ON THE POISSON EQUATION FOR SINGULAR DIFFUSIONS [1]

SEYED N. HASHEMI
Department of Computer Science, 560 Machray Hall
University of Manitoba, 186 Dysart Road
Winnipeg, Manitoba R3T 2N2, Canada
telephone: (204) - 474 - 8313
fax: (204) - 474 - 7609
e-mail: shashemi@ee.umanitoba.ca

ANDREW J. HEUNIS [2]
Department of Electrical & Computer Engineering, 3115 EIT Centre
University of Waterloo, 200 University Avenue West
Waterloo, Ontario N2L 3G1, Canada
telephone: (519) - 885 - 1211 ext. 2083
fax: (519) - 746 - 3077
e-mail: heunis@kingcong.uwaterloo.ca

[1]Research supported by NSERC of Canada
[2]Corresponding Author

Abstract

Pardoux and Veretennikov [2001, Ann. Probab. 29, 1061-1085] establish existence (in the Sobolev sense) and uniqueness of solutions of a Poisson equation for the differential operator of an ergodic diffusion in $\Re^D$, for which the covariance term is strictly non-degenerate and uniformly bounded. Our goal is to establish a similar solvability result, but for the complementary case of singular diffusions, in which the covariance is not necessarily of full-rank. In return for abandoning strict positive-definiteness of the covariance, we postulate second-order smoothness of the coefficients of the diffusion, and, to secure ergodicity, we postulate a "stability" condition on the eigenvalues of the symmetrized Jacobian matrix of the drift. We establish existence of solutions of the Poisson equation in the classical sense, and use this to characterize limits in a time-scales problem arising from perturbation of a stochastic differential equation by a rescaled singular diffusion; this is motivated by a stochastic averaging principle of Liptser and Stoyanov [1990, Stochastics & Stochastics Reports, 32, 145-163].

Abbreviated Title: On the Poisson equation

AMS Subject Classifications: 60H30, 60J45, 60J60, 35J15

Key Words: Poisson equation, singular diffusions, multiple time-scales

1 Introduction

Pardoux and Veretennikov [15] establish conditions for existence of solutions $\Theta : \Re^D \to \Re$ of the Poisson partial differential equation

$$\mathcal{A}\Theta(y) = h(y), \qquad y \in \Re^D, \tag{1.1}$$

in which $\mathcal{A}$ is the linear second-order differential operator for an $\Re^D$-valued diffusion $\{\xi(t)\}$ given by the Itô stochastic differential equation

$$d\xi(t) = b(\xi(t))\, dt + \sigma(\xi(t))\, d\beta(t), \tag{1.2}$$

namely

$$\mathcal{A}\Theta(y) := \sum_{i=1}^{D} b^i(y)(\partial_{y^i}\Theta)(y) + \frac{1}{2}\sum_{i,j=1}^{D}(\sigma\sigma')^{ij}(y)(\partial_{y^iy^j}\Theta)(y), \qquad y \in \Re^D, \tag{1.3}$$

and $h : \Re^D \to \Re$ is a given function satisfying conditions that will shortly be outlined. Two basic hypotheses on the coefficients $b(\cdot)$ and $\sigma(\cdot)$ of (1.2) are postulated in [15] to ensure solvability of (1.1), namely (I) the matrix $\sigma\sigma'(y)$ is *strictly* positive-definite and bounded in the sense that there are constants $0 < \lambda_- < \lambda_+ < \infty$ and $\Lambda \in (0, \infty)$ such that

$$\lambda_- \le (\sigma\sigma'(y)y/|y|, y/|y|) \le \lambda_+, \quad \mathrm{Trace}\{\sigma\sigma'(y)\} \le \Lambda, \quad \text{for all non-zero } y \in \Re^D, \tag{1.4}$$

[see Condition (A_σ) in ([15], p.1062)]; and (II) there are constants $\alpha \in [-1, \infty)$ and $r \in (0, \infty)$ such that

$$(b(y), y/|y|) \le -r|y|^\alpha, \qquad \text{for all large } \ y, \tag{1.5}$$

[see Condition (A_b) in ([15], p.1063)]. The latter is basically a recurrence condition of Has'minskii type, which ensures that the transition probability of the Markov process defined by (1.2) has a unique invariant probability measure $\bar{m}$. Subject to these conditions it is shown in ([15], Theorem 1 and Theorem 2, p.1066) that, corresponding to each Borel-measurable and polynomially-bounded $h : \Re^D \to \Re$ centered by $\bar{m}$ (i.e. $\int_{\Re^D} h \; d\bar{m} = 0$), there exist continuous functions $\Theta : \Re^D \to \Re$ in the Sobolev class $W^2_{p,\,loc}(\Re^D)$ (for any $p \in (1, \infty)$), which are polynomially-bounded to an order determined by the polynomial-boundedness assumed for the function $h(\cdot)$, and which solve the Poisson equation (1.1) in the Sobolev sense. Moreover, the difference of any two such solutions is a constant value, thus solutions of (1.1) are unique to within a constant.

Solvability of (1.1) is important for characterizing possible limits in asymptotic problems which involve multiple time-scales. In problems of this kind one typically has an ordinary or stochastic differential equation in which a "fast process" $\{\xi(\tau/\varepsilon),\ \tau \in [0, \infty)\}$, arising from time-rescaling the solution of (1.2) by a *small* parameter $\varepsilon > 0$, is present as a perturbation. If solutions Θ of (1.1) exist and are suitably bounded and regular for an appropriate class of mappings h, then martingale methods can often be used to characterize asymptotic properties of the solution of the perturbed equation as $\varepsilon \to 0$. For example, Pardoux and Veretennikov ([15], Section 4) use solvability of (1.1) to characterize the limiting distribution (when $\varepsilon \to 0$) of the process $\{X^\varepsilon(\tau)\}$ given by the perturbed stochastic differential equation

$$dX^\varepsilon(\tau) = F(X^\varepsilon(\tau), Y^\varepsilon(\tau))d\tau + \varepsilon^{-1/2}G(X^\varepsilon(\tau), Y^\varepsilon(\tau))d\tau + H(X^\varepsilon(\tau), Y^\varepsilon(\tau))dB^\varepsilon(\tau), \tag{1.6}$$

subject to $X^\varepsilon(0) =$ nonrandom $\ x_0$, for $Y^\varepsilon(\tau) := \xi(\tau/\varepsilon)$ and $B^\varepsilon(\tau) := \varepsilon^{1/2}\beta(\tau/\varepsilon)$.

Our goal is to study solvability of the Poisson equation (1.1), but under hypotheses which are different from those postulated by Pardoux and Veretennikov [15]. In particular, we abandon the strict nondegeneracy and boundedness condition (1.4), and deal with the singular case where the matrix $\sigma(y)$ can be rank-deficient for some $y \in \Re^D$. Our main hypotheses (Conditions 2.2 and 2.5 to follow) are second-order smoothness of the drift $b(\cdot)$ and covariance $\sigma(\cdot)$ in (1.2), as well as a condition on the eigenvalues of the Jacobian matrix $\partial b(y)$ of the drift, which will ensure that the Markov process defined by (1.2) has a unique invariant probability measure $\bar{m}$, and that (1.1) is solvable (in the classical sense) for polynomially-bounded and second-order smooth mappings $h : \Re^D \to \Re$ which are centered by $\bar{m}$. We shall then use this result to characterize the asymptotic distribution in a multiple time-scales problem arising as follows:

Fix nonrandom $x_0 \in \Re^d$, $y_0 \in \Re^D$, fix some $\Re^M$-valued standard Wiener process $\{w(t)\}$ on the same probability space $(\Omega, \mathcal{F}, P)$ on which the standard $\Re^N$-valued Wiener process $\{\beta(t)\}$ in (1.2) is defined, such that $\{\beta(t)\}$ and $\{w(t)\}$ are independent, and let $\{\xi(t, y_0)\}$ be the solution of (1.2) on $(\Omega, \mathcal{F}, P)$ subject to the initial condition $\xi(0) = y_0$ (in Section 2 we shall postulate standard Lipschitz-continuous conditions for the coefficients $b(\cdot)$ and $\sigma(\cdot)$). For $\varepsilon \in (0, 1)$ put

$$Y^\varepsilon(\tau) := \xi(\tau/\varepsilon, y_0), \qquad W^\varepsilon(\tau) := \varepsilon^{1/2} w(\tau/\varepsilon), \tag{1.7}$$

and let $\{X^\varepsilon(\tau)\}$ and $\{\bar{X}^\varepsilon(\tau)\}$ be solutions of the respective stochastic integral equations

$$X^\varepsilon(\tau) = x_0 + \int_0^\tau F(X^\varepsilon(s), Y^\varepsilon(s))\, ds + \int_0^\tau G(X^\varepsilon(s))\, dW^\varepsilon(s), \tag{1.8}$$

$$\bar{X}^\varepsilon(\tau) = x_0 + \int_0^\tau \bar{F}(\bar{X}^\varepsilon(s))\, ds + \int_0^\tau G(\bar{X}^\varepsilon(s))\, dW^\varepsilon(s). \tag{1.9}$$

The drift $\bar{F}(x)$ in (1.9) is defined in terms of the drift $F(x, y)$ in (1.8) by

$$\bar{F}(x) := \int_{\Re^D} F(x, y)\, d\bar{m}(y), \qquad x \in \Re^d, \tag{1.10}$$

in which $\bar{m}$ is the unique invariant probability measure for the Markov process defined by (1.2) (in Section 2 we shall postulate conditions on the functions $F(x, y)$ and $G(x)$ which are enough to ensure that the SDE's corresponding to (1.8), (1.9), have the properties of existence and pathwise-uniqueness). Given an arbitrary $T \in (0, \infty)$, the multiple time-scales problem is to characterize the asymptotic distribution of the process $\{Z^\varepsilon(\tau)\}$ given by

$$Z^\varepsilon(\tau) := \varepsilon^{-1/2} \left[X^\varepsilon(\tau) - \bar{X}^\varepsilon(\tau)\right], \qquad 0 \le \tau \le T, \tag{1.11}$$

as $\varepsilon \to 0$. Our motivation for studying this problem is from a work of Liptser and Stoyanov [14], who use results on stationary mixing processes to determine the asymptotics of $\{Z^\varepsilon(\tau)\}$ given by (1.7) - (1.11), when $\{\xi(t)\}$ in (1.7) is a given strictly stationary and strong mixing process on $(\Omega, \mathcal{F}, P)$, independent of the Wiener process $\{w(t)\}$, with one-dimensional marginal distribution $\bar{m}$. When the process $\{\xi(t)\}$ in (1.7) is a diffusion given by (1.2) with $\sigma(y)$ being rank-deficient for some $y \in \Re^D$, then it generally fails to have the usual mixing properties. One therefore needs an approach different from that of [14] to characterize the weak limit of $\{Z^\varepsilon(\tau)\}$ as $\varepsilon \to 0$; we shall use solvability of the Poisson equation (1.1) for this purpose.

In Section 2 we formulate conditions on the drift and diffusion terms of (1.2) (see Condition 2.2 and Condition 2.5) and state the main results on solvability of the Poisson equation (1.1) subject to these conditions (see Theorem 2.8 and Theorem 2.12). Then we postulate conditions on (1.7) to (1.9) (see Condition 2.14 and Condition 2.15) and characterize the weak limit of $\{Z^\varepsilon(\tau),\ \tau \in [0, T]\}$ as $\varepsilon \to 0$ (see Theorem 2.20). Sections 3 and 4 are devoted to establishing the results stated in Section 2, and Section 5 is a collection of technical results which are needed for the remaining sections; these results are stated so as to be freely referenced at any point in the course of reading the paper.

2 Notation, Conditions and Main Results

Notation 2.1 For reference we collect here much of the notation used throughout this work:

(i) $\Re^d$ is the space of real d-dimensional column vectors with Euclidean norm $|x| := [\sum_{i=1}^d (x^i)^2]^{1/2}$ and inner product $(x, y) := \sum_{i=1}^d x^i y^i$ for all $x, y \in \Re^d$. Let e^i denote the usual i-th canonical basis vector of $\Re^d$. Write $S_R^d := \{x \in \Re^d : |x| \leq R\}$ for each $R \in [0, \infty)$. Let $\Re^{d\times D}$ denote the space of real d by D matrices with operator norm $\|A\| := \max_{x\in\Re^D, |x|=1} |Ax|$ for all $A \in \Re^{d\times D}$. Write A' or $(A)'$ for the transpose of a matrix A; $\mathrm{Tr}\{A\}$ for the trace of a square matrix A; $\Lambda_{\min}\{A\}$, $\Lambda_{\max}\{A\}$, for the minimum and the maximum eigenvalues respectively of a real symmetric matrix A; and $A^{1/2}$ for the (unique) real symmetric positive-semidefinite square root of a real symmetric positive-semidefinite matrix A.

(ii) For a mapping $x \to \psi(x) : \Re^d \to \Re^{D\times N}$ whose scalar entries $\psi^{k,n}(\cdot)$ are C^1-mappings on $\Re^d$ for all $k = 1, \ldots, D$, $n = 1, \ldots, N$, write $\partial_{x^i}\psi(x)$ to denote the $D \times N$-matrix with (k, n)-entry given by $\partial_{x^i}\psi^{k,n}(x)$, for $x \in \Re^d$, $i = 1, 2, \ldots, d$.

(iii) For a mapping $(x, y) \to \psi(x, y) : \Re^d \times \Re^D \to \Re^s$, whose scalar entries $\psi^k(\cdot)$ are C^1-mappings on $\Re^d \times \Re^D$, write $\partial_x\psi(x, y)$ $[\partial_y\psi(x, y)]$ for the $s \times d$ $[s \times D]$ Jacobian matrix with (k, n)-entry given by $\partial_{x^n}\psi^k(x, y)$ $[\partial_{y^n}\psi^k(x, y)]$.

(iv) For a C^2-mapping $(x, y) \to \psi(x, y) : \Re^d \times \Re^D \to \Re$, write $\partial_{xy}\psi(x, y)$ $[\partial_{xx}\psi(x, y)$, $\partial_{yy}\psi(x, y)]$ for the $d \times D$ $[d \times d,\ D \times D]$ Hessian matrix with (k, n)-entry given by $\partial_{x^k y^n}\psi(x, y)$ $[\partial_{x^k x^n}\psi(x, y),\ \partial_{y^k y^n}\psi(x, y)]$.

(v) Fix some constant $T \in (0, \infty)$, and write $C_{\Re^d}[0, T]$ for the space of all $\Re^d$-valued continuous functions on the interval $[0, T]$ with metric derived from the usual supremum norm, put $\Omega^* := C_{\Re^d}[0, T] \times C_{\Re^d}[0, T]$, and let $\mathcal{B}^*$ be the Borel σ-algebra on Ω^* (with the product metric on Ω^*).

The following Conditions 2.2 and 2.5 will always be in force for (1.2):

Condition 2.2 $\{\beta(t),\ t \in [0, \infty)\}$ is an $\Re^N$-valued standard Wiener process on the complete probability space $(\Omega, \mathcal{F}, P)$. The mappings $b^k : \Re^D \to \Re$ and $\sigma^{k,n} : \Re^D \to \Re$ are C^2-functions for each $k = 1, 2, \ldots, D$ and $n = 1, 2, \ldots, N$, and the partial derivatives $\partial_{y^i} b^k(y)$, $\partial_{y^i y^j} b^k(y)$, $\partial_{y^i}\sigma^{k,n}(y)$ and $\partial_{y^i y^j}\sigma^{k,n}(y)$ are uniformly bounded with respect to $y \in \Re^D$.

Remark 2.3 Put

$$\mu_0 := \left\{ \sum_{i=1}^{D} \sup_{y\in\Re^D} \left\|\partial_{y^i}\sigma(y)\right\|^2 \right\}^{1/2},$$

(recall Notation 2.1(i)(ii)). One sees from Condition 2.2 that $\mu_0 < \infty$, and an easy calculation using the Mean Value Theorem and Cauchy-Schwarz inequality shows that

$$\|\sigma(y_1) - \sigma(y_2)\| \leq \mu_0 |y_1 - y_2|, \qquad y_1, y_2 \in \Re^D.$$

Remark 2.4 In light of the global Lipschitz continuity of the functions $\sigma(\cdot)$ and $b(\cdot)$ following from Remark 2.3 and Condition 2.2, we denote by $\{\xi(t, y),\ t \in [0, \infty)\}$ the pathwise-unique strong solution on $(\Omega, \mathcal{F}, P)$ of (1.2) subject to the nonrandom initial condition $\xi(0) = y \in \Re^D$.

Condition 2.5 For some constant $r \in [3/2, \infty)$, and some symmetric strictly positive-definite matrix $Q \in \Re^{D\times D}$, we have

$$(2.12)\quad \mu_1 := \sup_{y\in\Re^D} \Lambda_{\max}\{QJ(y) + J'(y)Q\} < \min\left[(1 - 2r - D), -D\frac{\Lambda_{\max}\{Q\}}{\Lambda_{\min}\{Q\}}\right] \mu_0^2 \Lambda_{\max}\{Q\},$$

where $J(y) := \partial_y b(y)$ is the $D \times D$ Jacobian matrix of the drift term $b(y)$ in (1.2), that is $\partial_{y^n} b^k(y)$ is the (k, n)-entry of $J(y)$ for each $n, k = 1, 2, \ldots, D$.

Remark 2.6 Condition 2.5 is the central hypothesis on the basis of which we establish solvability of the Poisson equation (1.1), and includes the case of singular diffusions, in which the matrix $\sigma(y)$ is rank-deficient for some $y \in \Re^D$. This is basically a "stability" condition which postulates that one must compensate for rapid variations in $\sigma(\cdot)$ (indicated by a large value for the Lipschitz-constant μ_0 in Remark 2.3) by having all eigenvalues of the symmetric matrix $QJ(y) + J'(y)Q$ sufficiently far to the left of the real axis uniformly in $y \in \Re^D$. Condition 2.5 is motivated by Theorem 4.1 on p.593 of Bhattacharya and Waymire [5] on invariant distributions for the diffusion given by (1.2), with $\sigma(y)$ possibly rank-deficient for some y, which says that if

$$\sup_{y\in\Re^D} \Lambda_{\max}\{J(y) + J'(y)\} < -D\,\mu_0^2, \tag{2.13}$$

then the Markov transition probability determined by (1.2) has a unique invariant probability (this result is further developed in Basak [1] and Basak and Bhattacharya [2]). If we take $Q = I$ (the $D \times D$ unit matrix) in (2.12) then Condition 2.5 reduces to

$$\sup_{y\in\Re^D} \Lambda_{\max}\{J(y) + J'(y)\} < (1 - 2r - D)\mu_0^2, \tag{2.14}$$

for a constant $r \in [3/2, \infty)$, which is clearly a more stringent requirement than (2.13), since it insists that the eigenvalues of $[J(y) + J'(y)]$ be even further to the left on the real line. As will be seen, this stronger condition ensures not only existence of a unique invariant probability measure for the diffusion (1.2) but also solvability of the Poisson equation (1.1) (in the classical sense), and it will further be seen that the larger the parameter r in Condition 2.5, the larger is the class of functions h for which (1.1) is solvable. The positive definite matrix Q in Condition 2.5 is an additional degree of freedom which, in principle, makes the condition easier to verify than if we just restrict attention to the case $Q := I$, that is, postulate (2.14) for some $r \geq 3/2$.

Remark 2.7 From now on Condition 2.2 and Condition 2.5 will always be in force. For later use we note from Proposition 5.1(ii) that the Markov transition probability defined by (1.2) has a unique invariant probability measure $\bar{m}$ on $\Re^D$, and $\bar{m}$ satisfies the integrability condition

$$\int_{\Re^D} |y|^{2r}\, d\bar{m}(y) < \infty, \tag{2.15}$$

in which r is the constant stipulated by Condition 2.5.

Our basic result on solvability of (1.1) is as follows:

Theorem 2.8 *Suppose that Condition 2.2 and Condition 2.5 hold, and let $\bar{m}$ be the invariant probability measure in Remark 2.7. Suppose that $h : \Re^D \to \Re$ is a C^2-function such that*

$$|\partial_{y^i} h(y)| + |\partial_{y^i y^j} h(y)| \leq C_1[1 + |y|^{q_1}], \qquad y \in \Re^D, \tag{2.16}$$

for some constants $C_1 \in [0, \infty)$ and $q_1 \in [0, r]$ (where r is given by Condition 2.5). Then

(i) the integral $\bar{h} := \int_{\Re^D} h\, d\bar{m}$ exists in $\Re$;
(ii) for $\tilde{h}(y) := h(y) - \bar{h}$, $y \in \Re^D$, the integral

$$\Theta(y) := \int_0^\infty E\tilde{h}(\xi(t, y))\, dt, \tag{2.17}$$

exists in $\Re$ for each $y \in \Re^D$ (recall Remark 2.4), and $\Theta : \Re^D \to \Re$ is a C^2-function such that

$$\mathcal{A}\Theta(y) = -\tilde{h}(y), \qquad y \in \Re^D, \tag{2.18}$$

$$|\partial_{y^i}\Theta(y)| + |\partial_{y^iy^j}\Theta(y)| \leq C_2[1 + |y|^{q_1}], \quad y \in \Re^D, \tag{2.19}$$

for a constant $C_2 \in [0, \infty)$. Moreover, solutions of (2.18) are unique in the following sense: if $\hat{\Theta} : \Re^D \to \Re$ is a C^2-mapping such that

$$|\partial_{y^i}\hat{\Theta}(y)| + |\partial_{y^iy^j}\hat{\Theta}(y)| \leq C_3[1 + |y|^{q_1}] \quad \text{and} \quad \mathcal{A}\hat{\Theta}(y) = -\tilde{h}(y), \quad y \in \Re^D, \tag{2.20}$$

for a constant $C_3 \in [0, \infty)$, then $\Theta(y) - \hat{\Theta}(y) = c$, all $y \in \Re^D$, for a constant $c \in \Re$.

Remark 2.9 Bhattacharya [4] establishes solvability for operator equations of Poisson-type which arise as follows: one is given a transition probability function $p(t, x, dy)$ on a measurable space $(S, \mathcal{B})$, with an ergodic invariant probability measure $\bar{m}$, for which the corresponding contraction semigroup on $L^2(S, \bar{m})$ has the infinitesimal generator $\hat{A} : \mathbb{D}(\hat{A}) \subset L^2(S, \bar{m}) \to L^2(S, \bar{m})$ ([4], p.187-188). The range of $\hat{A}$ is characterized by a functional-analytic approach ([4], Proposition 2.3, p.191), thus establishing a solvability theory for the Poisson equation $\hat{A}\Theta = h$. When $S := \Re^D$, and the transition probability function arises from (1.2) subject to a stability hypothesis similar to Condition 2.5, Basak ([1], Theorem 3.3(a)) explicitly identifies a subset of the range of $\hat{A}$ comprising Hölder-continuous functions. Unfortunately, the elegant solvability results of [4] and [1] are not directly applicable in our context, since we shall require solutions of the Poisson equation (1.1) with guaranteed smoothness and polynomial-boundedness properties when we study the multiple time-scales problem given by (1.7) - (1.11), whereas [4] and [1] establish solutions of an "extended version " of (1.1), which are members of $L^2(S, \bar{m})$, but for which these properties are otherwise unknown.

Remark 2.10 To characterize the asymptotic distribution of $\{Z^\varepsilon(\tau)\}$ given by (1.7) to (1.11) we shall actually need an extension of Theorem 2.8, which establishes solvability of the Poisson equation in the variable $y \in \Re^D$ when an additional parameter $x \in \Re^d$ is present. We first extend the second-order differential operator in (1.3) as follows: If $\Theta : \Re^d \times \Re^D \to \Re$ is a mapping with $\Theta(x, \cdot)$ being a C^2-function on $\Re^D$ for each $x \in \Re^d$, put

$$\mathcal{A}\Theta(x, y) := \sum_{i=1}^{D} b^i(y)(\partial_{y^i}\Theta)(x, y) + \frac{1}{2}\sum_{i,j=1}^{D}(\sigma\sigma')^{i,j}(y)(\partial_{y^iy^j}\Theta)(x, y), \quad (x, y) \in \Re^d \times \Re^D.$$

We next establish the existence and (x, y)-smoothness properties of a function $\Theta : \Re^d \times \Re^D \to \Re$ which solves the "x-parametrized" Poisson equation

$$\mathcal{A}\Theta(x, y) = -\tilde{h}(x, y), \qquad (x, y) \in \Re^d \times \Re^D, \tag{2.21}$$

in which $\tilde{h} : \Re^d \times \Re^D \to \Re$ is the "centered version" of a function $h : \Re^d \times \Re^D \to \Re$, namely

$$\tilde{h}(x, y) := h(x, y) - \bar{h}(x), \quad (x, y) \in \Re^d \times \Re^D, \tag{2.22}$$

for

$$\bar{h}(x) := \int_{\Re^D} h(x, y)\, d\bar{m}(y), \quad x \in \Re^d, \tag{2.23}$$

(with $\bar{m}$ being the invariant probability given by Remark 2.7). As we shall see, solutions of the parametrized Poisson equation (2.21) are of the form

$$\Theta(x, y) := \int_0^\infty E\tilde{h}(x, \xi(t, y))\, dt, \quad (x, y) \in \Re^d \times \Re^D. \tag{2.24}$$

Prior to addressing solvability of the "x-parametrized" Poisson equation (2.21) we give a basic result establishing conditions on $h : \Re^d \times \Re^D \to \Re$ which ensure existence, continuity, and x-smoothness of the integrals in (2.23) and (2.24):

Proposition 2.11 *Suppose Condition 2.2 and Condition 2.5.*
(I) Fix a mapping $h : \Re^d \times \Re^D \to \Re$, which is continuous in $(x,y) \in \Re^d \times \Re^D$, and y-locally Lipschitz continuous in the following sense: there is a constant $q_2 \in [1, 2r]$ (r being the constant in Condition 2.5), and, for each $R \in [0,\infty)$, there is a constant $C_1(R) \in [0,\infty)$ such that

$$(2.25)\quad |h(x,y_1) - h(x,y_2)| \le C_1(R)|y_1 - y_2|[1 + |y_1|^{(q_2-1)} + |y_2|^{(q_2-1)}], \quad x \in S_R^d, \quad y_1, y_2 \in \Re^D,$$

(recall Notation 2.1(i) for S_R^d). Then the integral in (2.23) exists in $\Re$ for each $x \in \Re^d$, and defines a continuous mapping on $\Re^d$; the integral in (2.24) exists in $\Re$ for each $(x,y) \in \Re^d \times \Re^D$, and defines a continuous mapping on $\Re^d \times \Re^D$; and for a constant $C \in [0,\infty)$ we have

$$(2.26)\qquad |\Theta(x,y)| \le CC_1(R)[1 + |y|^{q_2}], \quad (x,y) \in S_R^d \times \Re^D, \quad R \in [0,\infty).$$

(II) Fix a mapping $h : \Re^d \times \Re^D \to \Re$ satisfying the conditions of (I), and for which, in addition, the partial x-derivative functions $(\partial_{x^i} h)(x,y)$ exist, are continuous in $(x,y) \in \Re^d \times \Re^D$, and are y-locally Lipschitz continuous in the following sense: there is a constant $q_3 \in [1, 2r]$, and, for each $R \in [0,\infty)$, there is a constant $C_2(R) \in [0,\infty)$ such that

$$(2.27)\quad |\partial_{x^i} h(x,y_1) - \partial_{x^i} h(x,y_2)| \le C_2(R)|y_1 - y_2|[1 + |y_1|^{(q_3-1)} + |y_2|^{(q_3-1)}], \quad x \in S_R^d, \quad y_1, y_2 \in \Re^D.$$

Then the partial x-derivative functions $(\partial_{x^i}\tilde{h})(x,y)$ and $(\partial_{x^i}\Theta)(x,y)$ exist and are continuous in $(x,y) \in \Re^d \times \Re^D$ with

$$(2.28)\qquad \partial_{x^i}\Theta(x,y) = \int_0^\infty E[(\partial_{x^i}\tilde{h})(x, \xi(t,y))]\, dt, \quad (x,y) \in \Re^d \times \Re^D,$$

and there is a constant $C \in [0,\infty)$ such that, for each $R \in [0,\infty)$,

$$(2.29)\qquad |\partial_{x^i}\Theta(x,y)| \le CC_2(R)[1 + |y|^{q_3}], \quad (x,y) \in S_R^d \times \Re^D.$$

The next result extends Theorem 2.8 to the "x-parametrized" Poisson equation (2.21):

Theorem 2.12 *Suppose Condition 2.2 and Condition 2.5.*
(I) Fix a mapping $h : \Re^d \times \Re^D \to \Re$ such that
(a) $h(x,y)$ is continuous in $(x,y) \in \Re^d \times \Re^D$, and the partial y-derivatives $(\partial_{y^i} h)(x,y)$ exist and are continuous in $(x,y) \in \Re^d \times \Re^D$;
(b) there is a constant $q_1 \in [0, r]$ (where r is given by Condition 2.5), and, for each $R \in [0,\infty)$, there is a constant $C_1(R) \in [0,\infty)$ such that

$$(2.30)\qquad |\partial_{y^i} h(x,y)| \le C_1(R)[1 + |y|^{q_1}], \quad (x,y) \in S_R^d \times \Re^D;$$

(c) there is a constant $c \in [0,\infty)$, and, for each $R \in [0,\infty)$, there is a constant $C_2(R) \in [0,\infty)$ such that

$$(2.31)\qquad |\partial_{y^i} h(x_1,y) - \partial_{y^i} h(x_2,y)| \le C_2(R)|x_1 - x_2|[1 + |y|^c], \quad x_1, x_2 \in S_R^d, \quad y \in \Re^D.$$

Then the integral in (2.23) exists in $\Re$ for each $x \in \Re^d$, and gives a continuous mapping on $\Re^d$; the integral in (2.24) exists in $\Re$ for each $(x,y) \in \Re^d \times \Re^D$, and gives a continuous mapping on

$\Re^d \times \Re^D$; the partial y-derivatives $(\partial_{y^i}\Theta)(x,y)$ exist and are continuous in $(x,y) \in \Re^d \times \Re^D$; and there is a constant $C \in [0,\infty)$ such that, for each $R \in [0,\infty)$, one has

$$(2.32) \qquad |\partial_{y^i}\Theta(x,y)| \le CC_1(R)[1+|y|^{q_1}], \qquad (x,y) \in S_R^d \times \Re^D.$$

(II) Suppose that the mapping $h(x,y)$, in addition to satisfying (a), (b), (c), is such that
(d) the second partial y-derivatives $(\partial_{y^iy^j}h)(x,y)$ exist and are continuous in $(x,y) \in \Re^d \times \Re^D$, and, for each $R \in [0,\infty)$, one has

$$(2.33) \qquad |(\partial_{y^iy^j}h)(x,y)| \le C_1(R)[1+|y|^{q_1}], \quad (x,y) \in S_R^d \times \Re^D,$$

$$(2.34) \quad |(\partial_{y^iy^j}h)(x_1,y) - (\partial_{y^iy^j}h)(x_2,y)| \le C_2(R)|x_1-x_2|[1+|y|^c], \quad x_1,x_2 \in S_R^d, \quad y \in \Re^D,$$

(compare with (2.30), (2.31)). Then the second partial y-derivatives $(\partial_{y^iy^j}\Theta)(x,y)$ exist and are continuous in $(x,y) \in \Re^d \times \Re^D$; there is a constant $C \in [0,\infty)$ such that, for each $R \in [0,\infty)$, one has the upper-bound

$$(2.35) \qquad |(\partial_{y^iy^j}\Theta)(x,y)| \le CC_1(R)[1+|y|^{q_1}], \qquad (x,y) \in S_R^d \times \Re^D;$$

and $\Theta(x,y)$ satisfies the parametrized Poisson equation (2.21). Moreover, solutions of (2.21) are unique in the following sense: if, for each $x \in \Re^d$, $\hat{\Theta}(x,\cdot)$ is a C^2-mapping on $\Re^D$ such that $\mathcal{A}\hat{\Theta}(x,y) = -\tilde{h}(x,y)$, all $y \in \Re^D$, and

$$(2.36) \qquad |(\partial_{y^i}\hat{\Theta})(x,y)| + |(\partial_{y^iy^j}\hat{\Theta})(x,y)| \le C(x)[1+|y|^{q_1}], \qquad y \in \Re^D,$$

for a constant $C(x) \in [0,\infty)$, then $\Theta(x,y) - \hat{\Theta}(x,y) = c(x)$, $y \in \Re^D$, for a constant $c(x) \in \Re$.

Remark 2.13 Theorem 2.8 is an immediate consequence of Theorem 2.12 when we restrict attention to the case where h is a function of $y \in \Re^D$ only, with no dependence on $x \in \Re^d$. We next address the second goal of the present work, namely to characterize the asymptotic distribution (as $\varepsilon \to 0$) of $\{Z^\varepsilon(\tau)\}$ determined by (1.7) to (1.11). To this end we postulate the following Conditions 2.14 and 2.15 on (1.7) - (1.10):

Condition 2.14 The scalar entries $G^{k,n}(\cdot)$ of the mapping $G : \Re^d \to \Re^{d\times M}$ in (1.8) and (1.9), are C^2-functions on $\Re^d$, with uniformly bounded first and second derivatives. The mapping $(x,y) \to F(x,y) : \Re^d \times \Re^D \to \Re^d$ in (1.8) and (1.10) is such that the partial derivative functions

$$\partial_{x^i}F, \ \partial_{x^ix^j}F, \ \partial_{x^ix^jx^k}F, \ \partial_{x^iy^j}F, \ \partial_{x^ix^jy^k}F, \ \partial_{x^iy^jy^k}F, \ \partial_{x^ix^jx^ky^l}F, \ \partial_{x^ix^jy^ky^l}F, \ \partial_{y^i}F, \ \partial_{y^iy^j}F,$$

exist and are continuous on $\Re^d \times \Re^D$. Moreover, there are constants $C_1, r_1 \in [0,\infty)$, such that

$$(2.37) \qquad |H(x,y)| \le C_1[1+|y|^{r_1}], \qquad (x,y) \in \Re^d \times \Re^D,$$

where $H(x,y)$ denotes each of $\partial_{x^i}F(x,y)$, $\partial_{x^ix^j}F(x,y)$, $\partial_{x^ix^jx^k}F(x,y)$, $\partial_{x^iy^j}F(x,y)$, $\partial_{x^ix^jy^k}F(x,y)$, $\partial_{x^iy^jy^k}F(x,y)$, $\partial_{x^ix^jx^ky^l}F(x,y)$, and $\partial_{x^ix^jy^ky^l}F(x,y)$. For each $R \in [0,\infty)$, there is a constant $C(R) \in [0,\infty)$ such that

$$(2.38) \qquad |(\partial_{y^i}F)(x,y)| + |(\partial_{y^iy^j}F)(x,y)| \le C(R)[1+|y|^{r_1}], \qquad (x,y) \in S_R^d \times \Re^D.$$

Condition 2.15 The initial values $y_0 \in \Re^D$ and $x_0 \in \Re^d$ in (1.7), (1.8), and (1.9), are fixed nonrandom points, and $\{w(t),\ t \in [0,\infty)\}$ in (1.7) is an $\Re^M$-valued standard Wiener process on $(\Omega, \mathcal{F}, P)$ which is independent of the $\Re^N$-valued standard Wiener process $\{\beta(t),\ t \in [0,\infty)\}$ postulated in Condition 2.2.

Next, we define an SDE with the property of uniqueness-in-law, which characterizes the weak limit of $\{Z^\varepsilon(\tau)\}$ as $\varepsilon \to 0$. To this end we need the observations in Remarks 2.16 to 2.19:

Remark 2.16 Suppose the Conditions 2.2, 2.5, 2.14, and 2.15, and suppose furthermore that

$$r > \max\left[\frac{3}{2}(1+r_1),\ 1+2r_1\right], \tag{2.39}$$

where r, r_1, are the constants stipulated in Condition 2.5 and Condition 2.14 respectively. From Proposition 5.5 we see that the integral in (1.10) exists for each $x \in \Re^d$, and defines a C^3-mapping $\bar{F} : \Re^d \to \Re^d$ such that $\sup_x |\partial_{x^i}\bar{F}(x)| < \infty$ (the measure $\bar{m}$ in (1.10) is the unique invariant probability given by Remark 2.7). For

$$\tilde{F}(x,y) := F(x,y) - \bar{F}(x), \quad (x,y) \in \Re^d \times \Re^D, \tag{2.40}$$

it follows from Proposition 5.6(i) that the integral

$$\Phi^n(x,y) := \int_0^\infty E\tilde{F}^n(x,\xi(t,y))\,dt, \quad n = 1,2,\ldots,d, \tag{2.41}$$

exists and defines a function continuous in $(x,y) \in \Re^d \times \Re^D$ (recall Remark 2.4); and, for

$$a^{m,n}(x,y) := \tilde{F}^m(x,y)\Phi^n(x,y) + \tilde{F}^n(x,y)\Phi^m(x,y), \quad (x,y) \in \Re^d \times \Re^D, \tag{2.42}$$

it follows from Proposition 5.7(i) that the integral

$$\bar{a}^{m,n}(x) := \int_{\Re^D} a^{m,n}(x,y)\,d\bar{m}(y), \quad m,n = 1,2,\ldots,d, \tag{2.43}$$

exists and defines a symmetric positive semidefinite $d \times d$ matrix for each $x \in \Re^d$, the $\bar{a}^{m,n}(\cdot)$ are C^2-functions on $\Re^d$, and there is a constant $C \in [0,\infty)$ such that

$$|\bar{a}^{m,n}(x)| \le C[1+|x|^2], \quad x \in \Re^d. \tag{2.44}$$

Remark 2.17 One sees from (2.37) (with $H(x,y) := (\partial_{x^i}F)(x,y)$) and the Mean Value Theorem that

$$|F(x_1,y) - F(x_2,y)| \le C_1[1+|y|^{r_1}]|x_1 - x_2|, \quad x_1, x_2 \in \Re^d,\ y \in \Re^D. \tag{2.45}$$

It follows easily from (2.45), and the fact that $G(\cdot)$ is globally Lipschitz continuous on $\Re^d$ (by Condition 2.14), that (1.8) has a pathwise-unique strong solution $\{X^\varepsilon(\tau)\}$ on $(\Omega,\mathcal{F},P)$ for each $\varepsilon \in (0,1)$. Since $\bar{F}(\cdot)$ is globally Lipschitz continuous on $\Re^d$ (by Remark 2.16), for each $\varepsilon \in (0,1)$ there is similarly a pathwise-unique strong solution $\{\bar{X}^\varepsilon(\tau)\}$ of (1.9).

Remark 2.18 Since $\bar{a}(\cdot)$ is a C^2-mapping (see Remark 2.16), it follows from Theorem V(12.12)(ii) of Rogers and Williams ([16], p.134) that $\bar{a}^{1/2}(\cdot)$ is locally Lipschitz continuous on $\Re^d$ (see Notation 2.1(i)). In the following we shall write $\partial_x G(x)[z]$ to denote the d by M matrix whose (i,j)-element is given by

$$(\partial_x G(x)[z])^{i,j} := \sum_{k=1}^{d}(\partial_{x^k}G^{i,j})(x)z^k, \quad x,z \in \Re^d. \tag{2.46}$$

Fix independent standard Wiener processes $\{\hat{W}_1(\tau)\}$ and $\{\hat{W}_2(\tau)\}$, which are $\Re^M$ and $\Re^d$-valued respectively, on some complete probability space $(\hat{\Omega},\hat{\mathcal{F}},\hat{P})$. Since the mappings $\bar{F}(\cdot)$ and $G(\cdot)$

are globally Lipschitz continuous on $\Re^d$ (by Condition 2.14 and Remark 2.16), while $\partial_{x^k} G(\cdot)$ and $\partial_{x^k}\bar{F}(\cdot)$ are uniformly bounded and $\bar{a}(\cdot)$ is quadratically bounded on $\Re^d$ (see Remark 2.16), it follows easily from standard results on Itô SDE's (e.g. Theorem 5.3.7 and Theorem 5.3.11 of Ethier and Kurtz [8], p.297 and p.300) that the system of equations

$$\begin{aligned}
(2.47)\ \ d\hat{X}(\tau) &= \bar{F}(\hat{X}(\tau))d\tau + G(\hat{X}(\tau))d\hat{W}_1(\tau), \\
(2.48)\ \ d\hat{Z}(\tau) &= (\partial_x\bar{F})(\hat{X}(\tau))\hat{Z}(\tau)d\tau + (\partial_x G(\hat{X}(\tau))[\hat{Z}(\tau)])d\hat{W}_1(\tau) + \bar{a}^{1/2}(\hat{X}(\tau))d\hat{W}_2(\tau),
\end{aligned}$$

is *pathwise exact* (in the sense of Definition V(9.4) of Rogers and Williams [16], p.124), and therefore has the property of uniqueness-in-distribution (by Theorem V(17.1) on p.151 of [16]).

Remark 2.19 Fix some constant $T \in (0,\infty)$. For each $\varepsilon \in (0,1)$ let P^ε be the probability law on $(\Omega^*, \mathcal{B}^*)$ (see Notation 2.1(v)) of the process $\{(X^\varepsilon(\tau), Z^\varepsilon(\tau)),\ \tau \in [0,T]\}$ (see Remark 2.17), and, in light of Remark 2.18, let P^* be the uniquely defined probability law on $(\Omega^*, \mathcal{B}^*)$ of the solution $\{(\hat{X}(\tau), \hat{Z}(\tau),\ \tau \in [0,T]\}$ of (2.47), (2.48), with initial condition $\hat{X}(0) = x_0$, $\hat{Z}(0) = 0$.

With these preliminaries in place, the limiting distribution of $\{Z^\varepsilon(\tau)\}$ can be characterized:

Theorem 2.20 *Suppose Condition 2.2, Condition 2.5, Condition 2.14, and Condition 2.15, and suppose that (2.39) holds for the constant r in Condition 2.5 and the constant r_1 in Condition 2.14. Then, with reference to Remark 2.19, the measure P^ε converges weakly to P^* as $\varepsilon \to 0$.*

Remark 2.21 The significance of postulating (2.39) in Theorem 2.20 is as follows: if r_1 in Condition 2.14 is large, so that $F(x,y)$ and its derivatives increase rapidly with increasing y, then (2.39) stipulates correspondingly larger values of the constant r in Condition 2.5, ensuring enough "stability" in (1.2) to get weak convergence of $(X^\varepsilon, Z^\varepsilon)$ to a limiting distribution.

Remark 2.22 From Theorem 2.20 and (1.11) it follows that $X^\varepsilon(\cdot) - \bar{X}^\varepsilon(\cdot)$ converges in distribution to the (nonrandom) limit which is identically zero; hence the convergence is also in probability, that is, for each $\delta \in (0,\infty)$,

$$\lim_{\varepsilon\to 0} P\{\sup_{\tau\in[0,T]} |X^\varepsilon(\tau) - \bar{X}^\varepsilon(\tau)| \geq \delta\} = 0.$$

3 Proofs of Proposition 2.11 and Theorem 2.12

Remark 3.1 For a constant $p \in [1,\infty)$ and Borel-measurable mapping $g : \Re^D \to \Re$, put

$$[g(\cdot)]_p := \sup_{y_1\neq y_2} \frac{|g(y_1) - g(y_2)|}{|y_1 - y_2|[1 + |y_1|^{p-1} + |y_2|^{p-1}]} \in [0,\infty].$$

Proof of Proposition 2.11: (I) From (2.25) one sees that, for each $R \in [0,\infty)$, there is a constant $C(R) \in [0,\infty)$ such that

$$(3.49)\qquad |h(x,y)| \leq C(R)[1 + |y|^{q_2}], \quad (x,y) \in S_R^d \times \Re^D.$$

By (3.49) and Remark 2.7 one sees that the integral in (2.23) exists for each $x \in \Re^d$. Since $h(\cdot, y)$ is continuous on $\Re^d$ for each y, and the right-hand side of (3.49) is $\bar{m}$-integrable in y (Remark 2.7), it follows from the Dominated Convergence Theorem that $\bar{h}$ is continuous on $\Re^d$. As for existence and continuity of the integral (2.24), from (2.25) and Remark 3.1 we get $[h(x,\cdot)]_{q_2} \leq C_1(R) < \infty$, for all $x \in S_R^d$ and each $R \in [0,\infty)$, so that Proposition 5.3 (with $g(\cdot) := h(x,\cdot)$ for arbitrary $x \in S_R^d$ and $p := q_2$) establishes that the expectation $Eh(x, \xi(t,y))$ exists, with

$$(3.50)\ |E\tilde{h}(x,\xi(t,y))| \leq CC_1(R)\exp\{-\gamma t\}[1+|y|^{q_2}], \quad (t,x,y) \in [0,\infty) \times S_R^d \times \Re^D, \quad R \in [0,\infty),$$

for some constants $C, \gamma \in (0,\infty)$. Thus the integral in (2.24) exists for each $(x,y) \in \Re^d \times \Re^D$, and (2.26) follows from (3.50). We next show that $\Theta(\cdot)$ is continuous on $\Re^d \times \Re^D$: fix a sequence (x_n, y_n) in $\Re^d \times \Re^D$ converging to (x,y). From (3.50) there are constants $C, \gamma \in (0,\infty)$ such that $|E[\tilde{h}(x_n, \xi(t,y_n))]| \leq C\exp\{-\gamma t\}$, for all $t \in [0,\infty)$, $n = 1,2,3,\ldots$ Thus, it is enough to show

$$\lim_{n\to\infty} E[\tilde{h}(x_n,\xi(t,y_n))] = E[\tilde{h}(x,\xi(t,y))], \tag{3.51}$$

for each $t \in [0,\infty)$ in order to conclude that $\Theta(\cdot)$ is continuous (by Dominated Convergence). From (2.25) and Cauchy-Schwarz, for each $t \in [0,\infty)$ and $n = 1,2,\ldots$, we have

$$\begin{aligned} |E[\tilde{h}(x_n,\xi(t,y_n)) - \tilde{h}(x_n,\xi(t,y))]| &= |E[h(x_n,\xi(t,y_n)) - h(x_n,\xi(t,y))]| \\ &\leq CE^{1/2}\left[|\xi(t,y_n) - \xi(t,y)|^2\right] E^{1/2}\left[1 + |\xi(t,y_n)|^{2(q_2-1)} + |\xi(t,y)|^{2(q_2-1)}\right], \end{aligned} \tag{3.52}$$

with $C \in [0,\infty)$ being a constant. For arbitrary $t \in [0,\infty)$, we have $\sup_n E[|\xi(t,y_n)|^{2(q_2-1)}] < \infty$ (see e.g. 5.3.15 of Karatzas and Shreve [12], p.306) and $\lim_{n\to\infty} E[|\xi(t,y_n) - \xi(t,y)|^2] = 0$ (see e.g. Lemma 14.22 of Elliott [7], p.193), thus, in view of (3.52),

$$\lim_{n\to\infty} |E[\tilde{h}(x_n,\xi(t,y_n)) - \tilde{h}(x_n,\xi(t,y))]| = 0. \tag{3.53}$$

Moreover, $\lim_{n\to\infty} E[\tilde{h}(x_n,\xi(t,y))] = E[\tilde{h}(x,\xi(t,y))]$ (as follows easily from (2.22), (3.49), Dominated Convergence, the fact that $x \to \tilde{h}(x,y)$ is continuous, and the fact that $E|\xi(t,y)|^{q_2} < \infty$), and this, together with (3.53), gives (3.51). Thus $\Theta(\cdot)$ defined by (2.24) is continuous on $\Re^d \times \Re^D$.

(II) In view of (2.25) and (2.27), for each $R \in [0,\infty)$ there is a constant $C(R) \in [0,\infty)$ such that

$$\max\{|h(x,y)|, |\partial_{x^i} h(x,y)|\} \leq C(R)[1 + |y|^{(q_2 \vee q_3)}], \quad (x,y) \in S_R^d \times \Re^d. \tag{3.54}$$

From (3.54) and Proposition 5.4, one sees that $\bar{h}(\cdot)$ (recall (2.23)) is a C^1-function on $\Re^d$, thus the partial derivative $(\partial_{x^i}\tilde{h})(x,y)$ exists and is continuous on $\Re^d \times \Re^D$ (see (2.22)), and

$$\int_{\Re^D} (\partial_{x^i}\tilde{h})(x,y)\, d\bar{m}(y) = 0, \quad x \in \Re^d. \tag{3.55}$$

In view of (2.27) and Remark 3.1, for arbitrary $R \in [0,\infty)$ we have $[(\partial_{x^i}\tilde{h})(x,\cdot)]_{q_3} \leq C_2(R) < \infty$, for all $x \in S_R^d$, thus, from (3.55) and Proposition 5.3 (with $g(\cdot) := (\partial_{x^i}\tilde{h})(x,\cdot)$ and $p := q_3$), we find

$$|E[(\partial_{x^i}\tilde{h})(x,\xi(t,y))]| \leq CC_2(R)\exp\{-\gamma t\}[1 + |y|^{q_3}], \tag{3.56}$$

for some constants $C, \gamma \in (0,\infty)$, and all $(t,x,y) \in [0,\infty) \times S_R^d \times \Re^d$, $R \in [0,\infty)$. Then, by (2.24), the Mean Value Theorem, and the fact that $x \to \tilde{h}(x,y)$ is a C^1-function on $\Re^d$, we find

$$\begin{aligned} \Theta(x + \alpha e^i, y) - \Theta(x,y) &= \int_0^\infty E\left[\int_0^\alpha (\partial_{x^i}\tilde{h})(x + \eta e^i, \xi(t,y))\, d\eta\right] dt \\ &= \int_0^\alpha \left[\int_0^\infty E[(\partial_{x^i}\tilde{h})(x + \eta e^i, \xi(t,y))]\, dt\right] d\eta, \end{aligned} \tag{3.57}$$

where the second equality in (3.57) follows from (3.56) and Fubini's Theorem. It remains to show that the mapping $(x,y) \to E[(\partial_{x^i}\tilde{h})(x,\xi(t,y))]$ is continuous on $\Re^d \times \Re^D$ for each $t \in [0,\infty)$.

Indeed, when this is established, then we see from (3.56) and Dominated Convergence that $(x, y) \to \int_0^\infty E[(\partial_{x^i}\tilde{h})(x, \xi(t, y))]\, dt$ is continuous on $\Re^d \times \Re^D$, and hence (2.28) follows from (3.57) and the Fundamental Theorem of Calculus, while (2.29) follows from (2.28) and (3.56). Therefore, fix some $t \in [0, \infty)$ and some sequence (x_n, y_n) in $\Re^d \times \Re^D$ which converges to (x, y). Using the sensitivity of the mapping $y \to (\partial_{x^i}h)(x, y)$ given by (2.27) and an argument identical to that which led to (3.53), we find

$$\lim_{n\to\infty} |E[(\partial_{x^i}\tilde{h})(x_n, \xi(t, y_n)) - (\partial_{x^i}\tilde{h})(x_n, \xi(t, y))]| = 0. \tag{3.58}$$

Moreover, $\lim_{n\to\infty} E[(\partial_{x^i}\tilde{h})(x_n, \xi(t, y))] = E[(\partial_{x^i}\tilde{h})(x, \xi(t, y))]$ (as follows from (3.54), Dominated Convergence, and the facts that $x \to (\partial_{x^i}\tilde{h})(x, y)$ is continuous and $E|\xi(t, y)|^{(q_2\vee q_3)} < \infty$). This fact, in conjunction with (3.58), establishes that $(x, y) \to E[(\partial_{x^i}\tilde{h})(x, \xi(t, y))]$ is continuous on $\Re^d \times \Re^D$. This concludes the proof of Proposition 2.11. □

Proof of Theorem 2.12: (I) From (2.30) and the Mean value Theorem, we see that a bound of the form (2.25) holds, with $q_2 := 1 + q_1$. Since $q_1 \in [0, r]$ and $r \geq 3/2$, we get $q_2 \in [1, 2r]$, so that Proposition 2.11(I) shows that the integrals in (2.23) and (2.24) exist and give continuous functions on $\Re^d$ and $\Re^d \times \Re^D$ respectively. We next show that $\partial_{y^i}\Theta(x, y)$ exists and is continuous in $(x, y) \in \Re^d \times \Re^D$, and (2.32) holds. To this end we need Remark 3.2 and Proposition 3.3:

Remark 3.2 We briefly recall the notion of an L_2-derivative of a random function on $\Re^D$, due to Gihman and Skorohod ([9], p.55). Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space, and the mappings $f, g : \Re^D \times \Omega \to \Re$ are $\mathfrak{B}(\Re^D) \otimes \mathcal{F}$-measurable such that, for some $y \in \Re^D$ and $i = 1, 2, \ldots, D$,

$$\lim_{\alpha\to 0} E\left\{|(1/\alpha)[f(y + e^i\alpha) - f(y)] - g(y)|^2\right\} = 0.$$

Then $g(y)$ is called the *partial derivative in the L_2-sense* of the random function f with respect to its i-th argument at the point $y \in \Re^D$. When there is no possibility of confusion with the notation for ordinary derivatives we shall denote the random variable $g(y)$ by $\partial_{y^i}f(y)$ (following the notation of [9]). By replacing f with $\partial_{y^i}f$ in the preceding, we can also define the notion of the *double* partial derivative $\partial_{y^iy^j}f(y)$ in the L_2-sense. The next result is essential for the proof of Theorem 2.12, and is established later in the present section:

Proposition 3.3 *Suppose Condition 2.2 and Condition 2.5. Then the partial L_2-derivatives $\partial_{y^i}\xi(t, y)$ and $\partial_{y^iy^j}\xi(t, y)$ exist (see Remark 2.4), and there are constants $C, \gamma \in (0, \infty)$ such that*

$$E[|\partial_{y^i}\xi(t, y)|^4] + E[|\partial_{y^iy^j}\xi(t, y)|^2] \leq C\exp\{-\gamma t\}, \qquad (t, y) \in \Re^D \times [0, \infty),$$

for each $i, j = 1, 2, \ldots, D$.

Remark 3.4 Since Condition 2.2 and Condition 2.5 ensure that the Markov process defined by (1.2) has a unique invariant probability measure $\bar{m}$, it is plausible that the sensitivity of $\xi(t, y)$ to the initial y should decay to zero with increasing t. Proposition 3.3 makes this explicit.

Continuing with the proof of Theorem 2.12, put

$$\theta(t, x, y) := E[\tilde{h}(x, \xi(t, y))], \qquad (t, x, y) \in [0, \infty) \times \Re^d \times \Re^D. \tag{3.59}$$

From the smoothness hypotheses on $b(\cdot)$ and $\sigma(\cdot)$ in Condition 2.2, the hypothesis that $y \to \tilde{h}(x, y)$ is a C^1-function with polynomially-bounded first y-derivative (see (2.30)), and Corollary 1 of

Gihman and Skorohod ([9], p.62), we see that the mapping $y \to \theta(t,x,y)$ is a C^1-function on $\Re^D$, with first y-derivatives given by formal application of the chain-rule to (3.59), namely

$$\partial_{y^i}\theta(t,x,y) = \sum_{k=1}^{D} E[(\partial_{\xi^k}\tilde{h})(x,\xi(t,y))(\partial_{y^i}\xi^k)(t,y)], \qquad (t,x,y) \in [0,\infty) \times \Re^d \times \Re^D. \tag{3.60}$$

From (2.24), (3.59), and the Mean Value Theorem, we then have

$$\Theta(x, y + \alpha e^i) - \Theta(x,y) = \int_0^\infty \left[\int_0^\alpha \partial_{y^i}\theta(t,x,y+\eta e^i)\, d\eta \right] dt, \quad \alpha \in \Re. \tag{3.61}$$

Since $\rho := 2q_1 \in [0, 2r]$, it follows from Remark 5.2, the fact that $\partial_{\xi^k}\tilde{h}(x,\xi) = \partial_{\xi^k} h(x,\xi)$ (see (2.22)), and the upper-bound on $\partial_{\xi^k} h(x,\xi)$ given by (2.30), that there is a constant $C \in [0,\infty)$ such that, for each $R \in [0,\infty)$, one has

$$E^{1/2}[|(\partial_{\xi^k}\tilde{h})(x,\xi(t,y))|^2] \le CC_1(R)[1 + |y|^{q_1}], \quad t \in [0,\infty), \quad (x,y) \in S_R^d \times \Re^D. \tag{3.62}$$

Now from the Cauchy-Schwarz inequality, (3.60), (3.62), and Proposition 3.3, it follows that there is are constants $C, \gamma \in (0,\infty)$, such that, for each $R \in [0,\infty)$,

$$|\partial_{y^i}\theta(t,x,y)| \le CC_1(R)[1 + |y|^{q_1}] \exp\{-\gamma t\}, \quad t \in [0,\infty), \quad (x,y) \in S_R^d \times \Re^D. \tag{3.63}$$

In view of (3.61), (3.63), and Fubini's Theorem, we get

$$\Theta(x, y + \alpha e^i) - \Theta(x,y) = \int_0^\alpha \Gamma(x, y + \eta e^i)\, d\eta, \quad (x,y) \in \Re^d \times \Re^D, \quad \alpha \in \Re, \tag{3.64}$$

for

$$\Gamma(x,y) := \int_0^\infty \partial_{y^i}\theta(t,x,y)\, dt, \quad (x,y) \in \Re^d \times \Re^D. \tag{3.65}$$

We next establish that $\Gamma(x,y)$ is continuous in $(x,y) \in \Re^d \times \Re^D$. For this it is enough to show that $\partial_{y^i}\theta(t,x,y)$ is continuous in $(x,y) \in \Re^d \times \Re^D$ for each fixed t, since it then follows from (3.63) and Dominated Convergence that Γ is continuous on $\Re^d \times \Re^D$. Fix a sequence $\{(x_n, y_n)\}$ in $\Re^d \times \Re^D$ converging to the limit (x,y), and suppose that sequence $\{x_n\}$ is within the sphere S_R^d for some $R \in [0,\infty)$. From the sensitivity of the mapping $x \to \partial_{\xi^k} h(x,\xi) \equiv \partial_{\xi^k}\tilde{h}(x,\xi)$ given by (2.31), together with the Cauchy-Schwarz inequality, the fact that $\sup_n E|(\partial_{y^i}\xi)(t,y_n)|^2 < \infty$ (see Proposition 3.3), and the fact that $\sup_n E|\xi(t,y_n)|^{2c} < \infty$ (see e.g. 5.3.15 of Karatzas and Shreve [12], p.306), one sees that there is a constant $C(R) \in [0,\infty)$ such that

$$|E[\partial_{\xi^k}\tilde{h}(x_n,\xi(t,y_n))\partial_{y^i}\xi^k(t,y_n)] - E[\partial_{\xi^k}\tilde{h}(x,\xi(t,y_n))\partial_{y^i}\xi^k(t,y_n)]| \le C(R)|x_n - x|,$$

thus, from (3.60), for each $t \in [0,\infty)$ we get

$$\lim_{n\to\infty} |\partial_{y^i}\theta(t,x_n,y_n) - \partial_{y^i}\theta(t,x,y_n)| = 0. \tag{3.66}$$

Since we have already noted that $y \to \theta(t,x,y)$ is a C^1-mapping on $\Re^D$, we certainly have the convergence $\lim_{n\to\infty} \partial_{y^i}\theta(t,x,y_n) = \partial_{y^i}\theta(t,x,y)$, and now it follows from (3.66) that $\partial_{y^i}\theta(t,x,y)$ is continuous in $(x,y) \in \Re^d \times \Re^D$ for each $t \in [0,\infty)$, as required to see that $\Gamma(x,y)$ in (3.65) is continuous in $(x,y) \in \Re^d \times \Re^D$. Then, from (3.64) with the Fundamental Theorem of Calculus, we see that $\partial_{y^i}\Theta(x,y)$ exists and is given by $\partial_{y^i}\Theta(x,y) = \Gamma(x,y)$, that is

$$\partial_{y^i}\Theta(x,y) = \int_0^\infty \partial_{y^i}\theta(t,x,y)\, dt, \quad (x,y) \in \Re^d \times \Re^D. \tag{3.67}$$

Thus, $\partial_{y^i}\Theta(x,y)$ is continuous in (x,y), and (2.32) follows from (3.67) and (3.63).

(II) Consider the second derivative $\partial_{y^iy^j}\Theta(x,y)$. From the second order smoothness hypotheses on $b(\cdot)$ and $\sigma(\cdot)$ in Condition 2.2, the hypothesis that $y \to \tilde{h}(x,y)$ is a C^2-function with polynomially-bounded first and second y-derivatives (recall (2.30) and (2.33)), and Corollary 1 of Gihman and Skorohod ([9], p.62), we see that the mapping $y \to \theta(t,x,y)$ is a C^2-function on $\Re^D$, with the second y-derivatives given by formal differentiation of (3.60), namely

$$\begin{aligned}(3.68)\qquad \partial_{y^iy^j}\theta(t,x,y) &= \sum_{k=1}^{D} E[(\partial_{\xi^k}\tilde{h})(x,\xi(t,y))(\partial_{y^iy^j}\xi^k)(t,y)] \\ &+ \sum_{l,k=1}^{D} E[(\partial_{\xi^k\xi^l}\tilde{h})(x,\xi(t,y))(\partial_{y^i}\xi^k)(t,y)(\partial_{y^j}\xi^l)(t,y)];\end{aligned}$$

moreover, from (3.67) with the Mean Value Theorem,

$$(3.69)\qquad \partial_{y^i}\Theta(x,y+\alpha e^j) - \partial_{y^i}\Theta(x,y) = \int_0^\infty \left[\int_0^\alpha \partial_{y^iy^j}\theta(t,x,y+\eta e^j)\, d\eta\right] dt, \quad \alpha \in \Re.$$

From (3.68) and Hölder's inequality,

$$\begin{aligned}(3.70)\, |\partial_{y^iy^j}\theta(t,x,y)| &\le \sum_{k=1}^{D} E^{1/2}[(\partial_{\xi^k}\tilde{h})^2(x,\xi(t,y))]E^{1/2}[|\partial_{y^iy^j}\xi^k(t,y)|^2] \\ &+ \sum_{l,k=1}^{D} E^{1/2}[(\partial_{\xi^k\xi^l}\tilde{h})^2(x,\xi(t,y))]E^{1/4}[|\partial_{y^i}\xi^k(t,y)|^4]E^{1/4}[|\partial_{y^j}\xi^l(t,y)|^4].\end{aligned}$$

In view of Remark 5.2 and the upper-bound on $\partial_{\xi^k\xi^l}h(x,\xi) \equiv \partial_{\xi^k\xi^l}\tilde{h}(x,\xi)$ given by (2.33), one sees that (3.62) still holds with $\partial_{\xi^k\xi^l}\tilde{h}(x,\xi)$ in place of $\partial_{\xi^k}\tilde{h}(x,\xi)$; and from this fact, together with (3.70), Proposition 3.3, and (3.62), it follows that, for some constants $C, \gamma \in (0,\infty)$,

$$(3.71)\qquad |\partial_{y^iy^j}\theta(t,x,y)| \le CC_1(R)[1+|y|^{q_1}]\exp\{-\gamma t\}, \quad t \in [0,\infty), \quad (x,y) \in S_R^d \times \Re^D,$$

for each $R \in [0,\infty)$. Next, observe that $\partial_{y^iy^j}\theta(t,x,y)$ is continuous in $(x,y) \in \Re^d \times \Re^D$ for each $t \in [0,\infty)$. Indeed, for a sequence $\{(x_n,y_n)\}$ converging to (x,y), one easily sees from (3.68), (2.31), (2.34), Proposition 3.3, and Hölder's inequality, that

$$(3.72)\qquad \lim_{n\to\infty} |\partial_{y^iy^j}\theta(t,x_n,y_n) - \partial_{y^iy^j}\theta(t,x,y_n)| = 0,$$

for each fixed t (by a calculation similar to that for (3.66)). Since we have seen that $y \to \partial_{y^iy^j}\theta(t,x,y)$ is a continuous mapping on $\Re^D$, we see from (3.72) that $\partial_{y^iy^j}\theta(t,x,y)$ is continuous in $(x,y) \in \Re^d \times \Re^D$ for each t. It follows from (3.71) and Dominated Convergence, that the mapping $(x,y) \to \int_0^\infty \partial_{y^iy^j}\theta(t,x,y)\, dt$ is continuous on $\Re^d \times \Re^D$. Thus, from (3.71), (3.69), the Fubini Theorem, and the Fundamental Theorem of Calculus, we see that $\partial_{y^iy^j}\Theta(x,y)$ exists and

$$(3.73)\qquad \partial_{y^iy^j}\Theta(x,y) = \int_0^\infty \partial_{y^iy^j}\theta(t,x,y)\, dt, \quad (x,y) \in \Re^d \times \Re^D.$$

Hence $\partial_{y^iy^j}\Theta(x,y)$ is continuous in $(x,y) \in \Re^d \times \Re^D$, and (2.35) is a consequence of (3.71).

We next show that $\Theta(x,y)$ given by (2.24) satisfies the parametrized Poisson equation (2.21). To this end fix some $x \in \Re^d$, and fix $R \in [0,\infty)$ such that $x \in S_R^d$. We have established that

$y \to \Theta(x,y)$ is a polynomially-bounded and C^2-mapping on $\Re^D$, with polynomially-bounded first and second y-derivatives (recall (2.32) and (2.35)), and so, from Gihman and Skorohod ([9], Theorem 3 and Remark 3, p.293), we get

$$\lim_{t_1 \downarrow 0} \frac{E\Theta(x,\xi(t_1,y)) - \Theta(x,y)}{t_1} = \mathcal{A}\Theta(x,y), \quad y \in \Re^D, \tag{3.74}$$

(for $\mathcal{A}\Theta(x,y)$ given by Remark 2.10). We next show that the left-hand side of (3.74) is also given by $-\tilde{h}(x,y)$. From (3.59) and (2.24) we have

$$\Theta(x,\xi(t_1,y)) = \int_0^\infty \theta(t,x,\xi(t_1,y))\, dt, \quad (t_1,y) \in [0,\infty) \times \Re^D. \tag{3.75}$$

Moreover, as noted at the start of the proof of Theorem 2.12, a bound of the form (2.25) holds with $q_2 := 1 + q_1 \in [1, 2r]$, thus, in the notation of Remark 3.1, we have $[h(x,\cdot)]_{1+q_1} < \infty$. Thus, from (3.59), (2.22), and Proposition 5.3 (with $p := 1 + q_1$, $g(\cdot) := h(x,\cdot)$), there are constants $C, \gamma \in (0,\infty)$ such that

$$|\theta(t,x,\xi(t_1,y))| \le C[h(x,\cdot)]_{1+q_1} \exp\{-\gamma t\}[1 + |\xi(t_1,y)|^{1+q_1}], \quad t, t_1 \in [0,\infty), \quad y \in \Re^D. \tag{3.76}$$

By (3.76) we get $\int_0^\infty E\, |\theta(t,x,\xi(t_1,y))|\, dt < \infty$ for each $(t_1,y) \in [0,\infty)$, and so, from (3.75), and the Fubini Theorem, we find

$$E\Theta(x,\xi(t_1,y)) = \int_0^\infty E\, \theta(t,x,\xi(t_1,y))\, dt, \quad (t_1,y) \in [0,\infty) \times \Re^D. \tag{3.77}$$

In view of (3.59) and the Markov property of (1.2) (see e.g. Elliott [7], Theorem 14.27, p.196), we have $\theta(t+t_1,x,y) = E\, \theta(t,x,\xi(t_1,y))$, and thus, using (3.77), (3.59) and (2.24), we find

$$\frac{E\, \Theta(x,\xi(t_1,y)) - \Theta(x,y)}{t_1} = \int_0^\infty \frac{\theta(t+t_1,x,y) - \theta(t,x,y)}{t_1}\, dt, \quad (t_1,y) \in (0,\infty) \times \Re^D. \tag{3.78}$$

In view of (3.59), the boundedness and second-order smoothness of $b(\cdot)$ and $\sigma(\cdot)$ (see Condition 2.2), and the fact that $h(x,\cdot)$ is a C^2-mapping on $\Re^D$ subject to the polynomial-order bounds (2.30) and (2.33), one sees from Gihman and Skorohod ([9], Theorem 5, p.297) that $t \to \theta(t,x,y)$ is a C^1-mapping, and

$$\partial_t \theta(t,x,y) = \mathcal{A}\theta(t,x,y). \tag{3.79}$$

In view of (3.79), (3.71), (3.63), with the linear growth of $b^i(y)$ and $\sigma^{i,j}(y)$ in y, we get

$$|\partial_t \theta(t,x,y)| \le C \exp\{-\gamma t\}[1 + |y|^{2+q_1}], \quad (t,y) \in [0,\infty) \times \Re^D, \tag{3.80}$$

for constants $C, \gamma \in (0,\infty)$. Now, we have seen that the mapping $t \to \theta(t,x,y)$ is a C^1-function, so that $\theta(t+t_1,x,y) - \theta(t,x,y) = t_1(\partial_t\theta)(s_1,x,y)$ for some $s_1 \in [t, t+t_1]$, thus, from (3.80),

$$\left| \frac{\theta(t+t_1,x,y) - \theta(t,x,y)}{t_1} \right| \le C \exp\{-\gamma t\}[1 + |y|^{2+q_1}], \tag{3.81}$$

for all $t \in [0,\infty)$, $t_1 \in (0,\infty)$, $y \in \Re^D$. In view of (3.81) and Dominated Convergence, we get

$$\begin{aligned} \lim_{t_1 \downarrow 0} \int_0^\infty \frac{\theta(t+t_1,x,y) - \theta(t,x,y)}{t_1}\, dt &= \int_0^\infty \partial_t \theta(t,x,y)\, dt \\ &= \lim_{T \to \infty} \theta(T,x,y) - \theta(0,x,y) = -\tilde{h}(x,y), \end{aligned} \tag{3.82}$$

where the final equality of (3.82) follows from (3.76) (with $t_1 := 0$) and (3.59). It follows from (3.82), (3.78) and (3.74), that $\Theta(x,y)$ satisfies the parametrized Poisson equation (2.21).

It remains to establish uniqueness of solutions of (2.21) in the sense stated. Fix some arbitrary $x \in \Re^d$, and put $\Psi(x,y) := (\Theta - \hat{\Theta})(x,y)$, $y \in \Re^D$. Then $\mathcal{A}\Psi(x,y) = 0$, for all $y \in \Re^D$, hence we can expand $t \to \Psi(x, \xi(t,y))$ by Itô's formula to get

$$(3.83) \qquad E[\Psi(x,\xi(t,y))] = \Psi(x,y), \qquad (t,y) \in [0,\infty) \times \Re^D,$$

(the polynomial bounds postulated for the first and second y-derivatives of $\Theta(x,y)$ and $\hat{\Theta}(x,y)$ ensure that the stochastic integrals in Itô's formula have expectations identically zero). Moreover, from the Mean Value Theorem with (2.32) and (2.36), we have $[\Psi(x,\cdot)]_{1+q_1} < \infty$, and we have seen that $1+q_1 \in [1,2r]$. Hence Proposition 5.3 shows that $\int_{\Re^D} \Psi(x,z)\, d\bar{m}(z)$ exists and

$$(3.84) \qquad \lim_{t\to\infty} E[\Psi(x,\xi(t,y))] = \int_{\Re^D} \Psi(x,z)\, d\bar{m}(z), \quad y \in \Re^D.$$

From (3.84), and taking $t \to \infty$ in (3.83), we get $\Psi(x,y) = \int_{\Re^D} \Psi(x,z) d\bar{m}(z)$, $y \in \Re^D$, as required. This concludes the proof of Theorem 2.12. □

Proof of Proposition 3.3: For later reference define the $D \times N$-matrix $B(\xi,\vartheta)$ by

$$(3.85) \qquad B^{k,n}(\xi,\vartheta) := \sum_{l=1}^{D} (\partial_{\xi^l}\sigma^{k,n})(\xi)\vartheta^l, \qquad \xi,\vartheta \in \Re^D.$$

Fix some $i = 1,2,\ldots,D$. From the smoothness of $b^k(\cdot)$ and $\sigma^{k,n}(\cdot)$ postulated by Condition 2.2, together with Gihman and Skorohod ([9], eq.(3), p.59), one sees that the partial L_2-derivatives $\partial_{y^i}\xi^k(t,y)$ exist (recall Remark 3.2), and, for each $y \in \Re^D$, the vector

$$(3.86) \qquad \vartheta(t,y) := (\partial_{y^i}\xi^1(t,y), \partial_{y^i}\xi^2(t,y), \ldots, \partial_{y^i}\xi^D(t,y))',$$

satisfies the SDE obtained by formal differentiation with respect to y^i of the SDE for $\xi(t,y)$ (recall Remark 2.4), namely

$$(3.87) \qquad \vartheta(t,y) = e^i + \int_0^t J(\xi(s,y))\, \vartheta(s,y)\, ds + \int_0^t B(\xi(s,y),\vartheta(s,y))\, d\beta(s),$$

(for $J(\xi) := \partial b(\xi)$). Since y is fixed throughout the remainder of the proof, we shall write $\xi(t)$ and $\vartheta(t)$ for $\xi(t,y)$ and $\vartheta(t,y)$ respectively. Put

$$(3.88) \qquad \varphi(\vartheta) := (\vartheta, Q\vartheta)^2, \qquad \vartheta \in \Re^D,$$

where Q is the $D \times D$ symmetric positive-definite matrix postulated in Condition 2.5. From (3.87) and the Itô formula,

$$(3.89) \quad \varphi(\vartheta(t)) = \varphi(e^i) + \sum_{j=1}^{D} \int_0^t \partial_{\vartheta^j}\varphi(\vartheta(s))\, d\vartheta^j(s) + \frac{1}{2}\sum_{j,k=1}^{D} \int_0^t \partial_{\vartheta^j\vartheta^k}\varphi(\vartheta(s))\, d\langle \vartheta^j, \vartheta^k\rangle(s),$$

while, from (3.88), (3.87), the symmetry of Q, and an easy computation, we find

$$(3.90) \sum_{j=1}^{D} \int_0^t \partial_{\vartheta^j}(\vartheta(s))\, d\vartheta^j(s) = 2\int_0^t (\vartheta(s), Q\vartheta(s))\vartheta'(s)\, [QJ(\xi(s)) + J'(\xi(s))Q]\, \vartheta(s)\, ds + M(t),$$

wherein $\{M(t)\}$ is a continuous local martingale of stochastic integrals with respect to $\{\beta(t)\}$ (adapted to the filtration $\mathcal{F}_t := \sigma\{\beta(s),\ 0 \le s \le t\} \vee P-$null events in $\mathcal{F}$). Since $b^k(y)$ and $\sigma^{k,n}(y)$ are globally Lipschitz continuous in y, it is easily seen that $\{M(t)\}$ is in fact a square-integrable *martingale*, thus, in particular, $EM(t) = 0$, $t \in [0, \infty)$. Again, from (3.87), (3.88),

$$\text{(3.91)} \quad \sum_{j,k=1}^{D} \int_0^t \partial_{\vartheta^j \vartheta^k} \varphi(\vartheta(s))\, d\langle \vartheta^j, \vartheta^k\rangle(s) \;=\; 4\int_0^t (\vartheta(s), Q\vartheta(s)) \mathrm{Tr}[(BB')(\xi(s), \vartheta(s))Q]\, ds \\ +\; 8\int_0^t \vartheta'(s)Q(BB')(\xi(s), \vartheta(s))Q\vartheta(s)\, ds.$$

Now put $z(t) := E[\varphi(\vartheta(t))]$, and combine (3.89), (3.90), and (3.91), to get

$$\text{(3.92)} \qquad \begin{aligned} \dot z(t) \;&=\; 2E\{(\vartheta(t), Q\vartheta(t))\, \vartheta'(t)[QJ(\xi(t)) + J'(\xi(t))Q]\vartheta(t)\} \\ &+\; 2E\{(\vartheta(t), Q\vartheta(t))\ \mathrm{Tr}[(BB')(\xi(t), \vartheta(t))Q]\} \\ &+\; 4E\{\vartheta'(t)[Q(BB')(\xi(t), \vartheta(t))Q]\vartheta(t)\}, \qquad z(0) = \varphi(e^i). \end{aligned}$$

We next upper-bound each term on the right hand side of (3.92). From Rayleigh's principle (see Theorem 4.2.2 of Horn and Johnson [11], p.176), with μ_1 given by (2.12), we get

$$\text{(3.93)} \qquad \vartheta'[QJ(\xi) + J'(\xi)Q]\vartheta \le \mu_1|\vartheta|^2, \qquad \xi, \vartheta \in \Re^D.$$

Moreover, from (3.85), Remark 2.3, and the Cauchy-Schwarz inequality, we have

$$\text{(3.94)} \qquad \|(BB')(\xi, \vartheta)\| \le \|B(\xi, \vartheta)\|^2 \le \mu_0^2|\vartheta|^2, \qquad \xi, \varphi \in \Re^D.$$

Again, by Rayleigh's Principle and symmetry of $Q^{1/2}$, for any $B \in \Re^{D\times N}$ we have

$$\text{(3.95)} \qquad \Lambda_{\max}\{Q^{1/2}(BB')Q^{1/2}\} \le \sup_{z\neq 0} \frac{\|BB'\|\, |Q^{1/2}z|^2}{|z|^2} \le \|(BB')\|\, \Lambda_{\max}\{Q\}.$$

Since $\mathrm{Tr}[A] \le D\Lambda_{\max}\{A\}$ for real $D \times D$ symmetric matrices A, from (3.94) and (3.95) we find

$$\text{(3.96)} \qquad \mathrm{Tr}[(BB')(\xi, \vartheta)Q] \le D\Lambda_{\max}\{Q^{1/2}(BB')(\xi, \vartheta)Q^{1/2}\} \le D\mu_0^2\Lambda_{\max}\{Q\}|\vartheta|^2,$$

$$\text{(3.97)} \quad \vartheta'Q(BB')(\xi, \vartheta)Q\vartheta \le \Lambda_{\max}\{Q^{1/2}BB'(\xi, \vartheta)Q^{1/2}\}|Q^{1/2}\vartheta|^2 \le \mu_0^2\Lambda_{\max}\{Q\}(\vartheta, Q\vartheta)|\vartheta|^2,$$

for all $\xi, \vartheta \in \Re^D$. From (3.97), (3.96), (3.93), and (3.92),

$$\text{(3.98)} \qquad \dot z(t) \le 2\alpha E[(\vartheta(t), Q\vartheta(t))\ |\vartheta(t)|^2], \quad \text{for} \quad \alpha := \mu_1 + (D+2)\mu_0^2\Lambda_{\max}\{Q\} < 0,$$

where the strict inequality in (3.98) follows since $r \ge 3/2$ in Condition 2.5. From (3.98) and Rayleigh's Principle, we find $\dot z(t) \le -\gamma z(t)$, $t \in [0, \infty)$, for $\gamma := (-2\alpha)/\Lambda_{\max}\{Q\} > 0$ and $z(0) = \varphi(e^i)$. Thus $z(t) \le \varphi(e^i)\exp\{-\gamma t\}$, $t \in [0, \infty)$. This, along with Rayleigh's Principle gives $E[|\vartheta(t, y)|^4] \le (\varphi(e^i)/\Lambda^2_{\min}\{Q\})\exp\{-\gamma t\}$, $(t, y) \in [0, \infty) \times \Re^D$, which establishes the exponential decay of $E[|\partial_{y^i}\xi(t, y)|^4]$ (recall (3.86)).

Next consider the double L_2-derivative $\partial_{y^i y^j}\xi(t, y)$: From the second-order smoothness of $b^k(y)$ and $\sigma^{k,n}(y)$ given by Condition 2.5, one sees from Gihman and Skorohod ([9], eq.(4), p.60) that this second L_2-derivative exists and is given by formal differentiation of (3.87) with respect to parameter y^j. That is, for fixed arbitrary $i, j = 1, 2, \ldots, D$ and $y \in \Re^D$, the $\Re^D$-vector

$$\text{(3.99)} \qquad \chi(t, y) := (\partial_{y^i y^j}\xi^1(t, y), \partial_{y^i y^j}\xi^2(t, y), \ldots, \partial_{y^i y^j}\xi^D(t, y))',$$

satisfies the SDE (recall (3.85))

$$\chi(t,y) = \int_0^t J(\xi(s,y))\chi(s,y)\,ds + \int_0^t B(\xi(s,y),\chi(s,y))\,d\beta(s) + \int_0^t A(s,y)\,ds + \int_0^t G(s,y)\,d\beta(s), \tag{3.100}$$

where the $\Re^D$-vector $A(t,y)$ and $D \times N$-matrix $G(t,y)$ are defined respectively by

$$A^k(t,y) := [\partial_{y^j}\xi(t,y)]'[(\partial^2 b^k)(\xi(t,y))][\partial_{y^i}\xi(t,y)], \tag{3.101}$$

$$G^{k,n}(t,y) := [\partial_{y^j}\xi(t,y)]'[(\partial^2 \sigma^{k,n})(\xi(t,y))][\partial_{y^i}\xi(t,y)], \tag{3.102}$$

and $(\partial^2 b^k)(y)$, $(\partial^2\sigma^{k,n})(y)$ are the $D\times D$ symmetric Hessian matrices of the mappings b^k and $\sigma^{k,n}$ from $\Re^D$ into $\Re$. Since y is fixed throughout the proof, from now on we write $\xi(t)$, $\chi(t)$, $A(t)$, $B(t)$ and $G(t)$, for $\xi(t,y)$, $\chi(t,y)$, $A(t,y)$, $B(\xi(t,y),\chi(t,y))$ and $G(t,y)$ respectively. Put

$$\psi(\chi) := (\chi, Q\chi), \quad \chi \in \Re^D; \qquad q(t) := E[\psi(\chi(t))], \quad t\in[0,\infty). \tag{3.103}$$

Now use (3.100) and the Itô formula to expand $\psi(\chi(t))$, and then take expectations. After some elementary calculations this yields

$$\dot q(t) = E[\chi'(t)\,\{QJ(\xi(t)) + J'(\xi(t))Q\}\,\chi(t)] + 2E[\chi'(t)QA(t)] + E[\mathrm{Tr}\{(B(t)+G(t))(B(t)+G(t))'\,Q\}], \qquad q(0)=0. \tag{3.104}$$

By Rayleigh's principle and (3.103) we get

$$E[\chi'(t)\{QJ(\xi(t)) + J'(\xi(t))Q\}\chi(t)] \le \frac{\mu_1}{\Lambda\mathrm{max}\{Q\}} q(t), \tag{3.105}$$

where $\mu_1 < 0$ is given by (2.12). From (3.101), the uniform y-bound on the entries of the Hessian $\partial^2 b^k(y)$ (Condition 2.2), and Hölder's inequality, there is a constant $C \in [0,\infty)$, not depending on $(t,y) \in [0,\infty)\times\Re^D$, such that

$$E[\chi'(t)QA(t)] \le CE^{1/4}[|\partial_{y^i}\xi(t)|^4]E^{1/4}[|\partial_{y^j}\xi(t)|^4]E^{1/2}[|\chi(t)|^2]. \tag{3.106}$$

In view of (3.103), Rayleigh's principle, and the exponentially rapid decay of $E[|\partial_{y^i}\xi(t)|^4]$ already established, it follows from (3.106) that

$$E[\chi'(t)QA(t)] \le C\exp\{-\gamma t\}q^{1/2}(t), \tag{3.107}$$

for some constants $C,\gamma \in (0,\infty)$, not depending on (t,y). As for the third term on the right of (3.104), recalling $B(t) := B(\xi(t),\chi(t))$, we see from Rayleigh's principle, (3.96) with χ in place of ϑ, and (3.103), that

$$E[\mathrm{Tr}\{B(t)B'(t)Q\}] \le \frac{D\mu_0^2\Lambda\mathrm{max}\{Q\}}{\Lambda\mathrm{min}\{Q\}} q(t). \tag{3.108}$$

Again considering the third term on the right of (3.104), one sees from (3.85) with χ in place of ϑ, the uniform bounds postulated for $\partial_{y^i}\sigma^{k,n}(y)$ and $\partial_{y^iy^j}\sigma^{k,n}(y)$ (by Condition 2.2), and (3.102), that there is a constant $C\in[0,\infty)$, not depending on (t,y), such that $\mathrm{Tr}\{B(t)G'(t)Q\} \le C|\chi(t)|\;|\partial_{y^i}\xi(t)|\;|\partial_{y^j}\xi(t)|$. Then, exactly as at (3.106) and (3.107), there are constants $C,\gamma\in(0,\infty)$, not depending on (t,y), such that

$$E[\mathrm{Tr}\{B(t)G'(t)Q\}] \le C\exp\{-\gamma t\}q^{1/2}(t). \tag{3.109}$$

Still dealing with the third term of (3.104), from (3.102) and the uniform bounds on $\partial_{y^iy^j}\sigma^{k,n}(y)$, we have $\text{Tr}\{G(t)G'(t)Q\} \leq C\ |\partial_{y^i}\xi(t)|^2\ |\partial_{y^j}\xi(t)|^2$, for some constant $C \in [0,\infty)$, not depending on (t,y). Then from the Cauchy-Schwarz inequality, and the exponential decay of $E[|\partial_{y^i}\xi(t,y)|^4]$ already established, we can find constants $C,\gamma \in (0,\infty)$, not depending on (t,y), such that

$$E[\text{Tr}\{G(t)G'(t)Q\}] \leq C\exp\{-\gamma t\}. \tag{3.110}$$

Next, put $C_1 := -(\mu_1\Lambda_{\text{min}}\{Q\}+D\mu_0^2\Lambda_{\text{max}}^2\{Q\})/(\Lambda_{\text{max}}\{Q\}\Lambda_{\text{min}}\{Q\}) > 0$, the strict inequality following from Condition 2.5. Now combine (3.104), (3.105), (3.107), (3.108), and (3.110). This gives constants $C,\gamma \in (0,\infty)$ such that

$$\dot{q}(t) \leq -C_1 q(t) + C\exp\{-\gamma t\}q^{1/2}(t) + C\exp\{-\gamma t\}, \qquad t \in [0,\infty). \tag{3.111}$$

Now $0 \leq q^{1/2}(t) \leq 1 + q(t)$, thus one easily sees from (3.111), the fact that $q(0) = 0$, and an elementary calculation, that $q(t) \leq C\exp\{-\gamma t\}$, $t \in [0,\infty)$, for some constants $C,\gamma \in (0,\infty)$. In view of this, together with (3.103) and Rayleigh's Principle, we get the exponential decay of $E[|\partial_{y^iy^j}\xi(t,y)|^2]$, as required to establish Proposition 3.3. □

4 Proof of Theorem 2.20

We require some additional notation and terminology, set forth in Notation 4.1 and Remark 4.2:

Notation 4.1 (i) For a separable metric space E, let $\mathfrak{B}(E)$ denote the Borel σ-algebra on E, let $B(E)$ denote the set of all real-valued uniformly-bounded Borel-measurable mappings on E, and define the supremum norm on $B(E)$ by $\|\phi\| := \sup\{|\phi(x)| : x \in E\}$, $\forall\ \phi \in B(E)$. Also write $C(E)$ for the set of all real-valued continuous functions on E, put $\bar{C}(E) := B(E) \cap C(E)$, and write $C_c(E)$ for the set of all members of $\bar{C}(E)$ which have compact support. When E is also locally compact let $\hat{C}(E)$ denote the collection of all members of $\bar{C}(E)$ which vanish at infinity.

(ii) For a positive integer r, let $C^r(\Re^q)$ denote the collection of all members of $C(\Re^q)$ with continuous derivatives of each order, up to and including r. Let $C^\infty(\Re^q)$ denote the collection of all members of $C(\Re^q)$ with continuous derivatives of all orders. Put $C_c^\infty(\Re^q) := C_c(\Re^q) \cap C^\infty(\Re^q)$, and $C_c^r(\Re^q) := C_c(\Re^q) \cap C^r(\Re^q)$ for positive integers r.

(iii) When E is a separable metric space let $\mathcal{P}(E)$ denote the collection of all probability measures on the measurable space $(E, \mathfrak{B}(E))$ with the usual topology of weak convergence.

Remark 4.2 Suppose that E is a separable metric space:

(i) The set $M \subset B(E)$ *separates points in* E when the equality $f(x) = f(y)$, $\forall\ f \in M$, for some $x, y \in E$, implies $x = y$; and the set $M \subset B(E)$ *strongly separates points in* E when the convergence $\lim_n f(x_n) = f(x)$, $\forall\ f \in M$, for some $x, x_n \in E$, implies $\lim_n x_n = x$.

(ii) For $\phi, \phi_n \in B(E)$ write *b.p.* $-\lim_n \phi_n = \phi$ to indicate $\sup_n \|\phi_n\| < \infty$ and $\lim_n \phi_n(x) = \phi(x)$, $\forall\ x \in E$. A set $M \subset B(E) \times B(E)$ is called *b.p.-closed* when *b.p.* $-\lim_n \phi_n = \phi$ and *b.p.* $-\lim_n \psi_n = \psi$ for a sequence $\{(\phi_n, \psi_n)\} \subset M$ implies $(\phi,\psi) \in M$; and the *b.p.-closure* of a set $M \subset B(E) \times B(E)$ is defined to be the intersection of all b.p.-closed sets $M_\lambda \subset B(E) \times B(E)$ such that $M \subset M_\lambda$.

(iii) Let $\mathcal{A}$ be a linear operator from $\bar{C}(E)$ to $\bar{C}(E)$ with domain $\mathcal{D}(\mathcal{A}) \subset \bar{C}(E)$, and let $\mu \in \mathcal{P}(E)$. Then a *progressively measurable solution* of the martingale problem for $\mathcal{A}$ [for $(\mathcal{A},\mu)$] is some pair $\{(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{P}), (\tilde{X}_t)\}$, in which $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{P})$ is a filtered probability space and $\{\tilde{X}_t\}$ is an E-valued $\{\tilde{\mathcal{F}}_t\}$-progressively measurable process such that $g(\tilde{X}_t) - \int_0^t \mathcal{A}g(\tilde{X}_s)ds$ is a $\{\tilde{\mathcal{F}}_t\}$-martingale for each $g \in \mathcal{D}(\mathcal{A})$ [and $\tilde{P}\tilde{X}_0^{-1} = \mu$]. The martingale problem for $(\mathcal{A},\mu)$ has the property of *existence* when there exists some progressively measurable solution of the martingale problem for $(\mathcal{A},\mu)$, and has the property of *uniqueness* when, given any two progressively measurable solutions $\{(\tilde{\Omega},\tilde{\mathcal{F}},\{\tilde{\mathcal{F}}_t\},\tilde{P}),(\tilde{X}_t)\}$ and $\{(\bar{\Omega},\bar{\mathcal{F}},\{\bar{\mathcal{F}}_t\},\bar{P}),(\bar{X}_t)\}$ of the martingale problem for $(\mathcal{A},\mu)$, the E-valued processes $\tilde{X}$ and $\bar{X}$ necessarily have identical finite-dimensional distributions. The martingale problem for $(\mathcal{A},\mu)$ is called *well-posed* when it has the properties of both existence and uniqueness. Finally, the martingale problem for $\mathcal{A}$ is *well-posed* when the martingale problem for $(\mathcal{A},\mu)$ is well-posed for each $\mu \in \mathcal{P}(E)$.

The proof of Theorem 2.20 is based on the following convergence theorem:

Theorem 4.3 (Theorem 2.1 & Remark 2.2 of Bhatt and Karandikar [3]) *Suppose that E is a complete separable metric space, $\mu \in \mathcal{P}(E)$, $T \in (0,\infty)$ is a constant, and $\mathcal{A}$ is a linear operator from $\bar{C}(E)$ to $\bar{C}(E)$ with domain $\mathcal{D}(\mathcal{A}) \subset \bar{C}(E)$, such that the following conditions hold:*
(I) *There is a countable set $\{g_k\} \subset \mathcal{D}(\mathcal{A})$ such that (i) $\{(g,\mathcal{A}g):\ g \in \mathcal{D}(\mathcal{A})\}$ is a subset of the bp-closure of $\{(g_k,\mathcal{A}g_k)\}$, and (ii) the set $\{g_k\}$ strongly separates points in E.*
(II) *$\mathcal{D}(\mathcal{A})$ is an algebra, and, for each $x \in E$, we have $g(x) \neq 0$ for some $g \in \mathcal{D}(\mathcal{A})$).*
(III) *The martingale problem for $\mathcal{A}$ is well-posed (in the sense of Remark 4.2(iii)).*
(IV) *The martingale problem for $(\mathcal{A},\mu)$ has a solution $\{(\tilde{\Omega},\tilde{\mathcal{F}},\{\tilde{\mathcal{F}}_t\},\tilde{P}),(\tilde{X}_t),t\in[0,T]\}$ for which the process $\{\tilde{X}_t,\ t\in[0,T]\}$ has corlol (r.c.l.l.) paths.*
(V) *There is a sequence $\{X_n(t),\ t\in[0,T]\}$, $n=1,2,\dots$ of E-valued processes with corlol paths, each defined on a probability space $(\Omega_n,\mathcal{F}^n,P_n)$, such that $\{X_n(t),\ n=1,2,\dots\}$ is a tight sequence for each $t\in[0,T]$, and $X_n(0)$ converges weakly to μ with $n\to\infty$.*
(VI) *For each $g\in\mathcal{D}(\mathcal{A})$, there exist $\Re$-valued progressively measurable processes $\{(U_n(t),\mathcal{F}^n_t),\ t\in[0,T]\}$ and $\{(V_n(t),\mathcal{F}^n_t),\ t\in[0,T]\}$ on $(\Omega_n,\mathcal{F}^n,P_n)$, $n=1,2,\dots$, such that*

$$U_n(t) - \int_0^t V_n(s)\,ds, \qquad t\in[0,T], \quad \textit{is an } \{\mathcal{F}^n_t\} - \textit{martingale}, \tag{4.112}$$

$$\lim_n E_n\left[\sup_{t\in[0,T]} |U_n(t) - g(X_n(t))|\right] = 0, \tag{4.113}$$

$$\sup_n E_n\left[\left\{\int_0^T |V_n(s)|^p\,ds\right\}^{1/p}\right] < \infty, \quad \textit{for some} \quad p\in(1,\infty), \tag{4.114}$$

$$\lim_n E_n[|\mathcal{A}g(X_n(t)) - V_n(t)|] = 0, \qquad \textit{for each} \quad t\in[0,T]. \tag{4.115}$$

Then $\{X_n(t),\ t\in[0,T]\}$ converges weakly to $\{X(t),\ t\in[0,T]\}$ as $n\to\infty$.

Remark 4.4 We shall use Theorem 4.3 to prove Theorem 2.20. We note in passing that we are not by any means using the full power of this result, which is really intended for applications in which E is not necessarily locally compact. Nevertheless, even when E is just locally compact (as will be the case in the following), Theorem 4.3 seems to be decidedly easier to use than the more classical approaches for showing weak convergence to a Markov limit (see e.g. [8], Theorem 4.8.2 and Corollary 4.8.6, p.226 and p.231).

Motivated by (2.47) and (2.48), define the $2d$-dimensional vector $H(x,z)$ and $2d\times(M+d)$-matrix $\mathfrak{S}(x,z)$ by

$$H(x,z) := \begin{bmatrix} \bar{F}(x) \\ (\partial_x\bar{F})(x)z \end{bmatrix}, \qquad \mathfrak{S}(x,z) := \begin{bmatrix} G(x) & 0 \\ \partial_x G(x)[z] & \bar{a}^{1/2}(x) \end{bmatrix}, \qquad x,z\in\Re^d. \tag{4.116}$$

Then, we see that the linear second-order differential operator for the $\Re^{2d}$-valued diffusion given by (2.47), (2.48) is

$$
\begin{aligned}
(\mathcal{C}g)(x,z) &:= \sum_{i=1}^{2d} H^i(x,z)(\partial_i g)(x,z) + \frac{1}{2}\sum_{i,j=1}^{2d}(\mathfrak{S}\mathfrak{S}')^{i,j}(x,z)(\partial_{ij}g)(x,z) \\
&= (\partial_x g)(x,z)\bar{F}(x) + (\partial_z g)(x,z)(\partial_x \bar{F})(x)z \\
&+ \frac{1}{2}\left[\mathrm{Tr}\{(\partial_{xx}g)(x,z)(GG')(x)\} + 2\mathrm{Tr}\{(\partial_{xz}g)(x,z)(\partial_x G(x)[z])G'(x)\}\right. \\
&+ \left.\mathrm{Tr}\{(\partial_{zz}g)(x,z)\{\bar{a}(x) + (\partial_x G(x)[z])(\partial_x G(x)[z])'\}\}\right], \quad x,z \in \Re^d, \quad g \in C^2(\Re^{2d}).
\end{aligned}
\tag{4.117}
$$

Remark 4.5 We now define the various entities in Theorem 4.3: (a) Put $E := \Re^{2d}$, $\mu := \delta_{(x_0,0)}$ (the Dirac measure at $(x_0,0) \in \Re^{2d}$, for x_0 in Condition 2.15), $\mathcal{A} := \mathcal{C}$ (see (4.117)), and $\mathcal{D}(\mathcal{A}) \equiv \mathcal{D}(\mathcal{C}) := C_c^\infty(\Re^{2d})$. (b) Identify the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, the process $\{\tilde{X}_t,\ t \in [0,T]\}$, and the filtration $\{\tilde{\mathcal{F}}_t\}$ in Theorem 4.3(IV), with $(\Omega, \mathcal{F}, P)$ (see Conditions 2.2 and 2.15), the $\Re^{2d}$-valued continuous process $\{(\hat{X}(\tau), \hat{Z}(\tau)),\ \tau \in [0,T]\}$ given by the solution of the exact SDE (2.47), (2.48) with initial value $(\hat{X}(0), \hat{Z}(0)) = (x_0, 0)$, and the filtration of the Wiener process $(\hat{W}_1(t), \hat{W}_2(t))$ augmented with the null events of $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ (see Remark 2.18). (c) Fix some sequence $\{\epsilon_n\} \subset (0,1)$ such that $\epsilon_n \to 0$ with $n \to \infty$. Identify each $(\Omega_n, \mathcal{F}^n, P_n)$ in Theorem 4.3(V) with $(\Omega, \mathcal{F}, P)$ (in Conditions 2.2 and 2.15), and identify the process $\{X_n(t),\ t \in [0,T]\}$ in Theorem 4.3(V),(VI), with the $\Re^{2d}$-valued process $\{(X^{\epsilon_n}(\tau), Z^{\epsilon_n}(\tau)),\ \tau \in [0,T]\}$ on $(\Omega, \mathcal{F}, P)$ given by (1.7) to (1.11) (construction of the $\Re$-valued processes $\{U_n(t)\}$ and $\{V_n(t)\}$ in (VI) will be given later). We now verify (I) - (VI) of Theorem 4.3, from which Theorem 2.20 follows.

Verification of (I) and (II) in Theorem 4.3: The mappings in (4.116) are continuous in $(x,z) \in \Re^{2d}$, thus we have $\{(g, \mathcal{C}g):\ g \in \mathcal{D}(\mathcal{C})\} \subset \hat{C}(\Re^{2d}) \times \hat{C}(\Re^{2d})$ (recall Notation 4.1(i)). Moreover $\Re^{2d}$ is locally compact, thus the space $\hat{C}(\Re^{2d}) \times \hat{C}(\Re^{2d})$ is separable in the topology $\mathcal{T}$ generated by the usual supremum norm, hence there is a sequence $\{g_k\} \subset \mathcal{D}(\mathcal{C})$ such that $\{(g, \mathcal{C}g):\ g \in \mathcal{D}(\mathcal{C})\}$ is a subset of the $\mathcal{T}$-closure of $\{(g_k, \mathcal{C}g_k)\}$. Since the $\mathcal{T}$-closure of a subset of $\hat{C}(\Re^{2d}) \times \hat{C}(\Re^{2d})$ is included within the b.p.-closure of the set, we verify (I)(i). As for (I)(ii), this follows since $C_c^\infty(\Re^{2d})$ strongly separates points in $\Re^{2d}$ (see Remark 4.2(i)), and therefore $\{g_k\}$, which is a dense subset of $C_c^\infty(\Re^{2d})$ (in the topology of the supremum norm), also strongly separates points of $\Re^{2d}$. Since $C_c^\infty(\Re^{2d})$ is an algebra that vanishes nowhere, (II) is immediate.

Verification of (III) in Theorem 4.3: From Proposition 5.5 we see that the mapping $(x,z) \to H^i(x,z)$ given by (4.116) is locally Lipschitz continuous on $\Re^{2d}$; and, from the uniform bound for $\partial_x G(x)$ (see Condition 2.14), (2.46), and the local Lipschitz continuity of $\bar{a}^{1/2}(\cdot)$ (see Remark 2.18), it follows that the mapping $(x,z) \to \mathfrak{S}^{i,j}(x,z)$ is locally Lipschitz continuous on $\Re^{2d}$. Moreover, in view of Proposition 5.5 and (2.44), there is a constant $C \in [0,\infty)$ such that

$$
|H^i(x,z)| \le C[1 + |(x,z)|], \quad |(\mathfrak{S}\mathfrak{S}')^{i,j}(x,z)| \le C[1 + |(x,z)|^2], \quad (x,z) \in \Re^{2d}. \tag{4.118}
$$

Thus, we have verified bounds of the form (3.34) and (3.35) in Ethier and Kurtz ([8], p.300), and therefore Theorem 8.2.6 in Ethier and Kurtz ([8], p.374) establishes that the martingale problem for the linear operator $\mathcal{C}$ defined by (4.117) (with domain $\mathcal{D}(\mathcal{C}) := C_c^\infty(\Re^{2d})$) is well-posed in the sense of Remark 4.2(iii).

Verification of (IV) in Theorem 4.3: Immediate from Itô's formula and Remark 4.5(b).

It remains to verify (V) and (VI) of Theorem 4.3. As will become clear, the essential thing we shall need for this step is solvability of parametrized Poisson equations of the form (2.21) for appropriate choices of the mapping $h(x,y)$. However, some preliminary calculations are needed first. From (1.2), (1.7), and Condition 2.15, we have

$$(4.119)\quad Y^\varepsilon(\tau) = y_0 + \varepsilon^{-1}\int_0^\tau b(Y^\varepsilon(s))\,ds + \varepsilon^{-1/2}\int_0^\tau \sigma(Y^\varepsilon(s))\,dB^\varepsilon(s), \quad B^\varepsilon(\tau) := \varepsilon^{1/2}\beta(\tau/\varepsilon).$$

We next write an SDE representation for $Z^\varepsilon(\tau)$ given by (1.7) - (1.11), in terms of $X^\varepsilon(\tau)$ and $Y^\varepsilon(\tau)$, with $\bar{X}^\varepsilon(\tau)$ eliminated: From Proposition 5.5 we know that $\bar{F}(x)$ is certainly a C^2-function in $x \in \Re^d$, hence we have

$$(4.120)\qquad \bar{F}(x - \varepsilon^{1/2}z) = \bar{F}(x) - \varepsilon^{1/2}(\partial_x \bar{F})(x)z + \varepsilon I_1(\varepsilon, x, z),$$

$$(4.121)\qquad I_1^i(\varepsilon, x, z) := \int_0^1 (1-\zeta)z'(\partial_{xx}\bar{F}^i)(x - \zeta\varepsilon^{1/2}z)z\,d\zeta.$$

Likewise, since $G^{i,j}$ is a C^2-mapping (Condition 2.14), we have (recall (2.46))

$$(4.122)\qquad G(x - \varepsilon^{1/2}z) = G(x) - \varepsilon^{1/2}(\partial_x G)(x)[z] + \varepsilon I_2(\varepsilon, x, z),$$

where $I_2^{i,j}(\varepsilon, x, z)$ is defined in the same way as $I_1^i(\varepsilon, x, z)$, but with $G^{i,j}$ in place of $\bar{F}^i$. From (4.120) and (4.122), together with (1.11), (1.8), and (1.9), one sees that

$$\begin{aligned}(4.123)\quad Z^\varepsilon(\tau) &= \varepsilon^{-1/2}\int_0^\tau [F(X^\varepsilon(s), Y^\varepsilon(s)) - \bar{F}(X^\varepsilon(s))]ds + \int_0^\tau (\partial_x \bar{F})(X^\varepsilon(s))Z^\varepsilon(s)ds \\ &+ \int_0^\tau (\partial_x G)(X^\varepsilon(s))[Z^\varepsilon(s)]dW^\varepsilon(s) - \varepsilon^{1/2}\int_0^\tau I_1(\varepsilon, X^\varepsilon(s), Z^\varepsilon(s))\,ds \\ &- \varepsilon^{1/2}\int_0^\tau I_2(\varepsilon, X^\varepsilon(s), Z^\varepsilon(s))\,dW^\varepsilon(s).\end{aligned}$$

Now fix some $\varepsilon \in (0,1)$ and $g \in C_c^\infty(\Re^{2d})$, and put

$$\begin{aligned}(4.124)\quad \Xi_g^\varepsilon(x,y,z) &:= g(x,z) + \varepsilon^{1/2}\sum_{n=1}^d \Phi^n(x,y)(\partial_{z^n}g)(x,z) + \varepsilon\sum_{n=1}^d \Phi^n(x,y)(\partial_{x^n}g)(x,z) \\ &+ \frac{\varepsilon}{2}\sum_{m,n=1}^d \Psi^{m,n}(x,y)(\partial_{z^m z^n}g)(x,z), \qquad (x,y,z) \in \Re^d \times \Re^D \times \Re^d,\end{aligned}$$

where $\Phi^n(x,y)$ is defined by (2.41) and $\Psi^{m,n}(x,y)$ is defined by (5.156), (2.43) and (2.42). Next, observe from (1.8), (1.7), (4.119), and the independence of the Wiener processes $\{\beta(t)\}$ and $\{w(t)\}$ (Condition 2.15), that the semimartingales $\{X^\varepsilon(\tau)\}$ and $\{Y^\varepsilon(\tau)\}$ have co-quadratic variation equal to zero, namely $\langle X^\varepsilon, Y^\varepsilon\rangle \equiv 0$ a.s. In view of this fact, along with (1.8), (4.119), (4.123), and the smoothness properties of $\Phi^n(x,y)$ and $\Psi^{m,n}(x,y)$ established by Proposition 5.6(ii) and Proposition 5.7(ii)(iii), we can use Itô's formula to expand $\Xi_g^\varepsilon(X^\varepsilon(\tau), Y^\varepsilon(\tau), Z^\varepsilon(\tau))$, to get

$$(4.125)\ \Xi_g^\varepsilon(X^\varepsilon(\tau), Y^\varepsilon(\tau), Z^\varepsilon(\tau)) = \Xi_g^\varepsilon(x_0, y_0, 0) + \int_0^\tau (\mathcal{G}^\varepsilon \Xi_g^\varepsilon)(X^\varepsilon(s), Y^\varepsilon(s), Z^\varepsilon(s))ds + M_g^\varepsilon(\tau).$$

Here $\mathcal{G}^\varepsilon$ is the linear second-order differential operator for the triple of SDE's (1.8), (4.119), (4.123), which define the $\Re^{d+D+d}$-valued process $\{(X^\varepsilon(\tau), Y^\varepsilon(\tau), Z^\varepsilon(\tau)),\ \tau \in [0,T]\}$, and $\{M_g^\varepsilon(\tau)\}$ is a continuous local martingale relative to the filtration

$$(4.126)\qquad \mathcal{F}_\tau^\varepsilon := \sigma\{W^\varepsilon(s), B^\varepsilon(s),\ 0 \le s \le \tau\} \vee P\text{-null events in } \mathcal{F},$$

which is given by sums of stochastic integrals with respect to the standard Wiener processes $\{W^\varepsilon(\tau)\}$ and $\{B^\varepsilon(\tau)\}$. Actually, an easy but tedious calculation based on (4.124), the compact support of $g(x,z)$ and its partial derivatives (in $(x,z)\in\Re^d\times\Re^d$), and the polynomial bounds (in $(x,y)\in\Re^d\times\Re^D$) on the partial derivatives of $\Phi(x,y)$ and $\Psi(x,y)$ established in Proposition 5.6(ii) and Proposition 5.7(ii), shows that $\{(M_g^\varepsilon(\tau),\mathcal{F}_\tau^\varepsilon),\tau\in[0,T]\}$ is a square-integrable *martingale*, a fact that will be important later on. We next calculate the integrand in (4.125). Applying the second-order differential operator $\mathcal{G}^\varepsilon$ to Ξ_g^ε in (4.124), and arranging terms in increasing powers of ε, one sees from an easy, if lengthy, calculation, that (recall Notation 2.1(iii),(iv))

$$
\begin{aligned}
(\mathcal{G}^\varepsilon\Xi_g^\varepsilon)(x,y,z) &= \varepsilon^{-1}(\mathcal{A}\Xi_g^\varepsilon)(x,y,z)+\varepsilon^{-1/2}(\partial_z\Xi_g^\varepsilon)(x,y,z)[F(x,y)-\bar F(x)]\\
&+ \big[(\partial_x\Xi_g^\varepsilon)(x,y,z)F(x,y)+(\partial_z\Xi_g^\varepsilon)(x,y,z)(\partial_x\bar F)(x)z\\
&+ (1/2)\mathrm{Tr}\{(\partial_{xx}\Xi_g^\varepsilon)(x,y,z)(GG')(x)\}+\mathrm{Tr}\{(\partial_{xz}\Xi_g^\varepsilon)(x,y,z)(\partial_xG(x)[z])G'(x)\}\\
&+ (1/2)\mathrm{Tr}\{(\partial_{zz}\Xi_g^\varepsilon)(x,y,z)(\partial_xG(x)[z])(\partial_xG(x)[z])'\}\big]\\
&- \varepsilon^{1/2}\big[(\partial_z\Xi_g^\varepsilon)(x,y,z)I_1(\varepsilon,x,z)+\mathrm{Tr}\{(\partial_{xz}\Xi_g^\varepsilon)(x,y,z)I_2(\varepsilon,x,z)G'(x)\}\\
&+ \mathrm{Tr}\{(\partial_{zz}\Xi_g^\varepsilon)(x,y,z)(\partial_xG(x)[z])I_2'(\varepsilon,x,z)\}\big]\\
&+ (\varepsilon/2)\mathrm{Tr}\{(\partial_{zz}\Xi_g^\varepsilon)(x,y,z)(I_2I_2')(\varepsilon,x,z)\}.
\end{aligned}
\tag{4.127}
$$

Consider the first two terms on the right side of (4.127). Since g is a function of (x,z) in (4.124) and the second-order differential operator $\mathcal{A}$ involves first and second derivatives with respect to y only, we must have $(\mathcal{A}g)(x,y,z)\equiv 0$. Moreover, from Proposition 5.6(iii) and Proposition 5.7(iii), we have

$$
\text{(i) } \mathcal{A}\Phi^n(x,y)=-\tilde F^n(x,y),\qquad \text{(ii) } \mathcal{A}\Psi^{m,n}(x,y)=\bar a^{m,n}(x)-a^{m,n}(x,y),
\tag{4.128}
$$

for all $(x,y)\in\Re^d\times\Re^D$. Then, from (4.124) and (4.128)(i)(ii), we find

$$
\begin{aligned}
\varepsilon^{-1}(\mathcal{A}\Xi_g^\varepsilon)(x,y,z) &= \varepsilon^{-1/2}(\partial_zg)(x,z)[\bar F(x)-F(x,y)]+(\partial_xg)(x,z)[\bar F(x)-F(x,y)]\\
&+ \frac{1}{2}\mathrm{Tr}\{(\partial_{zz}g)(x,z)[\bar a(x)-a(x,y)]\}.
\end{aligned}
\tag{4.129}
$$

As for the second term on the right of (4.127), from (4.124), (2.42), and (2.40), we have

$$
\begin{aligned}
\varepsilon^{-1/2}(\partial_z\Xi_g^\varepsilon)(x,y,z)[F(x,y)-\bar F(x)] &= \varepsilon^{-1/2}(\partial_zg)(x,z)[F(x,y)-\bar F(x)]+\frac{1}{2}\mathrm{Tr}\{(\partial_{zz}g)(x,z)a(x,y)\}\\
&+ \varepsilon^{1/2}\sum_{i,n=1}^d\Phi^n(x,y)[F^i(x,y)-\bar F^i(x)](\partial_{x^nz^i}g)(x,z)\\
&+ \frac{\varepsilon^{1/2}}{2}\sum_{i,m,n=1}^d\Psi^{m,n}(x,y)[F^i(x,y)-\bar F^i(x)](\partial_{z^iz^mz^n}g)(x,z).
\end{aligned}
\tag{4.130}
$$

Upon combining (4.127), (4.129), (4.130), and (4.117), we find

$$
(\mathcal{G}^\varepsilon\Xi_g^\varepsilon)(x,y,z)=(\mathcal{C}g)(x,z)+\sum_{k=1}^4\varepsilon^{k/2}\Delta_k^g(\varepsilon,x,y,z),\quad (x,y,z)\in\Re^d\times\Re^D\times\Re^d,
\tag{4.131}
$$

for some mappings $\Delta_k^g : (0,1) \times \Re^d \times \Re^D \times \Re^d \to \Re$. These mappings are given by rather complicated expressions involving $I_1(\varepsilon, x, y, z)$, $I_2(\varepsilon, x, y, z)$, and $g(x,z)$, $\Phi(x,y)$, $\Psi(x,y)$ together with their partial derivatives. Using the bounds in Proposition 5.6(ii) and Proposition 5.7(ii), and the fact that $g(x,z)$ has compact support, it is easy (although tedious) to see that there is a constant $C(g) \in [0,\infty)$, depending only on the fixed $g \in C_c^\infty(\Re^2)$, such that

$$|\Delta_k^g(\varepsilon, x, y, z)| \le C(g)[1 + |y|^{3(1+r_1)}], \qquad (\varepsilon, x, y, z) \in (0,1) \times \Re^d \times \Re^D \times \Re^d, \tag{4.132}$$

for each $k = 1, \dots, 4$, where r_1 is the constant in Condition 2.14.

Remark 4.6 Observe that the right side of (4.131) involves only *non-negative* powers of ε, and the dominant term $(\mathcal{G}g)(x,z)$ (involving ε^0) is a function in (x,z) only, with no dependence in y. As will soon become clear, these are the keys to verifying (V) and (VI) of Theorem 4.3, and the form of (4.124) has been chosen with precisely this in mind. Indeed, in the preceding we have

(a) used the "first Poisson relation" (4.128)(i) to cancel terms in (4.127) which involve $\varepsilon^{-1/2}$;

(b) used (2.42) and the "second Poisson relation" (4.128)(ii) to cancel $a(x,y)$ from the terms on the right of (4.127) that involve ε^0, and thus ensure that the dominant term on the right of (4.131) is a function of (x,z) only. The form of (4.124) that we have used here is suggested by Kurtz ([13], Theorem 2.2, p.60).

Verification of (V) in Theorem 4.3: Fix a mapping $\phi : [0,\infty) \to [0,1]$ such that $\phi(r) := 1$ for all $r \in [0,1]$, $\phi(r) := 0$ for all $r \in [2,\infty)$, and $\phi(\cdot)$ has continuous derivatives of all orders, and define the functions $g_k : \Re^{2d} \to [0,1]$ by

$$g_k(x,z) := \phi\left(\frac{1}{k}\log[1 + |(x,z)|]\right), \qquad k = 1, 2, \dots, \qquad (x,z) \in \Re^{2d}, \tag{4.133}$$

where $|(x,z)|$ denotes the usual Euclidean norm of $(x,z) \in \Re^{2d}$. Then $g_k \in C_c^\infty(\Re^{2d})$, and it is easily seen, using (4.118) and (4.117), that

$$\lim_{k\to\infty}\left[\sup_{(x,z)\in\Re^{2d}} |(\mathcal{C}g_k)(x,z)|\right] = 0. \tag{4.134}$$

Moreover, we have observed (following (4.126)) that $\{M_{g_k}^{\epsilon_n}(\tau),\ \tau \in [0,T]\}$ is a *martingale* which is clearly null at $\tau = 0$, so that $E[M_{g_k}^{\epsilon_n}(\tau)] = 0$, $\tau \in [0,T]$. Then, upon taking expectations in (4.125) and using (4.124) and (4.131) with $g := g_k$ and $\varepsilon := \epsilon_n$, we find

$$\begin{aligned}
0 &= Eg_k(X_\tau^{\epsilon_n}, Z_\tau^{\epsilon_n}) + \epsilon_n^{1/2} E[\Phi(X_\tau^{\epsilon_n}, Y_\tau^{\epsilon_n})(\partial_z g_k)(X_\tau^{\epsilon_n}, Z_\tau^{\epsilon_n})] \\
&+ \epsilon_n E[\Phi(X_\tau^{\epsilon_n}, Y_\tau^{\epsilon_n})(\partial_x g_k)(X_\tau^{\epsilon_n}, Z_\tau^{\epsilon_n})] + \frac{\epsilon_n}{2} E[\Psi(X_\tau^{\epsilon_n}, Y_\tau^{\epsilon_n})(\partial_{zz} g_k)(X_\tau^{\epsilon_n}, Z_\tau^{\epsilon_n})] \\
&- g_k(x_0, 0) - \epsilon_n^{1/2}\Phi(x_0, y_0)(\partial_z g_k)(x_0, 0) - \epsilon_n \Phi(x_0, y_0)(\partial_x g_k)(x_0, 0) \\
&- \frac{\epsilon_n}{2}\Psi(x_0, y_0)(\partial_{zz} g_k)(x_0, 0) - \int_0^\tau \{E[(\mathcal{C}g_k)(X_s^{\epsilon_n}, Z_s^{\epsilon_n})] \\
&+ \sum_{l=1}^4 \epsilon_n^{l/2} E[\Delta_l^{g_k}(\epsilon_n, X_s^{\epsilon_n}, Y_s^{\epsilon_n}, Z_s^{\epsilon_n})]\,\}ds, \qquad \tau \in [0,T].
\end{aligned} \tag{4.135}$$

Remark 4.7 In order to save on notation in (4.135) we have put $X_\tau^{\epsilon_n}$ for $X^{\epsilon_n}(\tau)$ (and likewise for $Y_\tau^{\epsilon_n}$, $Z_\tau^{\epsilon_n}$), and have assumed $d = 1$. The general case just involves summation over the indices $m, n = 1, 2, \dots, d$ (see (4.124)).

Now fix some (small) $\eta \in (0,1)$. In view of (4.133) there is a positive integer k_0 such that $g_k(x_0,0)=1$ for all $k \geq k_0$, and, by (4.134), there is a positive integer $k_1 \geq k_0$, depending on η, such that

$$\int_0^T E[|(\mathcal{C}g_{k_1})(X_s^{\epsilon_n}, Z_s^{\epsilon_n})|]\, ds < \frac{\eta}{2}, \qquad n = 1,2,3,\ldots \tag{4.136}$$

From Proposition 5.6 (see (5.150)) and the fact that $g_{k_1}(x,z)$ has *compact* support in $(x,z) \in \Re^d \times \Re^D$, there is a constant $C \in [0,\infty)$, depending only on g_{k_1}, such that

$$|\Phi(x,y)(\partial_z g_{k_1})(x,z)| \leq C[1+|y|^{(1+r_1)}], \qquad (x,y,z) \in \Re^d \times \Re^D \times \Re^d. \tag{4.137}$$

In view of (1.7) and Remark 5.2, for each $\rho \in [0,2r]$ there is a constant $C(\rho) \in [0,\infty)$ such that

$$\sup_{\substack{\varepsilon \in (0,1) \\ \tau \in [0,T]}} E\left[|Y^\varepsilon(\tau)|^\rho\right] \leq C(\rho)[1+|y_0|^\rho], \tag{4.138}$$

Since $(1+r_1) \in [0,2r]$ (by (2.39)), from (4.138) and (4.137) we obtain

$$\sup_{\substack{\varepsilon \in (0,1) \\ \tau \in [0,T]}} E|\Phi(X_\tau^\varepsilon, Y_\tau^\varepsilon)(\partial_z g_{k_1})(X_\tau^\varepsilon, Z_\tau^\varepsilon)| < \infty, \tag{4.139}$$

and a bound identical to (4.139), but with $\partial_x g_{k_1}$ in place of $\partial_z g_{k_1}$, holds in exactly the same way. Using Proposition 5.7 (see (5.157)) and $2(1+r_1) \in [0,2r]$ (see (2.39)), we similarly get

$$\sup_{\substack{\varepsilon \in (0,1) \\ \tau \in [0,T]}} E|\Psi(X_\tau^\varepsilon, Y_\tau^\varepsilon)(\partial_{zz} g_{k_1})(X_\tau^\varepsilon, Z_\tau^\varepsilon)| < \infty, \tag{4.140}$$

and, using (4.132), (4.138), and the fact that $3(1+r_1) \in [0,2r]$ (by (2.39)), we find

$$\sup_{\substack{\varepsilon \in (0,1) \\ \tau \in [0,T]}} E|\Delta_l^{g_{k_1}}(\varepsilon, X_\tau^\varepsilon, Y_\tau^\varepsilon, Z_\tau^\varepsilon)| < \infty, \qquad l = 1,2,3,4. \tag{4.141}$$

Now fix an arbitrary $\tau \in [0,T]$. Since $\epsilon_n \to 0$, it follows from (4.135), the fact that $g_{k_1}(x_0,0)=1$, (4.136), and (4.139) - (4.141), that there is a positive integer n_1, depending on η, such that

$$|Eg_{k_1}(X_\tau^{\epsilon_n}, Y_\tau^{\epsilon_n}) - 1| < \eta, \qquad n \geq n_1. \tag{4.142}$$

Now put $R(\eta) := \exp\{2k_1(\eta)\} - 1$. From (4.133) and (4.142) we have $P[|(X_\tau^{\epsilon_n}, Z_\tau^{\epsilon_n})| > R(\eta)] < \eta$ for all $n \geq n_1(\eta)$. Now (V) follows from this fact and the arbitrary choice of $\eta \in (0,1)$.

Verification of (VI) in Theorem 4.3: Fix a mapping $g \in \mathcal{D}(\mathcal{C}) \equiv C_c^\infty(\Re^{2d})$, and put

$$U_n(\tau) := \Xi_g^{\epsilon_n}(X_\tau^{\epsilon_n}, Y_\tau^{\epsilon_n}, Z_\tau^{\epsilon_n}), \qquad V_n(\tau) := (\mathcal{G}^{\epsilon_n}\Xi_g^{\epsilon_n})(X_\tau^{\epsilon_n}, Y_\tau^{\epsilon_n}, Z_\tau^{\epsilon_n}), \qquad \tau \in [0,T]. \tag{4.143}$$

We have observed (following (4.126)) that $\{(M_g^\varepsilon(\tau), \mathcal{F}_\tau^\varepsilon), \tau \in [0,T]\}$ is a martingale for each $\varepsilon \in (0,1)$. Thus (4.112) follows from (4.125) and (4.143) (when we identify $\mathcal{F}_t^n$ with $\mathcal{F}_\tau^{\epsilon_n}$ given by (4.126)). Next consider (4.114). From the strict inequality in (2.39) there is some $p \in (1,\infty)$ such that $2r \geq 3p(1+r_1)$, and then it follows from (4.132) and (4.138) (with $\rho := 3p(1+r_1)$) that $E\left[|\Delta_l^g(\epsilon_n, X_\tau^{\epsilon_n}, Y_\tau^{\epsilon_n}, Z_\tau^{\epsilon_n})|^p\right] \leq C$ for all $l = 1,2,3,4$, $n = 1,2,\ldots$, $\tau \in [0,T]$, for some constant $C \in [0,\infty)$ depending only on g, p, y_0. In view of this bound, the definition of $V_n(\tau)$ (see (4.143)), (4.131), and the fact that $(\mathcal{C}g)(x,z)$ is uniformly bounded in $(x,z) \in \Re^{2d}$ (since $g(\cdot)$ has compact support), we get $E\left[|V_n(\tau)|^p\right] \leq C < \infty$, for all $n = 1,2,\ldots$ and $\tau \in [0,T]$, which, with Jensen's inequality and Fubini's Theorem, gives (4.114). As for (4.115), this follows from (4.131) and the fact that $\mathcal{A}g \equiv \mathcal{C}g$ (recall Remark 4.5). Finally, it remains to establish (4.113). To this end we require the following result, which is established later in the present section:

Lemma 4.8 *Suppose the hypotheses of Theorem 2.20. Then, for each $\rho \in [0, 2r]$, there is a constant $C(\rho) \in [0, \infty)$ such that*

$$E\left[\sup_{\tau\in[0,T]} |Y^\varepsilon(\tau)|^\rho\right] \leq \frac{C(\rho)}{\varepsilon}, \qquad \varepsilon \in (0,1).$$

From Proposition 5.6(i) and the fact that $g(x,z)$ has *compact* support in $(x,z) \in \Re^d \times \Re^D$, there is a constant $C \in [0,\infty)$, depending only on g, such that, for $i = 1, 2, \ldots, d$,

$$\text{(4.144)} \quad \sup_{\tau\in[0,T]} |\Phi^i(X_\tau^{\epsilon_n}, Y_\tau^{\epsilon_n})(\partial_{z^k} g)(X_\tau^{\epsilon_n}, Z_\tau^{\epsilon_n})|^3 \leq C\left[1 + \sup_{\tau\in[0,T]} |Y_\tau^{\epsilon_n}|^{3(1+r_1)}\right], \quad n = 1, 2, \ldots$$

In view of (4.144), the fact that $2r > 3(1 + r_1)$ (recall (2.39)), Lemma 4.8 [with $\rho := 3(1 + r_1)$], and $\epsilon_n \in (0,1)$, there is a constant $C \in [0,\infty)$ such that, for $i = 1, 2, \ldots, d$,

$$\text{(4.145)} \qquad E\left[\sup_{\tau\in[0,T]} |\Phi^i(X_\tau^{\epsilon_n}, Y_\tau^{\epsilon_n})(\partial_{z^i} g)(X_\tau^{\epsilon_n}, Z_\tau^{\epsilon_n})|\right] \leq \frac{C}{\epsilon_n^{1/3}}, \qquad n = 1, 2, \ldots$$

In exactly the same way, one finds a constant $C \in [0,\infty)$ such that (4.145) holds, but with $(\partial_{x^i} g)$ in place of $(\partial_{z^i} g)$; and, from Proposition 5.7(ii) along with the fact that $g(x,z)$ has compact support in $(x,z) \in \Re^{2d}$, we similarly see that there is a constant $C \in [0,\infty)$, depending only on g, such that

$$\text{(4.146)} \qquad E\left[\sup_{\tau\in[0,T]} |\Psi^{i,j}(X_\tau^{\epsilon_n}, Y_\tau^{\epsilon_n})(\partial_{z^i z^j} g)(X_\tau^{\epsilon_n}, Z_\tau^{\epsilon_n})|\right] \leq \frac{C}{\epsilon_n^{2/3}}, \qquad n = 1, 2, \ldots$$

for $i, j = 1, 2, \ldots, d$. Upon combining (4.124), (4.143), (4.145) (along with its analogue with $\partial_{x^i} g$ in place of $\partial_{z^i} g$), and (4.146), we see that $E\left[\sup_{\tau\in[0,T]} |U_n(\tau) - g(X_\tau^{\epsilon_n}, Z_\tau^{\epsilon_n})|\right] = O(\epsilon_n^{1/6})$ for all $n = 1, 2, \ldots$, which checks (4.113). We have now verified (VI) of Theorem 4.3. ☐

Proof of Lemma 4.8: For all $n = 0, 1, 2, \ldots$, we have (see (1.2))

$$\text{(4.147)} \quad \xi(t, y_0) = \xi(n, y_0) + \int_n^t b(\xi(s, y_0))\, ds + \int_n^t \sigma(\xi(s, y_0))\, d\beta(s), \qquad t \in [n, n+1].$$

From Condition 2.5 we have $2r > 2$, and therefore, from (4.147) and Lemma V(11.5) of Rogers and Williams ([16], p.129), there is a constant $C \in [0,\infty)$ such that

$$\text{(4.148)} \quad E\left[\sup_{t\in[n,n+1]} |\xi(t,y_0)|^{2r}\right] \leq C\left[E[|\xi(n,y_0)|^{2r}] + \int_n^{n+1} E[|b(\xi(s,y_0))|^{2r} + |\sigma(\xi(s,y_0))|^{2r}]\, ds\right],$$

for all $n = 1, 2, \ldots$ From Remark 5.2 we have $\sup_{n=0,1,2,\ldots} E[|\xi(n, y_0)|^{2r}] < \infty$, and, since $b(\cdot)$ and $\sigma(\cdot)$ are globally Lipschitz continuous, we similarly have $\sup_{s\in[0,\infty)} E[|b(\xi(s, y_0))|^{2r}] < \infty$ and $\sup_{s\in[0,\infty)} E[|\sigma(\xi(s, y_0))|^{2r}] < \infty$. In view of these facts together with (4.148) we then find that $E\left[\sup_{t\in[n,n+1]} |\xi(t, y_0)|^{2r}\right] < C$, for a constant $C \in [0,\infty)$ and all $n = 0, 1, 2, \ldots$. From this we obtain $E[\sup_{t\in[0,T\varepsilon^{-1}]} |\xi(t, y_0)|^{2r}] < C\varepsilon^{-1}$, $\varepsilon \in (0,1)$, for a constant $C \in [0,\infty)$. This, together with Jensen's inequality, gives the result. ☐

5 Miscellany

In this section we establish a miscellany of technical results that are used in the preceding sections.

Proposition 5.1 *Suppose Condition 2.2 and Condition 2.5. Then (i) there exists a constant $C \in [0, \infty)$ such that (recalling Remark 2.4) we have*

$$\sup_{t\in[0,\infty)} E[(\xi(t,y), Q\xi(t,y))^r] \leq (y, Qy)^r + C, \qquad y \in \Re^D; \tag{5.149}$$

(ii) there exists a unique invariant probability measure $\bar{m}$ on $\Re^D$ for the Markov transition probability defined by (1.2), and the integrability relation (2.15) holds.

Proof: From Conditions 2.2 and 2.5 it is easily seen that hypotheses (A_1), (A_2), and (A_{3r}) of Basak ([1], p.46, p.50) hold, with $C := Q$, $D := Q$, and $\beta := -\mu_1 + (1 - D)\mu_0^2\Lambda_{\max}\{Q\}$ (recall (2.12)). From Condition 2.5 we find $r - \beta/(2\Lambda_{\max}\{Q\}\mu_0^2) < 0$, hence we can take $\epsilon = 0$ in Proposition 3.1 and Lemma 3.2 of [1] to get (i) and (ii). $\square$

Remark 5.2 Suppose Condition 2.2 and Condition 2.5. Since $\Lambda_{\min}\{Q\} > 0$, it follows from (5.149), together with Rayleigh's principle and the Liapunov L_p-inequality (see Shiryaev [17], p.193) that, for each $\rho \in [0, 2r]$, there is a constant $C(\rho) \in [0, \infty)$ such that

$$E|\xi(t,y)|^\rho \leq C(\rho)[1 + |y|^\rho], \quad t \in [0,\infty), \quad y \in \Re^D.$$

The next result is essential for establishing Proposition 2.11 and Theorem 2.12:

Proposition 5.3 *Suppose Condition 2.2 and Condition 2.5, and let $\bar{m}$ be given by Remark 2.7. Then there are constants $C, \gamma \in (0, \infty)$ such that, for each $p \in [1, 2r]$ and mapping $g : \Re^d \to \Re$ with $[g(\cdot)]_p < \infty$ (recall Remark 3.1), the integrals $Eg(\xi(t,y))$ and $\int_{\Re^D} g\, d\bar{m}$ exist, and*

$$\left| Eg(\xi(t,y)) - \int_{\Re^D} g\, d\bar{m} \right| \leq C[g]_p \exp\{-\gamma t\}[1 + |y|^p], \;\; (t,y) \in [0,\infty) \times \Re^D.$$

Proof: The argument is identical to that used for ([10], Lemma 3.4). $\square$

Proposition 5.4 *Suppose Condition 2.2 and Condition 2.5, and let $\bar{m}$ be given by Remark 2.7. Suppose, also, that $h : \Re^d \times \Re^D \to \Re$ is Borel-measurable, that $x \to h(x,y)$ is a C^1-mapping on $\Re^d$ for each $y \in \Re^D$, and there are constants $q \in [0, 2r]$ and $C(R) \in [0, \infty)$ for each $R \in [0, \infty)$ such that $\max\{|h(x,y)|, |(\partial_{x^i} h)(x,y)|\} \leq C(R)[1 + |y|^q]$, $(x,y) \in S_R^d \times \Re^D$. Then the integral $\bar{h}(x) := \int_{\Re^D} h(x,y)\, d\bar{m}(y)$ exists for each $x \in \Re^d$ and defines a C^1-mapping $\bar{h}$ with $(\partial_{x^i}\bar{h})(x) = \int_{\Re^D} (\partial_{x^i} h)(x,y)\, d\bar{m}(y)$ for all $x \in \Re^d$.*

Proposition 5.4 is a straightforward consequence of Remark 2.7 (see (2.15)) and standard results on interchanging integrals and derivatives (see e.g. Lemma (1.7) of Durrett [6], Ch.4, p.129).

Proposition 5.5 *Suppose Condition 2.2, Condition 2.5, and Condition 2.14, and suppose that $r \geq (1 + r_1)/2$ (where r, r_1, are the constants in Conditions 2.5 and 2.14 respectively). Then the integral in (1.10) exists for each $x \in \Re^d$, giving a C^3-mapping from $\Re^d$ into $\Re^d$, and the partial-derivative functions $(\partial_{x^i}\bar{F})(x)$, $(\partial_{x^i x^j}\bar{F})(x)$, and $(\partial_{x^i x^j x^k}\bar{F})(x)$ are uniformly bounded in $x \in \Re^d$.*

Proposition 5.5 follows at once from Proposition 5.4 and Condition 2.14 The remaining two results of this section, namely Propositions 5.6 and 5.7, use Proposition 2.11 and Theorem 2.12 to establish solvability of the Poisson equations (4.128) as well as smoothness and growth properties of the functions $\Phi^n(x,y)$ and $\Psi^{m,n}(x,y)$. These properties are needed for Remark 2.16, as well as for the proof Theorem 2.20 (see lines (4.132), (4.137), (4.140), and (4.144)).

Proposition 5.6 *Suppose Conditions 2.2, 2.5, and 2.14, and suppose that (2.39) holds. Then: (i) the integral in (2.41) exists for each $(x,y) \in \Re^D$, the function $\Phi^n(x,y)$ is continuous in $(x,y) \in \Re^d \times \Re^D$, and there is a constant $C \in [0,\infty)$ such that*

$$|\Phi^n(x,y)| \le C[1+|x|][1+|y|^{(1+r_1)}], \quad (x,y) \in \Re^d \times \Re^D. \tag{5.150}$$

(ii) The partial derivative functions $(\partial_{x^i}\Phi^n)(x,y)$, $(\partial_{y^i}\Phi^n)(x,y)$, $(\partial_{y^iy^j}\Phi^n)(x,y)$, $(\partial_{x^ix^j}\Phi^n)(x,y)$, $(\partial_{x^ix^jx^k}\Phi^n)(x,y)$, $(\partial_{x^iy^j}\Phi^n)(x,y)$, $(\partial_{x^iy^jy^k}\Phi^n)(x,y)$, and $(\partial_{x^ix^jy^k}\Phi^n)(x,y)$, exist and are continuous on $\Re^d \times \Re^D$. Moreover, for each $R \in [0,\infty)$, there is a constant $C(R) \in [0,\infty)$ such that

$$|H(x,y)| \le C(R)[1+|y|^{r_1}], \quad (x,y) \in S^d_R \times \Re^D, \tag{5.151}$$

where $H(x,y)$ denotes each of $(\partial_{y^i}\Phi^n)(x,y)$, $(\partial_{y^iy^j}\Phi^n)(x,y)$, $(\partial_{x^iy^j}\Phi^n)(x,y)$, $(\partial_{x^iy^jy^k}\Phi^n)(x,y)$, $(\partial_{x^ix^jy^k}\Phi^n)(x,y)$, and there is a constant $C \in [0,\infty)$ such that

$$|J(x,y)| \le C[1+|y|^{(1+r_1)}], \quad (x,y) S^d_R \times \Re^D, \tag{5.152}$$

where $J(x,y)$ denotes each of $(\partial_{x^i}\Phi^n)(x,y)$, $(\partial_{x^ix^j}\Phi^n)(x,y)$, $(\partial_{x^ix^jx^k}\Phi^n)(x,y)$;
(iii) $\Phi^n(x,y)$ satisfies the parametrized Poisson equation (4.128)(i).

Proof: To establish (i) we verify the conditions of Proposition 2.11 with the identifications $h := F^n$, $q_2 = q_3 := 1 + r_1$, $\Theta := \Phi^n$. From (2.39) we know that $q_2, q_3 \in [1, 2r)$. In view of the upper bound on $(\partial_{y^i}F)(x,y)$ given by (2.38) and the Mean Value Theorem, we verify (2.25). Now Proposition 2.11(I) establishes that the integral in (2.41) exists and gives a continuous function $\Phi^n(x,y)$ on $\Re^d \times \Re^D$, and, for each $R \in [0,\infty)$, there is a constant $C(R) \in [0,\infty)$ such that

$$|\Phi^n(x,y)| \le C(R)[1+|y|^{(1+r_1)}], \quad (x,y) \in S^d_R. \tag{5.153}$$

Moreover, a bound of the form (2.27) follows from (2.37) (with $H := (\partial_{x^iy^j}F^n)$) and the Mean Value Theorem, and furtheremore $C_2(R) \equiv C_1$ for all $R \in [0,\infty)$, where C_1 is the constant in (2.37). Now Proposition 2.11(II) shows that the partial derivative function $(\partial_{x^i}\Phi^n)(x,y)$ exists and is continuous in $(x,y) \in \Re^d \times \Re^D$, with

$$(\partial_{x^i}\Phi^n)(x,y) = \int_0^\infty E[(\partial_{x^i}\tilde{F}^n)(x,\xi(t,y))]\,dt, \tag{5.154}$$

and establishes a bound of the form (5.152) with $J := (\partial_{x^i}\Phi^n)$. In view of this bound, together with (5.153) (with $x := 0$) and the Mean Value Theorem, we get existence of a constant $C \in [0,\infty)$ such that (5.150) holds, as required to establish (i).

We next proceed to (ii). The partial derivative $(\partial_{x^i}\Phi^n)(x,y)$ has already been dealt with in the preceding paragraph, hence we consider the partial derivatives $(\partial_{y^i}\Phi^n)(x,y)$ and $(\partial_{y^iy^j}\Phi^n)(x,y)$. To this end we verify the conditions of Theorem 2.12 with $h := F^n$ and $q_1 := r_1$. Clearly (a) of Theorem 2.12 holds, and, in view of (2.39), we certainly have $q_1 \in [0,r)$. In view of (2.38), we see that the upper-bounds (2.30) and (2.33) are verified. Moreover, from (2.37) with $H := (\partial_{x^iy^j}F^n)$ and $H := (\partial_{x^iy^jy^k}F^n)$, together with the Mean Value Theorem, we establish the upper-bounds (2.31) and (2.34), with $c := r_1$. Then, it follows from Theorem 2.12 that the partial derivative functions $(\partial_{y^i}\Phi^n)(x,y)$ and $(\partial_{y^iy^j}\Phi^n)(x,y)$ exist and are continuous in $(x,y)\in\Re^d\times\Re^D$, and the upper-bound (5.151) holds with $H:=(\partial_{y^i}\Phi^n)$ and with $H:=(\partial_{y^iy^j}\Phi^n)$. Moreover, (iii) is an immediate consequence of Theorem 2.12.

To complete the proof we must deal with the remaining partial derivatives in (ii). Consider the partial derivative $(\partial_{x^ix^j}\Phi^n)(x,y)$. Motivated by (5.154) we verify the conditions of Proposition 2.11 with $h:=(\partial_{x^i}F^n)$, $q_2=q_3:=1+r_1$, $\Theta:=(\partial_{x^i}\Phi^n)$. By (2.39) we have $q_2,q_3\in[1,2r)$; a bound of the form (2.25) follows from (2.37) (with $H:=(\partial_{x^iy^j}F^n)$) and the Mean Value Theorem, while a bound of the form (2.27) follows from (2.37) (with $H:=(\partial_{x^ix^jy^k}F^n)$) and the Mean Value Theorem; exactly as in the preceding paragraph, we have $C_2(R)\equiv C_1$ for all $R\in[0,\infty)$. Now Proposition 2.11 shows that the x_j-partial derivative of the function in (5.154), namely $(\partial_{x^ix^j}\Phi^n)(x,y)$, exists and is continuous in $(x,y)\in\Re^d\times\Re^D$, with

$$(\partial_{x^ix^j}\Phi^n)(x,y)=\int_0^\infty E[(\partial_{x^ix^j}\tilde{F}^n)(x,\xi(t,y))]\,dt, \tag{5.155}$$

and a bound of the form (5.152) holds with $J:=(\partial_{x^ix^j}\Phi^n)$. In the same way, motivated by (5.155), one deals with the third partial derivative $(\partial_{x^ix^jx^k}\Phi^n)(x,y)$ by verifying the conditions of Proposition 2.11 with $h:=(\partial_{x^ix^j}F^n)$, $q_2=q_3:=1+r_1$, $\Theta:=(\partial_{x^ix^j}\Phi^n)$; using the Mean Value Theorem and (2.37) (with $H:=(\partial_{x^ix^jy^k}F^n)$ and $H:=(\partial_{x^ix^jx^ky^l}F^n)$) we verify the bounds (2.25) and (2.27), and now Proposition 2.11 establishes that $(\partial_{x^ix^jx^k}\Phi^n)(x,y)$ exists and is continuous in $(x,y)\in\Re^d\times\Re^D$, and a bound of the form (5.152) holds with $J:=(\partial_{x^ix^jx^k}\Phi^n)$.

Next, the assertions in (ii) for the partial derivatives $(\partial_{x^iy^j}\Phi^n)(x,y)$ and $(\partial_{x^iy^jy^k}\Phi^n)(x,y)$ follow from Theorem 2.12(I)(II) with $h:=\partial_{x^i}F^n$, $q_1:=r_1$, $c:=r_1$, and $\Theta:=(\partial_{x^i}\Phi^n)$, for which (2.30), (2.31), (2.33) and (2.34) are easily verified by the Mean Value Theorem and (2.37) (where we identify H with $(\partial_{x^iy^j}F^n)$, $(\partial_{x^ix^jy^k}F^n)$, $(\partial_{x^iy^jy^k}F^n)$, and $(\partial_{x^ix^jy^ky^l}F^n)$ respectively). As for the assertions in (ii) concerning $(\partial_{x^ix^jy^k}\Phi^n)(x,y)$, these follow from Theorem 2.12(I) with $h:=\partial_{x^ix^j}F^n$, $q_1:=r_1$, $c:=r_1$ and $\Theta:=(\partial_{x^ix^j}\Phi^n)$, for which (2.30) and (2.31) are verified using the Mean Value Theorem and (2.37) (where we identify H with $(\partial_{x^ix^jy^k}F^n)$ and $(\partial_{x^ix^jx^ky^l}F^n)$ respectively). This completes the proof of (ii), and we have already established (iii). $\square$

Proposition 5.7 *Suppose Conditions 2.2, 2.5, and 2.14, and suppose that (2.39) holds. Then: (i) the integral in (2.43) exists for each* $x\in\Re^d$ *and defines a symmetric positive semidefinite* $d\times d$*-matrix,* $\bar{a}^{m,n}(x)$ *is a* C^2*-function in* $x\in\Re^d$*, and (2.44) holds for a constant* $C\in[0,\infty)$*.*
(ii) The integral defined by

$$\Psi^{m,n}(x,y):=\int_0^\infty E[\tilde{a}^{m,n}(x,\xi(t,y))]\,dt,\quad \tilde{a}^{m,n}(x,y):=a^{m,n}(x,y)-\bar{a}^{m,n}(x), \tag{5.156}$$

exists for each $(x,y)\in\Re^d\times\Re^D$*, and defines a continuous mapping on* $\Re^d\times\Re^D$*, and, for each* $R\in[0,\infty)$*, there is a constant* $C(R)\in[0,\infty)$ *such that*

$$|\Psi^{m,n}(x,y)|\le C(R)[1+|y|^{2(1+r_1)}],\qquad (x,y)\in S_R^d\times\Re^D. \tag{5.157}$$

Moreover, the partial derivative functions $(\partial_{x^i}\Psi^{m,n})(x,y)$ *and* $(\partial_{x^ix^j}\Psi^{m,n})(x,y)$ *exist and are continuous on* $\Re^d\times\Re^D$*, and, for each* $R\in[0,\infty)$*, there is a constant* $C(R)\in[0,\infty)$ *such that*

$$|(\partial_{x^i}\Psi^{m,n})(x,y)|+|(\partial_{x^ix^j}\Psi^{m,n})(x,y)|\le C(R)[1+|y|^{2(1+r_1)}],\qquad (x,y)\in S_R^d\times\Re^D. \tag{5.158}$$

(iii) The partial derivative functions $(\partial_{y^i}\Psi^{m,n})(x,y)$ *and* $(\partial_{y^iy^j}\Psi^{m,n})(x,y)$*, exist and are continuous on* $\Re^d\times\Re^D$*, for each* $R\in[0,\infty)$ *there is a constant* $C(R)\in[0,\infty)$ *such that*

$$|(\partial_{y^i}\Psi^{m,n})(x,y)|+|(\partial_{y^iy^j}\Psi^{m,n})(x,y)|\le C(R)[1+|y|^{1+2r_1}],\qquad (x,y)\in S_R^d\times\Re^D, \tag{5.159}$$

and $\Psi^{m,n}(x,y)$ *satisfies the parametrized Poisson equation (4.128)(ii).*

Proof: (i) From the uniform-boundedness of $\partial_{x^i}\bar{F}^n$ (see Proposition 5.5) there is a constant $C \in [0,\infty)$ such that $|\tilde{F}(x,y)| \leq C[1+|x|][1+|y|^{1+r_1}]$ for each $(x,y) \in \Re^d \times \Re^D$, and therefore in view of (5.150) and (2.42), there is a constant $C \in [0,\infty)$ such that

$$(5.160) \qquad |a^{m,n}(x,y)| \leq C[1+|x|^2][1+|y|^{2(1+r_1)}], \quad (x,y) \in \Re^d \times \Re^D.$$

From (2.39) we have $2(1+r_1) < 2r$. It follows from this, with (5.160) and Proposition 5.1 (see (2.15)), that the integral in (2.43) exists for each $x \in \Re^d$, and there is a constant $C \in [0,\infty)$ such that the quadratic bound (2.44) holds. We next show that $\bar{a}^{m,n}(\cdot)$ is a C^2-mapping on $\Re^d$: Computing the first and second x-derivatives of $a^{m,n}(x,y)$ in (2.42), and using the bounds given by (2.37), (5.150), and (5.152), together with the uniform boundedness of $(\partial_{x^i}\bar{F})$ and $(\partial_{x^ix^j}\bar{F})$ (see Proposition 5.5), one finds a constant $C(R) \in [0,\infty)$ for each $R \in [0,\infty)$ such that

$$(5.161) \quad \max\{|\partial_{x^i}a^{m,n}(x,y)|, |\partial_{x^ix^j}a^{m,n}(x,y)|\} \leq C(R)[1+|y|^{2(1+r_1)}], \quad (x,y) \in S_R^d \times \Re^D.$$

Now it follows from (5.161), (5.160), the fact that $2(1+r_1) < 2r$, and a two-fold application of Proposition 5.4, that $\bar{a}^{m,n}(\cdot)$ is a C^2-function. Moreover, $\bar{a}(x)$ is a symmetric matrix, and it remains to show that $\bar{a}(x)$ is positive semidefinite: Let $\{\bar{\xi}(t)\}$ be an $\Re^D$-valued strictly stationary Markov process defined by (1.2) with marginal distribution $\bar{m}$ given by Proposition 5.1, and, for $n, m = 1, 2, \dots, d$, $(t,x) \in [0,\infty) \times \Re^d$, put

$$(5.162) \qquad \Gamma^{m,n}(t,x) := E[\{\Phi^m(x,\bar{\xi}(0)) - \Phi^m(x,\bar{\xi}(t))\}\{\Phi^n(x,\bar{\xi}(0)) - \Phi^n(x,\bar{\xi}(t))\}].$$

We will show

$$(5.163) \qquad \bar{a}(x) = \lim_{t\downarrow 0} \frac{1}{t}\Gamma(t,x), \qquad x \in \Re^d.$$

Since $\Gamma(t,x)$ is positive semidefinite, it follows from (5.163) that $\bar{a}(x)$ is positive semidefinite. By strict stationarity of $\{\bar{\xi}(t)\}$ we have

$$(5.164) \qquad \begin{aligned} \Gamma^{m,n}(t,x) &= E[\Phi^m(x,\bar{\xi}(0))\{\Phi^n(x,\bar{\xi}(0)) - \Phi^n(x,\bar{\xi}(t))\}] \\ &+ E[\Phi^n(x,\bar{\xi}(0))\{\Phi^m(x,\bar{\xi}(0)) - \Phi^m(x,\bar{\xi}(t))\}], \end{aligned}$$

and, from the Markov property of $\{\bar{\xi}(t)\}$, we also have

$$(5.165) \qquad E[\Phi^m(x,\bar{\xi}(0))\Phi^n(x,\bar{\xi}(t))] = \int_{\Re^D} \Phi^m(x,y)E[\Phi^n(x,\xi(t,y))]\, d\bar{m}(y).$$

By the smoothness of $\Phi^n(x,\cdot)$ established in Proposition 5.6, we can expand $\Phi^n(x,\xi(t,y))$ by Itô's formula, and, upon taking expectations, get

$$(5.166) \qquad E[\Phi^n(x,\xi(t,y))] = \Phi^n(x,y) + \int_0^t E[(\mathcal{A}\Phi^n)(x,\xi(t,y))]\, dt,$$

(from the polynomial-boundedness of $y \to (\partial_{y^i}\Phi^n)(x,y)$ established in Proposition 5.6, it is seen that the stochastic integral with respect to $\{\beta(t)\}$ appearing in Itô's formula is in fact a square-integrable martingale and null at the origin). Now combine (5.166) and (5.165) to get

$$(5.167) \qquad \begin{aligned} E[\Phi^m(x,\bar{\xi}(0))\{\Phi^n(x,\bar{\xi}(0)) &- \Phi^n(x,\bar{\xi}(t))\}] \\ &= -\int_{\Re^D} \Phi^m(x,y)\left(\int_0^t E[\mathcal{A}\Phi^n(x,\xi(s,y))]ds\right) d\bar{m}(y). \end{aligned}$$

Now $y \to (\mathcal{A}\Phi^n)(x,y)$ is polynomially-bounded (as follows from Proposition 5.6), and therefore, by standard results on SDE's (see e.g. 5.3.15 on p.306 of Karatzas and Shreve [12]) we have $E[\max_{0\le t\le T} |\mathcal{A}\Phi^n(x,\xi(t,y))|] < \infty$, $T < \infty$. Thus, from Dominated Convergence, the function $s \to E[\mathcal{A}\Phi^n(x,\xi(s,y))]$ is continuous. Thus, from (5.167), together with Remark 5.2, the polynomial bounds on the first and second y-derivatives of $\Phi^n(x,y)$ given by (5.151), and Dominated Convergence, we easily obtain

$$(5.168)\quad \lim_{t\downarrow 0} \frac{1}{t} E[\Phi^m(x,\bar{\xi}(0))\{\Phi^n(x,\bar{\xi}(0)) - \Phi^n(x,\bar{\xi}(t))\}] = -\int_{\Re^D} \Phi^m(x,y)(\mathcal{A}\Phi^n)(x,y)\, d\bar{m}(y).$$

In view of (2.42) and Proposition 5.6(iii) we have

$$a^{m,n}(x,y) = -\Phi^m(x,y)(\mathcal{A}\Phi^n)(x,y) - \Phi^n(x,y)(\mathcal{A}\Phi^m)(x,y),$$

and (5.163) follows from this fact, along with (5.168), (5.164) and (2.43).

(ii) We verify the conditions of Proposition 2.11 with $h := a^{m,n}$ and $q_2 = q_3 := 2(1+r_1)$. From Condition 2.14, (2.42), Proposition 5.6(i)(ii), and the chain rule, we get an upper-bound of the form $\max\{|\partial_{x^i} h(x,y)|, |\partial_{x^i y^j} h(x,y)| \le C(R)[1+|y|^{1+2r_1}]$, $(x,y) \in S_R^d \times \Re^D$. Now the upper bounds (2.25) and (2.27) follow from the Mean Value Theorem. Proposition 2.11(I) then establishes that the integral (5.156) exists and defines a continuous function on $\Re^d \times \Re^D$, and, for each $R \in [0,\infty)$, there is a constant $C(R) \in [0,\infty)$ such that (5.157) holds. On the other hand, Proposition 2.11(II) establishes that the partial derivative $(\partial_{x^i}\Psi^{m,n})(x,y)$ exists and is continuous in $(x,y) \in \Re^d \times \Re^D$, that

$$(5.169)\qquad (\partial_{x^i}\Psi^{m,n})(x,y) = \int_0^\infty E[(\partial_{x^i}\tilde{a}^{m,n})(x,\xi(t,y))]\, dt, \qquad (x,y) \in \Re^d \times \Re^D,$$

and that $(\partial_{x^i}\Psi^{m,n})(x,y)$ is subject to an upper-bound of the form (5.158). As for the second partial derivative $\partial_{x^i x^j}\Psi^{m,n})(x,y)$, we again verify the conditions of Proposition 2.11, but with $h = (\partial_{x^i} a^{m,n})$, $q_2 = q_3 = 2(1+r_1)$, $\Theta = (\partial_{x^i}\Psi^{m,n})$. From Condition 2.14, (2.42), and Proposition 5.6(i)(ii), we get $\max\{|\partial_{x^i} h(x,y)|, |\partial_{x^i y^j} h(x,y)| \le C(R)[1+|y|^{1+2r_1}]$, $(x,y) \in S_R^d \times \Re^D$. Now (2.25) and (2.27) follow from the Mean Value Theorem, and, from Proposition 2.11(II) and (5.169), one sees that the partial derivative function $(\partial_{x^i x^j}\Psi^{m,n})(x,y)$ exists and is continuous in $(x,y) \in \Re^d \times \Re^D$, with an upper-bound of the form (5.158).

(iii) We verify the conditions of Theorem 2.12 with $h := a^{m,n}$, $q_1 := 1+2r_1$, $c := 1+2r_1$. From (2.42) and Proposition 5.6, one sees that (a) of Theorem 2.12 holds. From (2.42), the upper-bounds (5.151) and (5.152) given by Proposition 5.6(ii), and repeated use of the chain rule and Mean Value Theorem, it is easy (but tedious) to verify the upper bounds (2.30), (2.31), (2.33), and (2.34), as required to use Theorem 2.12 to get (iii). □

References

[1] Basak, G.K. (1991) "A class of limit theorems for singular diffusions", J. Multivariate Analysis **39**, 44-59.

[2] Basak, G.K. and Bhattacharya, R.N. (1992) "Stability in distribution for a class of singular diffusions", Ann. Probab. **20**, 312-321.

[3] Bhatt, A.G. and Karandikar, R.L. (1993) "Weak convergence to a Markov process: the martingale approach", Prob. Theor. Relat. Fields **96**, 335-351.

[4] Bhattacharya, R.N. (1982) "On the functional central limit theorem and the law of the iterated logarithm for Markov processes", Z. Wahrsch. verw. Geb. **60**, 185-201.

[5] Bhattacharya R.N. and Waymire E.C. (1990) Stochastic Processes with Applications (Wiley).

[6] Durrett, R. (1996) Stochastic Calculus: A Practical Introduction (CRC Press).

[7] Elliott, R.J. (1982) Stochastic Calculus and Applications (Springer-Verlag).

[8] Ethier, S.N. and Kurtz T.G. (1986) Markov Processes - Characterization and Convergence (Wiley).

[9] Gihman, I.I. and Skorohod, A.V. (1972) Stochastic Differential Equations (Springer-Verlag).

[10] Heunis, A.J. (2003) "Strong invariance principle for singular diffusions", Stoch. Proc. Appl. **104**, 57-80.

[11] Horn, R. and Johnson C.R. (1985) Matrix Analysis (Cambridge University Press).

[12] Karatzas, I. and Shreve, S.E. (1991) Brownian Motion and Stochastic Calculus, 2nd Ed. (Springer-Verlag).

[13] Kurtz, T.G. (1973) "A limit theorem for perturbed operator semigroups with applications to random evolutions", J. Funct. Analysis **12**, 55-67.

[14] Liptser, R. and Stoyanov, J. (1990) "Stochastic version of the averaging principle for diffusion type processes", Stochastics and Stochastics Reports **32**, 145-163.

[15] Pardoux, E. and Veretennikov, A.Yu. (2001) "On the Poisson equation and diffusion approximation. I", Ann. Probab. **29**, 1061-1085.

[16] Rogers, L.C.G. and Williams, D. (2000) Diffusions, Markov Processes and Martingales: Volume 2 Itô Calculus (Cambridge University Press).

[17] Shiryaev, A.N. (1996) Probability, 2nd. Ed. (Springer-Verlag).