ is the error in interpolation of the α multinomial. Specifically,*

$$g_{L,\alpha}(t) = t^\alpha - \sum_{\vartheta\in\Theta}\vartheta^\alpha\ell_\vartheta(t).$$

Proof. Since, $T_\alpha(e_t(s)) = t^\alpha$,

$$\begin{aligned} g_{L,\alpha}(t) &= T_\alpha\big(e_t(s) - \sum_{\vartheta\in\Theta}\ell_\vartheta(t)e_\vartheta(s)\big) = T_\alpha\big(e_t(s)\big) - \sum_{\vartheta\in\Theta}\ell_\vartheta(t)T_\alpha\big(e_\vartheta(s)\big) \\ &= t^\alpha - \sum_{\vartheta\in\Theta}\vartheta^\alpha\ell_\vartheta(t) \end{aligned}$$

$\square$

Corollary 1. *For all $\alpha\in\mathbb{Z}_+^d$, $g_{L,\alpha}\in I(\Theta)$.*

Proof. By Theorem 1, $g_{L,\alpha}$ is a polynomial. We just need to show that $g_{L,\alpha}(\theta) = 0\ \forall\theta \in \Theta$. This follows immediately, since it is the error in interpolation. I.e.,

$$g_{L,\alpha}(\theta) = \theta^\alpha - \sum_{\vartheta\in\Theta} \vartheta^\alpha \ell_\vartheta(\theta) = \theta^\alpha - \theta^\alpha = 0$$

$\square$

Corollary 2. *Let* $\Pi_L := \operatorname{span}\{\ell_\vartheta : \theta \in \Theta\}$, *then* $\operatorname{span}\left\{g_{L,\alpha} : \alpha \in \mathbb{Z}_+^d\right\} \oplus \Pi_L = \Pi(\mathbb{R}^d)$. *I.e.,*

$$\forall p \in \Pi(\mathbb{R}^d),\ \exists g \in \operatorname{span}\left\{g_{L,\alpha} : \alpha \in \mathbb{Z}_+^d\right\}\ \textit{ and } q \in \Pi_L \textit{ such that }\ p = g + q.$$

Proof. By Theorem 1, any multinomial is in the $\operatorname{span}\{g_{L,\alpha} : \alpha \in \mathbb{Z}_+^d\} \oplus \Pi_L$, therefore, so is $\Pi(\mathbb{R}^d)$. $\square$

Theorem 2. *The set* $\left\{g_{L,\alpha} : \alpha \in \mathbb{Z}_+^d\right\}$ *covers the ideal* $I(\Theta)$. *I.e.,*

$$\operatorname{span}\{g_{L,\alpha} : \alpha \in \mathbb{Z}_+^d\} = I(\Theta).$$

Proof. For all $p \in I(\Theta)$, we know $p \in \Pi(\mathbb{R}^d)$. Therefore, by corollary 2 there exists a sequence $\{c_\alpha : \alpha \in \mathbb{Z}_+^d\}$ such that

$$p = \sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha g_{L,\alpha} + \sum_{\vartheta\in\Theta} p(\vartheta)\ell_\vartheta = \sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha g_{L,\alpha}$$

$\square$

We can use these ideas to construct the ideal for our interpolation space. Returning to our example, $\Theta = \{(1,0),(0,1),(-1,0),(0,-1)\}$, let

$$\begin{aligned} L &= \left\{\ell_{(1,0)}, \ell_{(0,1)}, \ell_{(-1,0)}, \ell_{(0,-1)}\right\} \\ &= \frac{1}{4}\{x^2 - y^2 + 2x + 1, -x^2 + y^2 + 2y + 1, x^2 - y^2 - 2x + 1, \\ &\qquad -x^2 + y^2 - 2y + 1\}. \end{aligned}$$

This is the Lagrange basis associate with the Least space computed in Section 3. By Theorem 1,

$$g_{L,\alpha}(t) = t^\alpha - (1,0)^\alpha \ell_{(1,0)}(t) - (0,1)^\alpha \ell_{(0,1)}(t) - (-1,0)^\alpha \ell_{(-1,0)}(t) - (0,-1)^\alpha \ell_{(0,-1)}(t).$$

Evaluating each monomial α, $|\alpha| = 2$, to the points in Θ gives

$$\begin{aligned} \left((1,0)^{(2,0)}, (0,1)^{(2,0)}, (-1,0)^{(2,0)}, (0,-1)^{(2,0)}\right) &= (1,0,1,0) \\ \left((1,0)^{(1,1)}, (0,1)^{(1,1)}, (-1,0)^{(1,1)}, (0,-1)^{(1,1)}\right) &= (0,0,0,0) \\ \left((1,0)^{(0,2)}, (0,1)^{(0,2)}, (-1,0)^{(0,2)}, (0,-1)^{(0,2)}\right) &= (0,1,0,1). \end{aligned}$$

Using Theorem 1, we can determine generators for $I(\Theta)$:

$$\begin{aligned}
g_{L,(2,0)}(x,y) &= x^2 - \ell_{(1,0)}(x,y) - \ell_{(-1,0)}(x,y) = \frac{1}{2}(x^2+y^2-1) \\
g_{L,(1,1)}(x,y) &= xy \\
g_{L,(0,2)}(x,y) &= y^2 - \ell_{(0,1)}(x,y) - \ell_{(0,-1)}(x,y) = \frac{1}{2}(x^2+y^2-1).
\end{aligned}$$

In particular, note that the $g_{L,\alpha}$ need not be linearly independent or even unique.

We now have a mechanism for constructing the ideal that is sensitive to the choice of the interpolation space. The ideal is an invariant of the interpolation problem, completely independent of the interpolation space. There is no single interpolation space associated with an ideal; indeed, for any Π_L and any $q \in I(\Theta)$, $\Pi_L + q$ is also a correct interpolation space. However, for any correct interpolation space, Π_L, there is one and only one set of Lagrange polynomials L, therefore we use the Lagrange polynomials as the identifier for the interpolation space. Connecting $I(\Theta)$ and Π_L, which are fundamental to the interpolation problem, is one of the strengths of g_L.

Returning to the error formula for general projection operator from the beginning of this section, we see that the error has a simple representation, which we will repeatedly leverage.

Lemma 1. *Let $p \in \Pi$, with $p(t) = \sum_{\alpha \in \mathbb{Z}_+^d} c_\alpha t^\alpha$, then*

$$(p - P_L(p))(t) = \sum_{\alpha \in \mathbb{Z}_+^d} c_\alpha g_{L,\alpha}(t). \tag{3}$$

Proof.

$$\begin{aligned}
p(t) - P_L(p)(t) &= \sum_{\alpha \in \mathbb{Z}_+^d} c_\alpha t^\alpha - \sum_{\vartheta \in \Theta} \sum_{\alpha \in \mathbb{Z}_+^d} c_\alpha \vartheta^\alpha \ell_\vartheta(t) \\
&= \sum_{\alpha \in \mathbb{Z}_+^d} c_\alpha t^\alpha - \sum_{\alpha \in \mathbb{Z}_+^d} c_\alpha \sum_{\vartheta \in \Theta} \vartheta^\alpha \ell_\vartheta(t) \\
&= \sum_{\alpha \in \mathbb{Z}_+^d} c_\alpha \left(t^\alpha - \sum_{\vartheta \in \Theta} \vartheta^\alpha \ell_\vartheta(t) \right) \\
&= \sum_{\alpha \in \mathbb{Z}_+^d} c_\alpha g_{L,\alpha}.
\end{aligned}$$

□

The importance of the error in the projection of the multinomials, $g_{L,\alpha}$, has already been noted by other authors, e.g., see [2], [7] and [6]. However, one difference in our approach is the use of (1). While we primarily use it as a notational convenience for simultaneous operations on all the $g_{L,\alpha}$, we will also, implicitly and explicitly, use the linear independence of Exp_Θ.

Lemma 2. *$\forall t \in \mathbb{R}^d \backslash \Theta$, there exists α such that $g_{L,\alpha}(t) \neq 0$.*

Proof. This trivially follows from the fact that for a fixed $t \notin \Theta$, $e_t(s) \notin \operatorname{span}\{e_\theta(s) : \theta \in \Theta\}$. □

5 A constructive method for determining a Lagrange basis

In the previous section, we provided a mechanism for computing generators for the ideal, $I(\Theta)$, when one is given a Lagrange basis. In this section, we will provide a mechanism to generate a Newton or Lagrange basis from an ordered point set. To do this, we will build the basis by sequentially processing the points in Θ.

Assume that there is an ordering to Θ, and let $\Theta_i = \{\theta_1, \ldots, \theta_i\}$ and $L_i = \{\ell_{1,i}, \ldots, \ell_{i,i}\}$ with L_i Lagrange for Θ_i. The set $\left\{g_{L_i,\alpha} : \alpha \in \mathbb{Z}_+^d\right\}$ covers the ideal $I(\Theta_i)$, so it provides a reasonable place to look for the next Newton polynomial. Indeed, by appropriate bookkeeping, one can maintain a Lagrange basis as one works through the point set.

Theorem 3. *Let $L_{i-1} = \{\ell_{1,i-1}, \ldots, \ell_{i-1,i-1}\}$ be a Lagrange basis for Θ_{i-1}. Given any $\ell_{i,i}(t) \in I(\Theta_{i-1})$ such that $\ell_{i,i}(\theta_i) = 1$, one can construct a Lagrange basis $L_i = \{\ell_{1,i}, \ldots, \ell_{i,i}\}$ for Θ_i, with*

$$\ell_{j,i}(t) = \ell_{j,i-1}(t) - \ell_{j,i-1}(\theta_i)\ell_{i,i}(t), \qquad \forall j < i. \tag{4}$$

Additionally, one can construct $g_{L_i}(t,s)$ by

$$g_{L_i}(t,s) = g_{L_{i-1}}(t,s) - g_{L_{i-1}}(\theta_i, s)\ell_{i,i}(t). \tag{5}$$

Proof. 1. By Lemma 2, $\ell_{i,i}$ exists and is constructable from $\operatorname{span}\left\{g_{L_{i-1},\alpha}\right\}$. Since $\ell_{i,i}$ is Lagrange,

$$\ell_{j,i-1}(\theta_k) - \ell_{i,i}(\theta_k)\ell_{j,i-1}(\theta_i) = \begin{cases} \ell_{j,i-1}(\theta_j) = 1 & \text{if } k = j, \\ \ell_{j,i-1}(\theta_i) - \ell_{j,i-1}(\theta_i) = 0 & \text{if } k = i, \\ \ell_{j,i-1}(\theta_k) = 0. & \text{otherwise.} \end{cases}$$

Thus, (4) holds.

2. Recall from definition,

$$g_{L_{i-1}}(t,s) = e_t(s) - \sum_{j=1}^{i-1} \ell_{j,i-1}(t) e_{\theta_j}(s)$$

We need to show

$$g_{L_i}(t,s) = e_t(s) - \sum_{j=1}^{i} \ell_{j,i}(t) e_{\theta_j}(s)$$

with $\ell_{j,i}$ as in (4) and Lagrange.

$$
\begin{aligned}
&g_{L_{i-1}}(t,s) - \ell_{i,i}(t) g_{L_{i-1}}(\theta_i, s) \\
&\quad = e_t(s) - \sum_{j=1}^{i-1} \ell_{j,i-1}(t) e_{\theta_j}(s) - \ell_{i,i}(t) e_{\theta_i}(s) + \sum_{j=1}^{i-1} \ell_{i,i}(t) \ell_{j,i-1}(\theta_i) e_{\theta_j}(s) \\
&\quad = e_t(s) - \left(\sum_{j=1}^{i-1} \left(\ell_{j,i-1}(t) - \ell_{i,i}(t) \ell_{j,i-1}(\theta_i) \right) e_{\theta_j}(s) \right) - \ell_{i,i}(t) e_{\theta_i}(s) \\
&\quad = e_t(s) - \sum_{j=1}^{i} \ell_{j,i}(t) e_{\theta_j}(s) = g_{L_i}(t,s).
\end{aligned}
$$

with the last equality holding since L_i is Lagrange.

□

Corollary 3. *The set $\{\ell_{i,i} : i = 1, \ldots, \#\Theta\}$ is a Newton Basis for interpolation on Θ.*

Proof. By construction, $\ell_{i,i} \in I(\Theta_{i-1})$ and $\ell_{i,i}(\theta_i) = 1$. □

This suggests that the construction of Newton polynomials for nested interpolation schemes may be represented by the following algorithm.

Algorithm 1.

```
Initialize g(t,s) = [t^α : α ∈ Z^d_+]
for i = 1 to #Θ
     pick θ_i ∈ Θ\{θ_1, ..., θ_{i-1}}
     compute g(θ_i, s) = [g_α(θ_i) : α ∈ Z^d_+]
     select ℓ_{i,i} = Σ_{α∈Z^d_+} c_α g_α such that ℓ_{i,i}(θ_i) = 1
     update g(t,s)
end
```

Note that by Lemma 2, there is a $\{c_\alpha\}$ such that only finitely many c_α's are nonzero (e.g., $\ell_{j,j}$ can be chosen to be a polynomial). We will deal with ways to make this choice in Section 8.

To demonstrate Algorithm 1, again consider $\Theta = \{(1,0), (0,1), (-1,0), (0,-1)\}$. When $i = 0$, this is similar to the start of the Least construction

$$g_{L_0}(t,s) = [1,\ t_x,\ t_y,\ t_x^2,\ t_x t_y,\ t_y^2,\ \ldots\ ].$$

With $\theta_1 = (1,0)$, select $\ell_{1,1}(t) = 1$, then

$$g_{L_1}(t,s) = g_{L_0}(t,s) - \ell_{1,1}(t) g_{L_0}((1,0), s)$$

$$\begin{array}{rcl} g_{L_0}((1,0),s) & = & [\ 1, \quad 1, \quad 0, \quad 1, \quad 0, \quad 0, \quad \dots] \\ \therefore \quad g_{L_1}(t,s) & = & [\ 0, \quad t_x-1, \quad t_y, \quad t_x^2-1, \quad t_xt_y, \quad t_y^2, \quad \dots] \end{array}$$

With $\theta_2 = (0,1)$, select $\ell_{2,2}(t) = (t_x - 1)/(-1) = 1 - t_x$, then

$$g_{L_2}(t,s) = g_{L_1}(t,s) - \ell_{2,2}(t) g_{L_1}((0,1),s)$$

$$\begin{array}{rcl} g_{L_1}((0,1),s) & = & [\ 0, \quad -1, \quad 1, \quad -1, \quad 0, \quad 1, \quad \dots] \\ \therefore \quad g_{L_2}(t,s) & = & [\ 0, \quad 0, \quad t_x+t_y-1, \quad t_x^2-t_x, \quad t_xt_y, \quad t_y^2+t_x-1, \quad \dots] \end{array}$$

With $\theta_3 = (-1,0)$, select $\ell_{3,3} = (t_x + t_y - 1)/(-2) = \frac{-1}{2}((t_x + t_y - 1)$, then

$$\begin{array}{rcl} g_{L_2}((-1,0),s) & = & [\ 0, \quad 0, \quad -2, \quad 2, \quad 0, \quad -2, \quad \dots] \\ \therefore \quad g_{L_3}(t,s) & = & [\ 0, \quad 0, \quad 0, \quad t_x^2+t_y-1, \quad t_xt_y, \quad t_y^2-t_y, \quad \dots] \end{array}$$

Finally, select $\ell_{4,4}(t) = \frac{-1}{2}(t_x^2 + t_y - 1)$, which results in a Newton Basis:

$$\left\{1, 1 - t_x, \frac{-1}{2}(t_x + t_y - 1), \frac{-1}{2}(t_x^2 + t_y - 1)\right\}.$$

If we continue the process once more, we have

$$\begin{array}{rcl} g_{L_3}((0,-1),s) & = & [\ 0, \quad 0, \quad 0, \quad -2, \quad 0, \quad 2, \quad \dots] \\ \therefore \quad g_{L_4}(t,s) & = & [\ 0, \quad 0, \quad 0, \quad 0, \quad t_xt_y, \quad t_x^2+t_y^2-1, \quad \dots] \end{array}$$

This gives us the set

$$\{t_xt_y, t_x^2 + t_y^2 - 1\}$$

which generates $I(\Theta)$.

Performing the back substitution from (4), we get the Lagrange basis of

$$\left\{\frac{1}{2}(t_x^2 + t_x), \frac{-1}{2}(t_x^2 - t_y - 1), \frac{1}{2}(t_x^2 - t_x), \frac{-1}{2}(t_x^2 + t_y - 1)\right\}.$$

The choice of $\ell_{j,j}(t)$ in this example was made for illustration of Theorem 3, but in general it is a poor method for choosing the basis function. In the next sections we will explore how one would make better choices.

Note also that Algorithm 1 is quite general, capturing any monotonic scheme. In particular, for any scheme S that uses a fixed basis, it is possible to construct a sequence of Newton polynomials that span the interpolation space. Since any $I(\Theta_i)$ is spanned by $\{g_{L_i,\alpha}\}$, a representation as in Algorithm 1 can be found. Further, monotonicity is one of the properties of the Least interpolation scheme, so one should be able construct an Algorithm 1 approach to its construction, as we will do in Section 8.3.

6 On errors and degeneracies

The algorithm used to compute Newton polynomials for monotonic (or nested) schemes outlines in the previous section requires the choice of $\ell_{j,j}$. To understand the consequences of this choice, we need to understand the connection between the geometry Θ and the space Π_L. Ultimately, we would like to choose Π_L such that the interpolation error

$$f - P_L f$$

is as small as possible for as wide a class of functions as possible. (Note that that the observation that the Newton approach interpolates error is not new; see, for example, [9].)

In Section 1, we constructed an example where approximation power was lost before correctness. To understand this loss of approximation power, we will consider the situation where one fixes the interpolation space and varies the geometry. Specifically, assume that one has processed $j-1$ points in Θ and is happy with the interpolation provided by L_{j-1}. Further, let us fix the choice of the last Newton polynomial in our basis, $\nu_j(t)$ (we work with the Newton polynomial rather than the Lagrange polynomial $\ell_{j,j}(t)$, since the scaling of $\ell_{j,j}(t)$ will change with θ_j). The following shows the relationship between interpolation by L_j and L_{j-1}, using the construction in Theorem 3.

$$\begin{aligned}
P_{L_j} f(t) &= f(\theta_j)\frac{\nu_j(t)}{\nu_j(\theta_j)} + \sum_{i=1}^{j-1} f(\theta_i)\ell_{i,j}(t) \\
&= f(\theta_j)\frac{\nu_j}{\nu_j(\theta_j)} + \sum_{i=1}^{j-1} f(\theta_i)\left(\ell_{i,j-1}(t) - \ell_{i,j-1}(\theta_j)\frac{\nu_j(t)}{\nu_j(\theta_j)}\right), \quad \text{by (4)} \\
&= \left[f(\theta_j) - \sum_{i=1}^{j-1} f(\theta_i)\ell_{i,j-1}(\theta_j)\right]\frac{\nu_j}{\nu_j(\theta_j)} + \sum_{i=1}^{j-1} f(\theta_i)\ell_{i,j-1} \\
&= (f - P_{L_{j-1}} f)(\theta_j)\frac{\nu_j}{\nu_j(\theta_j)} + P_{L_{j-1}} f, \qquad (6)
\end{aligned}$$

where $(f - P_{L_{j-1}} f)(\theta_j)$ and $P_{L_{j-1}} f$ are completely independent of the choice of ν_j. Note that $(f - P_{L_{j-1}} f)(\theta_j)$ is a difference operator that vanishes on $\Pi_{L_{j-1}}$.

With the relationship in (6), we can see that the quality of the interpolant is effected by the value $\nu_j(\theta_j)$. Namely, addition of this term can possibly make the interpolation worse.

Theorem 4. *Let Ω be a region where ν_j is bounded away from zero, $\nu_j(\theta) = 0$, and f be such that $(f - P_{L_{j-1}} f)(\theta) \neq 0$, then*

$$P_{L_j} f \to \infty \qquad \text{as} \qquad \theta_j \to \theta.$$

with

$$P_{L_j} f \sim (f - P_{L_{j-1}} f)(\theta_j)\frac{\nu_j}{\nu_j(\theta_j)}$$

for θ_j in a neighborhood of θ.

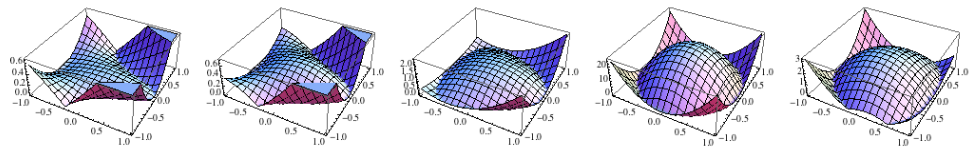

Figure 2: From the example in Figure 1, we plot the error function for interpolation of $t_x {t_y}^2$ by quadratics at the five Chebyshev points on the circle and the sixth point in $\{(0.4, 0.4), (0.5, 0.5), (0.6, 0.6), (0.7, 0.7), (0.8, 0.8)\}$. One can see the influence of $t_x^2 - t_y^2 - 1$ in all the error functions except the one for $(0.4, 0.4)$.

Proof. Since $\nu_j(\Omega) \neq 0$, $\forall \omega \in \Omega$

$$\begin{aligned}\frac{P_{L_j} f(\omega)}{\frac{\nu_j(\omega)}{\nu_j(\theta_j)}} &= \frac{\nu_j(\theta_j) P_{L_j} f(\omega)}{\nu_j(\omega)} = (f - P_{L_{j-1}} f)(\theta_j) + \frac{\nu_j(\theta_j) P_{L_{j-1}} f(\omega)}{\nu_j(\omega)} \\ &\to (f - P_{L_{j-1}} f)(\theta_j) \text{ as } \nu_j(\theta_j) \to 0.\end{aligned}$$

□

In the experiment shown in Figure 2, we see that the interpolant is dominated by a scaled version of $t_x^2 - t_y^2 - 1$, at least away from the other interpolation points. In the example, it is clear that the unit circle will be degenerate, but approximation power is lost well before we reach the circle. In general, identification of these degeneracies is not as obvious. Theorem 4 indicates that reduction in error requires sufficient flexibility in choosing $\nu_j(t)$ such that the variety of this Newton polynomial is away from the data point θ. While $\nu_j(t)$ must be in the $I(\{\theta_1, \ldots, \theta_{j-1}\})$, any scheme that limits the choice to a subset that introduces new points into the variety would not supply this flexibility. For example, schemes that construct $\nu_j(t)$ using a fixed monomial ordering or minimal schemes that select from

$$I(\{\theta_1, \ldots, \theta_{j-1}\}) \cap \Pi_k \quad \text{when} \quad I(I(\{\theta_1, \ldots, \theta_{j-1}\}) \cap \Pi_k) \neq I(\{\theta_1, \ldots, \theta_{j-1}\})$$

will admit degeneracies when there are additional points in its variety.

To avoid creating structural degeneracies in an interpolation scheme, one should select the Newton polynomial from a generating subset of $I(\{\theta_1, \ldots, \theta_{j-1}\})$.

For a further understanding of the error, we exploit the standard Taylor expansion.

Theorem 5. *Let $k = \deg L$ and $f \in C^{k+1}(\mathbb{R}^d)$ (i.e., a function with $k+1$ continuous derivatives), then there exists $\{R_\beta(t) : \beta \in \mathbb{Z}_+^d, |\beta| = k+1\}$ with*

$$\begin{aligned}(f - P_L f)(t) &= \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) g_{L,\alpha}(t) && (7) \\ &\quad + \sum_{\vartheta \in \Theta} \ell_\vartheta(t) \sum_{|\beta| = k+1} \frac{1}{\beta!} \left(t^\beta R_\beta(t) - \vartheta^\beta R_\beta(\vartheta) \right). && (8)\end{aligned}$$

Further, $R_\beta(t) = D^\beta f(r(t)t)$*, with* $0 \le r(t) \le 1$*.*

Proof. By the multivariate Taylor Theorem (see, e.g., [8]),

$$f(t) = \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f(0) t^\alpha + \sum_{|\beta|=k+1} \frac{1}{\beta!} R_\beta(t) t^\beta,$$

with $R_\beta(t) = D^\beta f(r(t)t)$, with $0 \le r(t) \le 1$.

By Lemma 1 and the definition of P_L,

$$\begin{aligned} (f - P_L f)(t) &= \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f(0) g_{L,\alpha}(t) + \sum_{|\beta|=k+1} \frac{1}{\beta!} R_\beta(t) t^\beta \\ &\qquad - \sum_{|\beta|=k+1} \sum_{\vartheta\in\Theta} \frac{\vartheta^\beta}{\beta!} R_\beta(\vartheta) \ell_\vartheta(t) \\ &= \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f(0) g_{L,\alpha}(t) + \sum_{\vartheta\in\Theta} \left(\sum_{|\beta|=k+1} \frac{1}{\beta!} R_\beta(t) t^\beta \right) \ell_\vartheta(t) \\ &\qquad - \sum_{|\beta|=k+1} \sum_{\vartheta\in\Theta} \frac{\vartheta^\beta}{\beta!} R_\beta(\vartheta) \ell_\vartheta(t), \end{aligned}$$

since $\sum_{\vartheta\in\Theta} \ell_\vartheta(t) = 1$. Collecting terms, we have

$$\begin{aligned} (f - P_L f)(t) &= \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f(0) g_{L,\alpha}(t) \\ &\qquad + \sum_{|\beta|=k+1} \frac{1}{\beta!} \sum_{\vartheta\in\Theta} \left(t^\beta R_\beta(t) - \vartheta^\beta R_\beta(\vartheta) \right) \ell_\vartheta(t). \end{aligned}$$

□

There are two parts of this error formulation. The first term

$$\sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f(0) g_{L,\alpha}(t)$$

is related to polynomial reproduction, i.e., how well we interpolate the monomials in Π_k. By a simple counting argument, there will only be a few linearly independent terms left in this sum. More precisely,

$$\dim \left\{ g_{L,\alpha} : \alpha \in \mathbb{Z}_+^d, |\alpha| \le k \right\} = \dim \Pi_k(\mathbb{R}^d) - \#\Theta.$$

The second term contains the factor

$$\sum_{|\beta|=k+1} \frac{1}{\beta!} \left(t^\beta R_\beta(t) - \vartheta^\beta R_\beta(\vartheta) \right),$$

which is a smoothness condition on the unknown function and is outside of our control. However, this term is multiplied by a Lagrange polynomial, and hence can be made worse if this polynomial is poorly behaved. This is further complicated by recalling the choice (with c_α having compact support),

$$\ell_{j,j}(t) = \sum_{\alpha} c_\alpha g_{L_{j-1},\alpha}(t). \tag{9}$$

With this, we are constructing our Lagrange polynomials from the error in interpolation of the monomials. Indeed, by Theorem 2, any iterative scheme must select the $j+1$ Lagrange from $I(\Theta_j)$ which is spanned by errors in the interpolation. Therefore, using tolerances to determine proximity of the interpolation point to the variety of a possible Newton polynomial can be challenging.

What should be pointed out is that any scheme that forces the selection of the Lagrange polynomial, for example by predetermining a monomial order or by enforcing minimal degree, will run afoul of robustness issues.

This combined with Theorems 4 and 5 suggests that an interpolation scheme constructed by sequentially selecting Lagrange polynomials must be robust in that selection. Any problems with a particular choice should show up already when investigating an appropriate subset of $g_{L,\alpha}$'s. The question is what subset? It is our contention that one can understand and control the error by investigating the error on a generating subset of the ideal $I(\Theta)$. We will show in the next section that such a generating subsequence must exist in

$$\left\{ g_{L,\alpha} : \alpha \in \mathbb{Z}_+^d, |\alpha| \leq k+1 \right\} = \Pi_{k+1} \cap I(\Theta)$$

and we will show how such a linearly independent generating subset can be computed.

7 On the construction of Π_L and generating subsets of $I(\Theta)$

Now that we understand how an inappropriately chosen Lagrange polynomial can lead to poor interpolation spaces, we will continue to explore how the choice of $\ell_{j,j}$ impacts Π_L. Explicitly, we will leverage (9) and investigate the impact of the choice of $\{c_\alpha\}$. After that, we will show how this understanding can be used to construct a minimal generating set for $I(\Theta_j)$ that balances the need for low degree and for robustness.

With the Lagrange normalization requirement

$$1 = \ell_{j,j}(\theta_j) = \sum_{\alpha \in \mathbb{Z}_+^d} c_\alpha g_{L_{j-1},\alpha}(\theta_j),$$

we can determine exactly how we are augmenting the interpolation space $\Pi_{L_{j-1}}$ by the choice in (9), namely by

$$\operatorname{span} \left\{ \sum_{\alpha \in \mathbb{Z}_+^d} c_\alpha t^\alpha \right\},$$

as shown in the following theorem.

Theorem 6. *Let* $\sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha g_{L_{j-1},\alpha}(\theta_j) = 1$ *and* $\ell_{j,j} = \sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha g_{L_{j-1},\alpha}(t)$*, then*

$$\Pi_{L_j} = \Pi_{L_{j-1}} \oplus \operatorname{span}\left\{\sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha t^\alpha\right\}.$$

Proof. Since $\dim \Pi_{L_j} = \dim \Pi_{L_{j-1}} + 1$, we need to show that $\sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha t^\alpha \in \Pi_{L_j}$ but $\sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha t^\alpha \notin \Pi_{L_{j-1}}$. For the second condition, using Lemma 1

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha t^\alpha - P_{L_{j-1}}\big(\sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha t^\alpha\big) &= \sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha g_{L_{j-1},\alpha} \\ &= \ell_{j,j}(t) \neq 0.\end{aligned}$$

Similarly, using Lemma 1, (5), and the assumed definition and normalization of $\ell_{j,j}$

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha t^\alpha - P_{L_j}\big(\sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha t^\alpha\big) &= \sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha g_{L_j,\alpha}(t) &(10)\\ &= \sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha\big(g_{L_{j-1},\alpha}(t) - \ell_{j,j}(t) g_{L_{j-1},\alpha}(\theta_j)\big) &(11)\\ &= \ell_{j,j}(t) - \ell_{j,j}(t)\sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha g_{L_{j-1},\alpha}(\theta_j) = 0. &(12)\end{aligned}$$

□

From lines (10) - (12), we see that

$$\sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha g_{L_j,\alpha}(t) = 0.$$

This means that the $g_{L_j,\alpha}$'s are linearly dependent, i.e., we have removed a polynomial from the ideal. We next conclude that this linear dependent relationship is maintained even as the $g_{L_j,\alpha}$'s are modified by later operations.

Corollary 4. *With the same conditions as in Theorem 6,*

$$\sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha g_{L_i,\alpha} = 0, \quad \forall i \geq j.$$

Proof. We will use induction. When $i = j$, the result follows from lines (10) - (12) above.

Assume true for i where $i \geq j$, then we need to show that the result is true for $i + 1$

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha g_{L_{i+1},\alpha}(t) &= \sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha\big(g_{L_i,\alpha}(t) - \ell_{i+1,i+1}(t) g_{L_i,\alpha}(\theta_i)\big) \\ &= \left(\sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha g_{L_i,\alpha}(t)\right) - \ell_{i+1,i+1}(t)\left(\sum_{\alpha\in\mathbb{Z}_+^d} c_\alpha g_{L_i,\alpha}(\theta_i)\right) = 0,\end{aligned}$$

by the inductive assumption. □

A generating subset for $I(\Theta)$ would be sufficient to avoid the type of structural degeneracies noted in the previous section. However, for efficient computations, we would still like to have control over the degree and complexity of the Lagrange polynomials. I.e., we would like to have our interpolation space as small as possible while having a generating set for the ideal, $I(\Theta)$.

Definition 5. *Let* $k_\theta := \min\{k : I(I(\Theta) \cap \Pi_k) = I(\Theta)\}$*, then we call a set*

$$G_\Theta := \{g_{L,\alpha} : \deg g_{L,\alpha} \leq k_\Theta\}$$

minimally complete *if* $I(G_\Theta) = I(\Theta)$.

de Boor and Ron have shown that the Least polynomial space Π_Θ has minimum degree [5]. We would like to construct an interpolant that only increases this degree by at most 1. As the following lemma shows, it is possible to find generating subsets of $I(\Theta)$ that are in $\Pi_{\deg \Pi_\Theta + 1}$.

Lemma 3. $k_\Theta \leq \deg \Pi_\Theta + 1$.

Proof. We need to show that

$$I(\Theta) = I(I(\Theta) \cap \Pi_{\deg \Pi_\Theta + 1}).$$

It is sufficient to show that $\forall k$

$$I(\Theta) \cap \Pi_k \subset I(I(\Theta) \cap \Pi_{\deg \Pi_\Theta + 1}).$$

This is trivially true when $k \leq \deg \Pi_\Theta + 1$. Assume it is true for some $k > \deg \Pi_\Theta + 1$ and note that for any β such that $|\beta| > \deg \Pi_\Theta$ and L the Lagrange basis for Π_Θ,

$$g_{L,\beta}(t) = t^\beta - P_{\Pi_\Theta}(t^\beta) \text{ with } \deg P_{\Pi_\Theta}(t^\beta) < |\beta|.$$

Pick α with $|\alpha| = \deg \Pi_\Theta + 1$ and $\beta - \alpha \in \mathbb{Z}_+^d$, then

$$g_{L,\beta}(t) - t^{\beta-\alpha} g_{L,\alpha}(t) \in I(\Theta) \cap \Pi_k.$$

By the inductive hypothesis, $g_{L,\beta} \in I(I(\Theta) \cap \Pi_{\deg \Pi_\Theta + 1})$. Since this is true for all k, and using Theorem 2, we have shown

$$I(\Theta) = I(I(\Theta) \cap \Pi_{\deg \Pi_\Theta + 1}).$$

□

Definition 6. *A Newton method is* **minimally complete** *if for each i, $\ell_{i,i}$ is selected from a minimally complete set. I.e., $\ell_{i,i} \in \operatorname{span}\{G_{\Theta_i}\}$.*

Theorem 7. *Any minimally complete Newton method has degree $\leq \deg \Pi_\Theta + 1$.*

Proof. This is trivially true for the empty set of points. Assume it is true for point sets with n points; that it is true for $n+1$ points follows directly from the inductive hypothesis, Theorem 3 and Lemma 3. $\square$

We will occasionally abuse notation and refer to $\alpha \in G_\Theta$ to reference the index of element $g_{L,\alpha} \in G_\Theta$. With this, we will let (9) have the form

$$\ell_{j,j}(t) = \sum_{\alpha \in G_{\Theta_{j-1}}} c_\alpha g_{L_{j-1},\alpha}(t). \tag{13}$$

The number of elements in G_Θ can be minimized by leveraging Corollary (4) and using a division algorithm similar to that used in the proof of Lemma 3 to only admit elements of $\{g_{L,\alpha} : |\alpha| = k_\Theta\}$ that are required for completeness. The resulting algorithm would have the following form:

Algorithm 2.

Starting with a G_{Θ_j} and k (initialized to $\{1\}$ and 0)
Select $\ell_{j,j}$ from G_{Θ_j}
Set k to $\min\{k, \deg \ell_{j,j} + 1\}$
Use Corollary (4) to drop one element from G_{Θ_j}, and call this $G_{\Theta_{j+1}}$
Use Theorem 3 to update all the elements of $G_{\Theta_{j+1}}$
For each $g_{L,\alpha}$ with $|\alpha| \leq k$ and not already added to $G_{\Theta_{j+1}}$
    if $g_{L,\alpha} \notin I(G_{\Theta_{j+1}})$, add it.

It is the last step that requires a division algorithm to solve the ideal membership problem. Leveraging a mechanism based on Gaussian Elimination by Segments, we have a solution to this problem that does not require introduction of a monomial ordering. This result will appear in a later paper. However, our algorithm does not require that construction; using any standard polynomial division algorithm will suffice (see, [1]).

8 On the construction of error sensitive interpolation schemes

Given the concept of minimal completeness and an understanding of Π_L, we now are ready to construct examples of minimally complete interpolation schemes using Theorem 3 and (13). We will construct two examples; an example that reduces the number of monomials in Π_L while controlling error, and a minimally complete analog of the Least interpolation method.

8.1 The monomial case

Assume that we want to minimize the number of terms in our Lagrange polynomials, thus reducing the cost of evaluation. The monomial schemes will construct Lagrange polynomials from $\#\Theta$ monomials. Here, (13) becomes

Monomial Simplification Choice:

$$\ell_{j,j}(t) = \frac{g_{L_{j-1},\alpha}(t)}{g_{L_{j-1},\alpha}(\theta_j)}$$

for some $\alpha \in G_{\Theta_{j-1}}$.

A possible method here would be a **Greedy Approach**. Namely, find

$$\max \left\{ |g_{L_{j-1},\alpha}(\theta_j)| : \alpha \in G_{\Theta_{j-1}} \right\}.$$

This has the advantage of simplicity, but there is no assurance of minimizing error.

For a more complete approach, we would need to find

$$\min_{\alpha \in G_{\Theta_{j-1}}} \max_{\beta \in G_{\Theta_{j-1}}} \left\| \left(g_{L_{j-1},\beta}(t) - \frac{g_{L_{j-1},\beta}(\theta_j)}{g_{L_{j-1},\alpha}(\theta_j)} g_{L_{j-1},\alpha}(t) : g_{L_{j-1},\alpha}(\theta_j) \neq 0 \right) \right\|_\infty .$$

While this can be computed exactly, in practice, maintaining an additional sampling of $g_{L,\alpha}(t)$, for example at the Vorinoi tessellation of Θ, should be sufficient to act as an estimate for this computation. Such a sampling would provide a better estimator of the optimal monomial to eliminate than would the single sample provided by the greedy approach.

8.2 The least square case

If the computation complexity of the resulting Lagrange polynomials is not a constraint, then assume a more general form for $\ell_{j,j}(t)$, but limit it in the following way:

Least Square Simplification Choice: Find a pointwise vector 2-norm minimum over $G_{\Theta_{j-1}}$. I.e., for fixed t, find $\ell_{j,j}(t)$ to minimize

$$\left\| G_{\Theta_{j-1}}(t) - G_{\Theta_{j-1}}(\theta_j)\ell_{j,j}(t) \right\|_2 ,$$

where $G_{\Theta_{j-1}}(t)$ is thought of as a vector.

The standard method to solve the problem of minimizing $G_{\Theta_{j-1}}(t) - \ell_{j,j}(t)G_{\Theta_{j-1}}(\theta_j)$ is least squares. I.e, $\ell_{j,j}(t)$ is chosen such that

$$\begin{aligned}
0 &= (G_{\Theta_{j-1}}(t) - \ell_{j,j}(t)G_{\Theta_{j-1}}(\theta_j)) \cdot G_{\Theta_{j-1}}(\theta_j) \\
\Longrightarrow \ell_{j,j}(t) &= \frac{G_{\Theta_{j-1}}(t) \cdot G_{\Theta_{j-1}}(\theta_j)}{G_{\Theta_{j-1}}(\theta_j) \cdot G_{\Theta_{j-1}}(\theta_j)} \\
&= \frac{1}{G_{\Theta_{j-1}}(\theta_j) \cdot G_{\Theta_{j-1}}(\theta_j)} \sum_{\alpha \in G_{\Theta_{j-1}}} g_{L_{j-1},\alpha}(\theta_j) g_{L_{j-1},\alpha}(t) \\
&= \frac{1}{\sum_{\beta \in G_{\Theta_{j-1}}} g^2_{L_{j-1},\beta}(\theta_j)} \sum_{\alpha \in G_{\Theta_{j-1}}} g_{L_{j-1},\alpha}(\theta_j) g_{L_{j-1},\alpha}(t)
\end{aligned}$$

Note the following:

- $\forall t \in \mathbb{R}^d$, $\sum_{\alpha \in G_{\Theta_{j-1}}} g^2_{L_{j-1},\alpha}(t) > 0$, therefore there are no degeneracies.
- The resulting method is stable under perturbation of θ_j, more precisely, the map

$$\mathbb{R}^d \mapsto \Pi(\mathbb{R}^d) : s \mapsto \frac{1}{\sum_{\beta \in G_{\Theta_{j-1}}} g^2_{L_{j-1},\beta}(s)} \sum_{\alpha \in G_{\Theta_{j-1}}} g_{L_{j-1},\alpha}(s) g_{L_{j-1},\alpha}(t)$$

  is continuous.
- One can continuously perturb any of the interpolation points, say point θ_i, by using $\ell_{i,\#\Theta}(t)$ and some subset of G_Θ, call the subset G_Θ^{new} such that $I(\{\ell_{i,\#\Theta} \cup G_\Theta^{\text{new}}\}) = I(\Theta \setminus \theta_i)$ using the map

$$\mathbb{R}^d \mapsto \Pi(\mathbb{R}^d) : s \mapsto \frac{1}{\ell^2_{i,\#\Theta}(s) + \sum_{g \in G_\Theta^{\text{new}}} g^2(s)} \left(\ell_{i,\#\Theta}(s)\ell_{i,\#\Theta}(t) + \sum_{g \in G_\Theta^{\text{new}}} g(s)g(t) \right).$$

  Therefore, this scheme is continuous with respect to any single data point.

For these reasons, we will call this method out specifically:

Definition 7. *The* **Complete Minimal method** *is the choice*

$$\ell_{j,j}(t) = \frac{1}{\sum_{\alpha \in G_{\Theta_{j-1}}} g^2_{L_{j-1},\alpha}(\theta_j)} \sum_{\alpha \in G_{\Theta_{j-1}}} g_{L_{j-1},\alpha}(\theta_j) g_{L_{j-1}}. \tag{14}$$

Note that this is an unweighted average of the elements of the ideal. While we use this unweighted average in this paper, using a weighted average is something we plan to explore in the future.

8.3 Complete Minimal method is the Least Method using minimally complete rather than minimal degree

More specifically, if we replace our requirement of minimally complete with the requirement of minimal degree, then the Complete Minimal interpolation method generates the Lagrange basis for the Least space, Π_Θ.

Definition 8. *Let* $p \in \Pi$, *then*

$$p_\uparrow(t) = \sum_{|\alpha| = \deg p} T_\alpha(p) t^\alpha / \alpha!,$$

is the **leading term** *of* p.

Lemma 4. *Let* $k_i = \min\{|\alpha| : g_{L_i,\alpha} \neq 0\}$,

$$\nu_{i+1}(t) = \sum_{|\alpha|=k_i} g_{L_i,\alpha}(\theta_{i+1}) g_{L_i,\alpha}(t), \tag{15}$$

then

$$\{(\nu_1)_\uparrow, \dots, (\nu_n)_\uparrow\}$$

is the Least Basis.

Proof. The Gram-Schmidt step in the Least computation is

$$w_{j,i+1} = w_{j,i} - \frac{v_i \cdot w_{j,i}}{v_i \cdot v_i} v_i,$$

where $v_i = (g_{L_{i-1},\alpha}(\theta_i) : |\alpha| = k_i)$ and $w_{j,i} = (g_{L_{i-1},\alpha}(\theta_j) : |\alpha| = k_i)$.

Select $\ell_{i,i}(t) = \nu_i(t)/\nu_i(\theta_i)$ and note

$$\ell_{i,i}(\theta_j) = \frac{\nu_i(\theta_j)}{\nu_i(\theta_i)} = \frac{v_i \cdot w_{j,i}}{v_i \cdot v_i}.$$

Then,

$$w_{j,i+1} = w_{j,i} - \ell_{i,i}(\theta_j) v_i.$$

Since $g_{L_0}(\Theta) = e_\Theta$, and each computation step is the same, the lemma holds. $\square$

Computationally, this result is interesting. It says that the entries of the matrix formed from e_Θ after the j-th step are $g_{L_j,\alpha}(\theta_i)$ for $i > j$. Thus, it is possible to compute the quantities needed to generate $g_{L_j,\alpha}(t)$ and $\ell_\vartheta(t)$ with a minimal number of polynomial evaluations. The result also indicates as the number of points increase, and the error in interpolation decreases, the values in the least matrix will become small making the use of tolerances to determine degeneracy difficult.

Finally, for completeness, we will show that by restricting to minimum degree, the Complete Minimal interpolation method yields the Least Basis.

Corollary 5. *With* ν_{i+1} *chosen as in (15),* $L(\Theta)$ *the corresponding Lagrange basis, and* Π_Θ *the Least interpolation space, then*

$$\Pi_{L(\Theta)} = \Pi_\Theta.$$

Proof. Let $K_i = \left\{\alpha \in \mathbb{Z}_+^d : |\alpha| = k_i, g_{L_i,\alpha}(\theta_{i+1}) \neq 0\right\}$, let $\|v_i\| = \sum_{\alpha \in K_i} \left(g_{L_i,\alpha}(\theta_{i+1})\right)^2$. With ν_{i+1} chosen as in (15),

$$\ell_{i+1,i+1}(t) = \sum_{\alpha \in K_i} \frac{g_{L_i,\alpha}(\theta_{i+1})^2}{\|v_i\|} \frac{g_{L_i,\alpha}(t)}{g_{L_i,\alpha}(\theta_{i+1})}.$$

Since

$$\sum_{\alpha \in K_i} \frac{g_{L_i,\alpha}(\theta_{i+1})^2}{\|v_i\|} = 1$$

and $g_{L_i,\alpha}(\theta_{i+1}) \neq 0$ for all $\alpha \in K_i$, we know from Theorem (6) that

$$\Pi_{L(\Theta)} = \operatorname{span}\left\{ \sum_{\alpha \in K_i} \frac{g_{L_i,\alpha}(\theta_{i+1})}{\|v_i\|} t^\alpha \right\} = \operatorname{span}\{(\nu_{i+1})_\uparrow\} = \Pi_\Theta.$$

□

9 Examples

We will now present thre examples. The first example, with three points, will show how the two methods from the previous section interact with the error. We then repeat the example from Section 1 using the Complete Minimal method and compare results. Finally, we look at an example with 19 points, looking to place the 20th. In that example, although there are still two polynomials to choose from in $\Pi_5(\mathbb{R}^2)$, certain locations will still yield poor results for minimal methods.

9.1 Three points

In this example, $\Theta = \{(-1,0),(1,0),(a,b)\}$. After computing the first two points, one has

$$\begin{aligned}
\Theta_2 &= \{(-1,0),(1,0)\} \\
L_2 &= \{\ell_{1,2}(x,y), \ell_{2,2}(x,y)\} = \{(1-x)/2, (1+x)/2\} \\
G_{L_2} &= \left\{0,0,y,x^2-1,xy,y^2,x^3-x,x^2y,xy^2,y^3,\ldots\right\} \\
G_{\Theta_2} &= \left\{y, x^2-1\right\}
\end{aligned}$$

We now need to pick our Lagrange polynomial for the point (a,b). Using the monomial simplification choice, one would have to pick from

$$\ell_{3,3}(x,y) \in \left\{\frac{y}{b}, \frac{x^2-1}{a^2-1}\right\},$$

and with the least squared choice,

$$\ell_{3,3}(x,y) = \frac{by + (a^2-1)(x^2-1)}{b^2 + (a^2-1)^2}.$$

Looking at G_{L_3}, with $\ell_{3,3}(x,y) = y/b$, we have

$$g_{(2,0),L_3}(x,y) = (x^2-1) - \frac{(a^2-1)}{b} y = \frac{1}{b}\left(b(x^2-1) - (a^2-1)y\right);$$

with $\ell_{3,3}(x,y) = (x^2-1)/(a^2-1)$, we have

$$g_{(0,1),L_3}(x,y) = y - \frac{b}{(a^2-1)}(x^2-1) = \frac{-1}{(a^2-1)}\left(b(x^2-1)-(a^2-1)y\right);$$

while with $\ell_{3,3}(x,y) = (by + (a^2-1)(x^2-1))/(b^2+(a^2-1)^2)$, we have

$$\begin{aligned} g_{(0,1),L_3}(x,y) &= y - \frac{b}{b^2+(a^2-1)^2}(by+(a^2-1)(x^2-1)) \\ &= \frac{(1-a^2)}{b^2+(a^2-1)^2}\left(b(x^2-1)-(a^2-1)y\right), \\ g_{(2,0),L_3}(x,y) &= (x^2-1) - \frac{(a^2-1)}{b^2+(a^2-1)^2}(by+(a^2-1)(x^2-1)) \\ &= \frac{b}{b^2+(a^2-1)^2}\left(b(x^2-1)-(a^2-1)y\right). \end{aligned}$$

We thus see that in the first case, the error is concentrated in the $g_{(2,0)}$ term; in the second case, the error is concentrated in the $g_{(0,1)}$ term; while in the third case the error is spread over the $g_{(0,1)}$ and $g_{(2,0)}$ terms.

9.2 Example: five fixed points with sixth point varying.

We repeat the example from the introduction (Figure 1), using the Complete Minimal method rather than interpolation by quadratics (Π_2). The minimally complete generating set for the ideal associated with the five fixed points consisted of one quadratic, namely $t_x^2 + t_y^2 - 1$, and two cubics. The Complete Minimal method leverages all three polynomials when selecting the sixth Lagrange polynomial, avoiding the degeneracies demonstrated with the quadratics. Figure 3 shows that the error is reduced for significantly more locations of the 6th point with the Complete Minimal method, and that even when the error increases, the increase is relatively small (bounded by 0.25).

Similarly, we repeat the plots from Figure 2 using the Complete Minimal method instead of the minimal quadratics. While the errors for the minimal scheme grew to over twenty times the value of $t_x t_y^2$, Figure 4 shows that the error functions are more stable for the Complete Minimal method as the sixth interpolation point changes.

9.3 Example: nineteen fixed points with twentieth point varying.

While the previous example showed that problems can occur when selecting 6 polynomials to interpolate 6 points from a polynomial space with 6 elements (Π_2), problems can occur in other cases. Consider the following example with 19 randomly chosen points. $\dim \Pi_5(\mathbb{R}^2) = 21$, and with this particular set of points (as with most randomly chosen point sets), the Minimally Complete generating set is comprised of 2 polynomials of degree 5 and 3 polynomials of degree 6. A minimal scheme would need to be of degree 5, so would

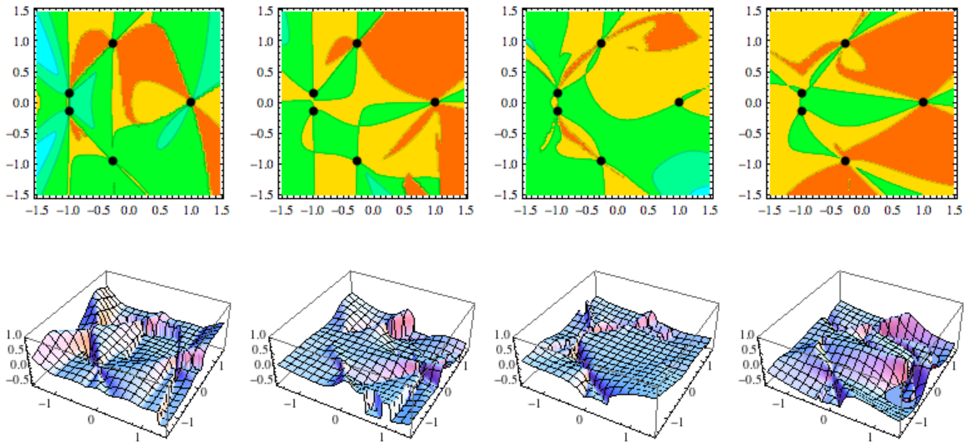

Figure 3: Experiment: Fix five points on the unit circle and allow the sixth point to vary on a 101x101 grid. For each point, compute how much the addition of the sixth point has improved the interpolant, i.e., the maximum error for interpolation of t_x^3, $t_x^2 y$, $t_x t_y^2$, and t_y^2 at six points minus the respective maximum error for interpolation by 5 fixed points. In the contour plot, brown and yellow are improvements, and all other colors are places where the interpolation with that point is worse, with green being close to zero.

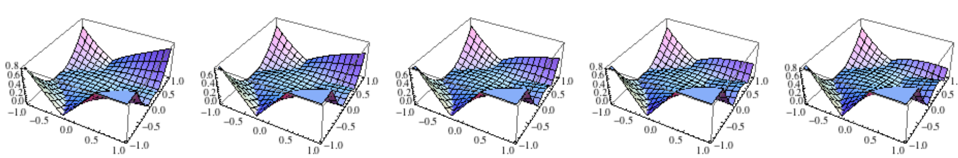

Figure 4: Similar to Figure 2, we plot the error function for interpolation of $t_x {t_y}^2$ by a Complete Minimal method at the five Chebyshev points on the circle and the sixth point in $\{(0.4, 0.4), (0.5, 0.5), (0.6, 0.6), (0.7, 0.7), (0.8, 0.8)\}$.

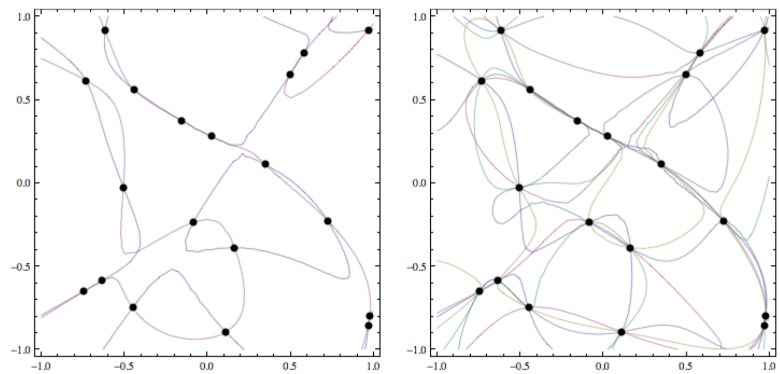

Figure 5: On the left are the two zero sets for the degree five generators, while on the right are the zero sets for each of the five Minimally Complete generators.

pick the Lagrange polynomial from a linear combination of the two degree five polynomials, while a Complete Minimal method would use all five polynomials.

Figure 5 shows the zero sets for the 2 polynomials of degree 5, and the zero set for all five polynomials in the Minimally Complete generating set.

Note while the variety for the ideal generated by the two polynomials of degree five includes the nineteen fixed points, it also has additional points. Again, we test the impact of adding a twentieth point from a 101 x 101 grid of points by computing the maximum error for interpolation of the degree six monomials. The maximum error is constrained to the convex hull of the original nineteen points. Rather than show a plot for each monomial, we ask what is the best the interpolation scheme can do (the minimum of the maximum values at the point) and what is the worst (the maximum of the maximum values at the point). These plots are given in Figure 6. Clearly, the Complete Minimal method has a more controlled error.

10 Conclusions

In this paper, we explored the strong connection between algebraic geometry and polynomial interpolation. In particular, we used algebraic geometry tools to develop a machinery for the analysis of nested polynomial interpolation schemes. The main practical result coming out of our analysis is that for robustness, one should choose the next basis function for polynomial interpolation from a complete generating set of the ideal. One consequence of this is one should not use a minimum degree scheme if one wants to control the error of an interpolation scheme.

This work began when we, the authors, were attempting to use the Least interpolation scheme to solve to a problem motivated by an industrial application. We noted that approximately ten percent of the time, the results of our interpolation was highly oscillatory.

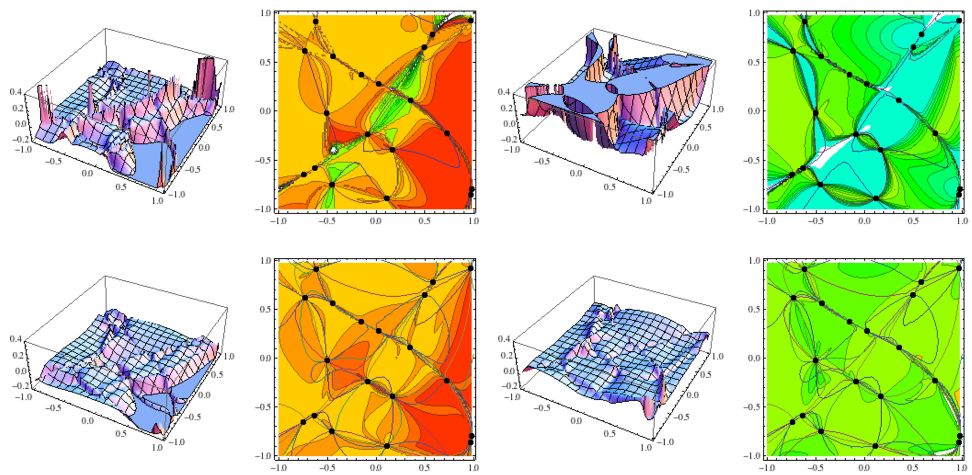

Figure 6: The best and worst case for interpolation of twenty points with nineteen fixed for a minimal scheme (top) and the Complete Minimal method (bottom).

At first we assumed we had a bug, then that we were running into a numerical issue when making the decision to increase degree. To analyze how to choose the tolerances, we programmed the Least into Mathematica and put a symbolic point into the computation. This lead to the discovery that the entries in the Least matrix were the errors in interpolation of the monomials, our $g_{L,\alpha}(\theta)$. Thus, we discovered that the problem we were attempting to address was mathematical rather than numerical in nature. The resulting analysis not only suggested a solution to this particular problem but also showed why the Least typically behaves so well (minimizing the error in pointwise least squared manner).

Our analysis can be applied to a wide range of polynomial interpolation schemes (in particular, it applies to any interpolation scheme that produces spaces that are nested as more points are added to the problem). It can be used to construct other robust schemes as we have done in Section 8. It shows that there is an interplay between robustness and complexity—restrictions like minimality can reduce the degrees of freedom that may be required to avoid robustness issues. We have concluded that the modified condition of **minimally complete** strikes the appropriate balance.

We purposely avoided dealing with collocation and interpolation of derivatives in this paper, although will make the unproven claim that the two methods in Section 8 are stable under collocation. We also have not addressed concrete implementation issues such as how to order the points and efficient solution of the ideal membership problem. While we show that the least square interpolation method is continuous as a function of a single interpolation point, θ, we have not addressed continuity when two or more points are modified simultaneously. Finally, while we have made some very general statements of error, we have not shown a detailed analysis of the error in the two schemes of Section 8. We will, however, make the following conjecture: The error can be characterized by the elements of G_Θ.

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