

# CONVEX DUALITY IN CONSTRAINED MEAN-VARIANCE PORTFOLIO OPTIMIZATION <sup>1</sup>

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# Summary

We apply conjugate duality to establish existence of optimal portfolios in an asset-allocation problem, with the goal of minimizing the variance of the final wealth which results from trading over a fixed finite horizon in a continuous-time complete market, subject to the constraints that the expected final wealth equal a specified target value, and the portfolio of the investor, defined by the dollar amount invested in each stock, takes values in a given closed convex set. The asset prices are modelled by Itô processes, for which the market parameters are random processes adapted to the information filtration available to the investor. We synthesize a dual optimization problem and establish a set of optimality relations, similar to the Euler-Lagrange and transversality relations of calculus of variations, giving necessary and sufficient conditions for the given optimization problem and its dual to each have a solution, with zero duality gap. We then resolve these relations to establish existence of an optimal portfolio.

**Abbreviated Title:** Convex duality

**AMS Subject Classifications:** 93E20, 91B28, 90A09, 49N15

**Key Words:** Convex analysis, duality synthesis, variational analysis

# 1 Introduction

In this work we study an asset-allocation problem, the goal of which is to minimize the variance of the final wealth which results from trading on a fixed finite horizon in a continuous-time complete market with random market parameters, subject to the constraints that the expected final wealth equal a specified target value and the portfolio of the investor (defined by the dollar amount in each stock) always takes values in a given closed convex set. This constraint is general enough to model a prohibition on short-selling of stock, incomplete markets, limits on the dollar amount allocated to each stock, and other trading restrictions. Our goal is to establish existence of an optimal portfolio and characterize it.

Problems of this kind belong to the general area of *mean-variance portfolio selection*, and their financial relevance, as compared with the more common objective of maximizing expected utility, has been discussed by Lim and Zhou [12] and Li, Zhou and Lim [13]. In fact [12] addresses this problem, but for *unconstrained* portfolios, using the methods of stochastic LQ control. The follow-up work [13] deals with a similar problem, but includes a no-short selling constraint; it is postulated that the market coefficients are *nonrandom*, and viscosity solutions of the (correspondingly nonrandom) Bellman equation are used to characterize the constrained optimal portfolio. The problem of interest here involves a combination of both random market parameters *and* general portfolio constraints. This rules out application of stochastic LQ theory, as in [12] (which relies on the absence of portfolio constraints), as well as the approach of [13] (for which the market parameters must be nonrandom).

In light of the preceding we turn to the use of *conjugate duality*. The goal is to formulate an associated “dual” optimization problem for which it is (hopefully) easy to directly establish existence of a solution, and then to construct an optimal portfolio in terms of the solution of the dual problem. Our approach is motivated by a recent work of Rogers [14] in which the central idea is to regard the dynamical relation satisfied by the wealth and the portfolio as itself *defining a constraint*, a point of view which then provides the key for synthesizing a dual optimization problem. We cannot in fact directly apply the method of Rogers [14], since this work does not address the problem of existence of optimal portfolios, but nevertheless the fundamental viewpoint of [14], namely that the wealth equation is a constraint, is essential to us. We shall account for this constraint in a way which is suggested by a work of Bismut [2] on stochastic convex control problems. The basic idea is to remove the portfolio “variable” to obtain a Bolza problem in the (stochastic) calculus of variations which amounts to minimization of a convex functional over a set of Itô processes large enough to include all of the possible wealth processes. Bismut [2] establishes a powerful duality theory for dealing with such stochastic Bolza problems, which we shall use to construct a dual optimization problem, together with optimality relations (similar to the Euler-Lagrange and transversality relations of calculus of variations) which are equivalent to the primal and dual problems being solvable with zero duality gap. We then use these relations to establish existence of an optimal portfolio and corresponding wealth process.

In Sections 2 to 4 we introduce the market model and formulate the problem of constrained mean-variance portfolio selection, and, in Sections 5 and 6, we use conjugate duality to construct the optimal portfolio and wealth process. Finally, in Section 7, we indicate how the

approach we have used for mean-variance minimization extends easily to problems of utility maximization.

## 2 Market Model

Throughout the sequel  $T \in (0, \infty)$  is a given constant, and  $\{W(t), t \in [0, T]\}$  is a given  $\mathbb{R}^N$ -valued standard Brownian motion on the complete probability space  $(\Omega, \mathcal{F}, P)$ . Put

$$(2.1) \quad \mathcal{F}_t := \sigma\{W(\tau), \tau \in [0, t]\} \vee \mathcal{N}(P),$$

in which  $\mathcal{N}(P)$  denotes the collection of all  $P$ -null events in  $(\Omega, \mathcal{F}, P)$ . We consider a market comprising  $N + 1$  assets traded continuously on the interval  $[0, T]$ , namely a *bond* with price  $\{S_0(t)\}$  given by

$$(2.2) \quad dS_0(t) = r(t)S_0(t) dt, \quad 0 \leq t \leq T, \quad S_0(0) = 1,$$

and  $N$  *stocks* with prices  $\{S_n(t)\}$ ,  $n = 1, 2, \dots, N$ , given by

$$(2.3) \quad dS_n(t) = S_n(t) \left[ b_n(t) dt + \sum_{m=1}^N \sigma_{nm}(t) dW_m(t) \right], \quad 0 \leq t \leq T,$$

the initial values  $S_n(0)$  being given strictly positive constants. We shall always postulate

**Condition 2.1.** In (2.2) and (2.3) the *interest rate*  $\{r(t)\}$ , the entries  $\{b_n(t)\}$  of the  $\mathbb{R}^N$ -valued process  $\{b(t)\}$  of *mean rates of return on stocks*, and the entries  $\{\sigma_{nm}(t)\}$  of the  $N \times N$  matrix-valued *volatility process*  $\{\sigma(t)\}$  are *uniformly bounded* and  $\{\mathcal{F}_t\}$ -progressively measurable scalar processes on  $\Omega \times [0, T]$ , and  $\{r(t)\}$  is non-negative. There is a constant  $\kappa \in (0, \infty)$  such that  $z' \sigma(\omega, t) \sigma'(\omega, t) z \geq \kappa \|z\|^2$  for all  $(z, \omega, t) \in \mathbb{R}^N \times \Omega \times [0, T]$ .

**Remark 2.2.** In view of Condition 2.1 and Karatzas and Shreve ([10], 5.8.1, p.372), there exists a constant  $\kappa_1 \in (0, \infty)$  such that  $\max\{\|(\sigma(\omega, t))^{-1}z\|, \|(\sigma'(\omega, t))^{-1}z\|\} \leq \kappa_1 \|z\|$  for all  $(z, \omega, t) \in \mathbb{R}^N \times \Omega \times [0, T]$ . This bound will often be used.

**Remark 2.3.** Define the usual *market price of risk*  $\theta(t) := (\sigma(t))^{-1}[b(t) - r(t)\mathbf{1}]$ , in which  $\mathbf{1} \in \mathbb{R}^N$  has all unit entries. From Condition 2.1 and Remark 2.2, we see that  $\{\theta(t)\}$  is uniformly bounded on  $\Omega \times [0, T]$ .

Given some  $x_0 \in \mathbb{R}$ , and some  $\{\mathcal{F}_t\}$ -progressively measurable process  $\pi : \Omega \times [0, T] \rightarrow \mathbb{R}^N$  satisfying  $\int_0^T \|\pi(t)\|^2 dt < \infty$  a.s., it follows that there exists a scalar-valued, continuous, and  $\{\mathcal{F}_t\}$ -progressively measurable process  $\{X^\pi(t), t \in [0, T]\}$  such that

$$(2.4) \quad dX^\pi(t) = \{r(t)X^\pi(t) + \pi'(t)\sigma(t)\theta(t)\} dt + \pi'(t)\sigma(t) dW(t), \quad X^\pi(0) = x_0,$$

which is unique (to within indistinguishability) and given by

$$(2.5) \quad X^\pi(t) = S_0(t) \left\{ x_0 + \int_0^t S_0^{-1}(\tau) \pi'(\tau) \sigma(\tau) \theta(\tau) d\tau + \int_0^t S_0^{-1}(\tau) \pi'(\tau) \sigma(\tau) dW(\tau) \right\}.$$

From now on we consider a small investor who trades in the market following a *self-funded strategy* from a given initial wealth  $x_0 \in (0, \infty)$ . If  $\pi_n(t)$ , the  $n$ -th entry of the  $\mathbb{R}^N$ -valued vector  $\pi(t)$ , is interpreted as the *dollar amount* invested in the stock with price  $S_n(t)$ ,  $n = 1, 2, \dots, N$ , then it follows from (2.2), (2.3) and Remark 1.3.3 of Karatzas and Shreve ([9], p.10), that  $X^\pi(t)$  gives the investor's wealth at instant  $t \in [0, T]$ .

### 3 A Class of Square-Integrable Itô Processes

We formulate the optimization problem in the following section, but must first define a class of square-integrable Itô processes which will be essential in all later developments.

Write  $\mathcal{F}^*$  for the  $\{\mathcal{F}_t\}$ -progressively measurable  $\sigma$ -algebra on  $\Omega \times [0, T]$ . The measure space  $(\Omega \times [0, T], \mathcal{F}^*, (P \otimes \lambda))$ , where  $\lambda$  stands for the Lebesgue measure (on the Borel  $\sigma$ -algebra on  $[0, T]$ ), is used extensively, and the qualifier “a.e.” always refers to the measure  $(P \otimes \lambda)$  on  $\Omega \times [0, T]$ . For example, if  $\pi$  is an  $\mathbb{R}^N$ -valued  $\mathcal{F}^*$ -measurable mapping on  $\Omega \times [0, T]$  and  $K \subset \mathbb{R}^N$ , then  $\pi(t) \in K$  a.e. means that  $\pi(\omega, t) \in K$  for  $(P \otimes \lambda)$ -almost all  $(\omega, t)$ . Similarly, the qualifier “a.s.” is always with reference to the probability  $P$  on  $\mathcal{F}$ . For any mapping  $\xi$  on  $\Omega \times [0, T]$  with values in some Euclidean space (the dimensionality of which will be clear from the context) we write  $\xi \in \mathcal{F}^*$  to indicate that  $\xi$  is  $\mathcal{F}^*$ -measurable. Motivated by Bismut ([2], p.386, p.390), put

$$\begin{aligned} L_{21} &:= \left\{ v : \Omega \times [0, T] \rightarrow \mathbb{R} \mid v \in \mathcal{F}^* \text{ and } \mathbb{E} \left( \int_0^T |v(t)| dt \right)^2 < \infty \right\}, \\ L_{22} &:= \left\{ \xi : \Omega \times [0, T] \rightarrow \mathbb{R}^N \mid \xi \in \mathcal{F}^* \text{ and } \mathbb{E} \int_0^T \|\xi(t)\|^2 dt < \infty \right\}, \\ \mathbb{B} &:= \mathbb{R} \times L_{21} \times L_{22}, \end{aligned}$$

in which  $\|\xi\|$  denotes the usual Euclidean length of  $\xi \in \mathbb{R}^N$ . Write  $X \in \mathbb{B}$  to indicate that  $\{(X(t), \mathcal{F}_t), t \in [0, T]\}$  is a continuous semimartingale of the form

$$(3.6) \quad X(t) = X_0 + \int_0^t \dot{X}(\tau) d\tau + \int_0^t \Lambda'_X(\tau) dW(\tau),$$

for some  $(X_0, \dot{X}, \Lambda_X) \in \mathbb{B}$ , and write  $X \equiv (X_0, \dot{X}, \Lambda_X)$  to indicate that (3.6) holds. In the expansion (3.6) it is clear that the integrands  $\dot{X}$  and  $\Lambda_X$  are uniquely determined a.e. on  $\Omega \times [0, T]$ . The set  $\mathbb{B}$  is essentially the collection of all *square-integrable* Itô processes with respect to the Brownian motion  $\{W(t)\}$ . From Doob's  $L^2$ -inequality we immediately have

$$(3.7) \quad \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X(t)|^2 \right] < \infty, \quad \text{for each } X \in \mathbb{B}.$$

Note from (2.4) that  $X^\pi$ , for any given  $\mathbb{R}^N$ -valued  $\pi \in \mathcal{F}^*$  for which the stochastic integration is defined, is an Itô process with respect to the Brownian motion  $\{W(t)\}$ . The next result gives conditions on  $\pi$  for membership of  $X^\pi$  in  $\mathbb{B}$ . The proof is elementary and is omitted.

**Proposition 3.1.** *Assume Condition 2.1 and suppose that  $\pi : \Omega \times [0, T] \rightarrow \mathbb{R}^N$  is  $\mathcal{F}^*$ -measurable and  $\int_0^T \|\pi(t)\|^2 dt < \infty$  a.s. Then  $X^\pi \in \mathbb{B}$  if and only if  $\pi \in L_{22}$ .*

## 4 The Optimization Problem

In order to formulate the optimization problem we postulate the following basic ingredients:

**Condition 4.1.** We are given a closed convex set  $K \subset \mathbb{R}^N$  with  $0 \in K$ , and an  $\mathcal{F}_T$ -measurable random variable  $a$  on  $(\Omega, \mathcal{F}, P)$  such that  $0 < \inf_{\omega \in \Omega} a(\omega) \leq \sup_{\omega \in \Omega} a(\omega) < \infty$ .

**Condition 4.2.** We are given a number  $d \in \mathbb{R}$ , together with  $\mathcal{F}_T$ -measurable square-integrable random variables  $c_0$  and  $c_1$  on  $(\Omega, \mathcal{F}, P)$ .

Put

$$(4.8) \quad \mathcal{A} := \{\pi \in L_{22} \mid \pi(t) \in K \text{ a.e.}\},$$

$$(4.9) \quad \hat{J}(\omega, x) := \frac{1}{2} [a(\omega)x^2 + 2c_0(\omega)x], \quad (\omega, x) \in \Omega \times \mathbb{R},$$

$$(4.10) \quad G(\pi) := \mathbb{E}[c_1 X^\pi(T)] - d, \quad \pi \in L_{22},$$

$$(4.11) \quad \hat{v} := \inf_{\substack{\pi \in \mathcal{A} \\ G(\pi)=0}} \mathbb{E}[\hat{J}(X^\pi(T))].$$

We regard  $\mathcal{A}$  as the set of *admissible portfolios*, while  $G(\pi) = 0$  represents a constraint on the terminal wealth. The problem of interest, which we denote by  $(\hat{\mathcal{P}})$ , is

$$(4.12) \quad \text{determine some } \hat{\pi} \in \mathcal{A} \text{ such that } G(\hat{\pi}) = 0 \text{ and } \hat{v} = \mathbb{E}[\hat{J}(X^{\hat{\pi}}(T))],$$

in the sense of demonstrating existence of  $\hat{\pi}$  and characterizing its dependence on the market parameters  $\{r(t)\}$ ,  $\{b(t)\}$ ,  $\{\sigma(t)\}$  and the information filtration  $\{\mathcal{F}_t\}$  available to the investor. We must also postulate  $0 \in \{G(\pi) \mid \pi \in \mathcal{A}\}$ , for otherwise the constraints on  $\pi$  in (4.11) are mutually contradictory and we will have  $\hat{v} = +\infty$ , rendering the problem (4.12) meaningless. In fact, we impose the following *constraint qualification*:

**Condition 4.3.** The constant  $d$ , the set  $K$  and the random variable  $c_1$  are such that the set  $\{G(\pi) \mid \pi \in \mathcal{A}\} \subset \mathbb{R}$  has a nonempty interior which includes 0 (see Remark 4.6).

**Example 4.4.**  $K = \mathbb{R}^N$  in (4.8) corresponds to the case of no constraints on the portfolio. On the other hand,  $K = [0, \infty)^N$  represents a short-selling prohibition on stocks, while the constraint set  $K = \{\pi \in [0, \infty)^N \mid \pi_{n+1} = \dots = \pi_N = 0\}$  represents the same prohibition, but in an incomplete market, for which the dimension  $N$  of the Brownian motion  $\{W(t)\}$  exceeds the number of stocks  $n$  available to the investor. Other examples can be similarly formulated.

**Remark 4.5.** The most important case of problem (4.12) occurs when  $a = 2$ ,  $c_0 = 0$ , and  $c_1 = 1$ , for then  $\mathbb{E}[\hat{J}(X^\pi(T))] - d^2 = \text{Var}(X^\pi(T))$  (the *variance* of the terminal wealth) when  $G(\pi) = 0$ . Now problem (4.12) amounts to minimizing this variance subject to the terminal wealth constraint  $\mathbb{E}[X^\pi(T)] = d$ , together with the portfolio constraint  $\pi \in \mathcal{A}$ . This is the problem of *constrained mean-variance portfolio selection*.

**Remark 4.6.** We show that Condition 4.3 holds in the case where  $c_1 \equiv 1$  in (4.10), the market model is “interesting” in the sense that  $\mathbb{E}[X^{\tilde{\pi}}(T)] > \mathbb{E}[x_0 S_0(T)]$  for some portfolio  $\tilde{\pi} \in \mathcal{A}$  (the problem (4.12) is pointless otherwise, since the best expected terminal wealth would be attained by just investing the entire fortune risk-free in the money-market), and the expected terminal wealth  $d$  in the constraint (4.10) is “reasonable” in a sense to be specified. From (4.8), convexity of  $K \subset \mathbb{R}^N$  (see Condition 4.1), and (2.5), it follows that  $\mathcal{R} := \{\mathbb{E}[X^\pi(T)] \mid \pi \in \mathcal{A}\} \subset \mathbb{R}$  is convex, hence an interval. Thus, the interior of  $\mathcal{R}$  is identical to  $\mathcal{I} := (\inf_{\pi \in \mathcal{A}} \mathbb{E}[X^\pi(T)], \sup_{\pi \in \mathcal{A}} \mathbb{E}[X^\pi(T)])$ , which is non-empty since  $0 \in K$  and the market model is “interesting”. Now it follows from (4.10) that Condition 4.3 holds provided that  $d$  is specified in the “reasonable range”  $d \in \mathcal{I}$ .

## 5 Partially Constrained Problem

Here we establish duality relations for a *partially constrained* optimization problem in which the terminal wealth condition  $G(\pi) = 0$  of (4.12) is discarded. In Section 6 these relations will then be used to establish existence for the fully constrained problem (4.12). We postulate

**Condition 5.1.** We are given a constant  $q \in \mathbb{R}$ , along with an  $\mathcal{F}_T$ -measurable square-integrable random variable  $c$  on  $(\Omega, \mathcal{F}, P)$ .

Recalling the random variable  $a$  and convex set  $K$  in Condition 4.1, the set  $\mathcal{A}$  in (4.8), put

$$(5.13) \quad \vartheta_{c,q} := \inf_{\pi \in \mathcal{A}} \mathbb{E}[J(X^\pi(T))], \quad \text{for} \quad J(\omega, x) := \frac{1}{2} [a(\omega)x^2 + 2c(\omega)x] + q.$$

The *partially constrained optimization problem*, which we denote by  $(\mathcal{P}_{c,q})$ , is:

$$(5.14) \quad \text{determine some } \bar{\pi} \in \mathcal{A} \text{ such that } \vartheta_{c,q} = \mathbb{E}[J(X^{\bar{\pi}}(T))].$$

**Remark 5.2.** We distinguish between the coefficients  $c_0$  and  $c$  in the linear terms of  $\hat{J}$  and  $J$  respectively (recall (4.9) and (5.13)), because, in Section 6, these coefficients will play somewhat different roles. It follows at once from the quadratic form of  $x \mapsto J(\omega, x)$  in (5.13), Conditions 4.1 and 5.1, and Proposition 3.1, that  $-\infty < \vartheta_{c,q} < +\infty$ .

### 5.1 Synthesis of a Dual Problem and Optimality Relations

**Remark 5.3.** Our goals are to reformulate problem (5.14) as a “primal” optimization problem over the set  $\mathbb{B}$  of Section 3 (see (5.23)), synthesize a “dual” optimization problem and corresponding Euler-Lagrange-Hamilton optimality relations (Proposition 5.8) by following an algorithmic approach motivated by Bismut [2], and establish existence for the dual problem (Proposition 5.12).

**Step I:** From Proposition 3.1 we know that  $X^\pi \in \mathbb{B}$  for each admissible  $\pi \in \mathcal{A}$ . We therefore express the value (5.13) as the infimum over the set  $\mathbb{B}$  of some appropriate mapping  $\Phi : \mathbb{B} \rightarrow (-\infty, \infty]$ , by introducing “penalty terms” on  $\mathbb{B}$ , which account for the initial-wealth

constraint  $X(0) = x_0$ , the portfolio constraint  $\pi(t) \in K$  a.e., and the “dynamical constraint” implicit in (2.4); these will be defined to give zero penalty when the constraints hold and “infinite” penalty otherwise. For each  $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{B}$  (recall Section 3), put

$$(5.15) \quad \mathcal{U}(X) := \{ \pi \in \mathcal{A} \mid \dot{X}(t) = r(t)X(t) + \pi'(t)\sigma(t)\theta(t) \text{ and } \Lambda_X(t) = \sigma'(t)\pi(t) \text{ a.e.} \}.$$

We then see the following: for each  $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{B}$  we have  $X(t) = X^\pi(t)$  a.e. for some  $\pi \in \mathcal{A}$  if and only if  $X_0 = x_0$  and  $\mathcal{U}(X) \neq \emptyset$ ; from this equivalence and (5.13), we obtain

$$(5.16) \quad \vartheta_{c,q} = \inf_{\substack{X \in \mathbb{B} \\ X_0 = x_0 \\ \mathcal{U}(X) \neq \emptyset}} \mathbb{E}[J(X(T))].$$

Now define a penalty function on  $\mathbb{B}$  giving zero penalty when the constraint  $\mathcal{U}(X) \neq \emptyset$  is satisfied, and infinite penalty otherwise. From Remark 2.2, for each  $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{B}$

$$(5.17) \quad \mathcal{U}(X) \neq \emptyset \iff \dot{X}(t) = r(t)X(t) + \Lambda'_X(t)\theta(t) \text{ and } [\sigma'(t)]^{-1}\Lambda_X(t) \in K \text{ a.e.}$$

Motivated by (5.17), define the mapping  $L : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \{0, \infty\}$  by

$$(5.18) \quad L(\omega, t, x, v, \xi) = \begin{cases} 0 & \text{if } v = r(\omega, t)x + \xi'\theta(\omega, t) \text{ and } [\sigma'(\omega, t)]^{-1}\xi \in K; \\ \infty & \text{otherwise.} \end{cases}$$

It is clear that  $L(t, X(t), \dot{X}(t), \Lambda_X(t))$  is  $\mathcal{F}^*$ -measurable, and, in view of (5.17) and (5.18),

$$(5.19) \quad \mathbb{E} \int_0^T L(t, X(t), \dot{X}(t), \Lambda_X(t)) dt = \begin{cases} 0 & \text{if } \mathcal{U}(X) \neq \emptyset; \\ \infty & \text{otherwise,} \end{cases}$$

for each  $X \in \mathbb{B}$ . We see that (5.19) establishes a penalty for the constraint  $\mathcal{U}(X) \neq \emptyset$  in (5.16). As for the initial-wealth constraint  $X_0 = x_0$  in (5.16), put

$$(5.20) \quad l_0(x) := \begin{cases} 0 & \text{if } x = x_0; \\ \infty & \text{otherwise,} \end{cases}$$

for each  $x \in \mathbb{R}$ . Now define

$$(5.21) \quad \Phi(X) := l_0(X_0) + \mathbb{E}[l_T(X(T))] + \mathbb{E} \int_0^T L(t, X(t), \dot{X}(t), \Lambda_X(t)) dt,$$

for each  $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{B}$ , where, for consistency of notation, we put

$$(5.22) \quad l_T(\omega, x) := J(\omega, x), \quad (\omega, x) \in \Omega \times \mathbb{R}.$$

Upon combining (5.16), (5.19), (5.20), (5.21), and (5.22), we obtain

$$(5.23) \quad \vartheta_{c,q} = \inf_{X \in \mathbb{B}} \Phi(X).$$

**Remark 5.4.** From (5.19) and (5.20) it is clear that  $\Phi(X)$  exists in  $(-\infty, \infty]$  for each  $X \in \mathbb{B}$ .

**Step II:** In this step we synthesize a “cost” functional  $\Psi : \mathbb{B} \rightarrow (-\infty, \infty]$  for an optimization problem which is dual to the primal problem (5.23). To this end define the convex conjugate functions:

$$\begin{aligned}
(5.24) \quad m_0(y) &:= l_0^*(y) := \sup_{x \in \mathbb{R}} \{xy - l_0(x)\} \\
m_T(\omega, y) &:= l_T^*(\omega, -y) := \sup_{x \in \mathbb{R}} \{x(-y) - l_T(\omega, x)\} \\
M(\omega, t, y, s, \gamma) &:= L^*(\omega, t, s, y, \gamma) := \sup_{\substack{x, v \in \mathbb{R} \\ \xi \in \mathbb{R}^N}} \{xs + vy + \xi' \gamma - L(\omega, t, x, v, \xi)\},
\end{aligned}$$

for each  $y \in \mathbb{R}$ ,  $s \in \mathbb{R}$ ,  $\gamma \in \mathbb{R}^N$ ,  $\omega \in \Omega$  and  $t \in [0, T]$ . From (5.20), (5.22), and (5.18), it is easy to explicitly calculate these conjugates, namely for each  $(\omega, y) \in \Omega \times \mathbb{R}$  we have

$$(5.25) \quad m_0(y) = x_0 y, \quad m_T(\omega, y) = \frac{(y + c(\omega))^2}{2a(\omega)} - q,$$

$$(5.26) \quad M(\omega, t, y, s, \gamma) = \begin{cases} \delta(-\sigma(t) [\theta(t)y + \gamma]) & \text{if } s + r(t)y = 0, \\ \infty & \text{otherwise,} \end{cases}$$

where  $\delta(\cdot)$  is the *support functional* of the set  $-K$ , defined by

$$(5.27) \quad \delta(z) := \sup_{\pi \in K} \{-\pi' z\}, \quad z \in \mathbb{R}^N.$$

For each  $Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{B}$ , define

$$(5.28) \quad \Psi(Y) := m_0(Y_0) + \mathbb{E}[m_T(Y(T))] + \mathbb{E} \int_0^T M(t, Y(t), \dot{Y}(t), \Lambda_Y(t)) dt.$$

**Remark 5.5.** Since  $\delta(\cdot)$  is lower semicontinuous on  $\mathbb{R}^N$ , it is easily seen from (5.26) that  $M(t, Y(t), \dot{Y}(t), \Lambda_Y(t))$  is  $\mathcal{F}^*$ -measurable for each  $Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{B}$ , and it is clear that  $\Psi(Y)$  exists in  $(-\infty, \infty]$  for each  $Y \in \mathbb{B}$ .

Next we require the following result from Bismut ([2], Proposition I-1, p.387):

**Proposition 5.6.** *For members  $X \equiv (X_0, \dot{X}, \Lambda_X)$  and  $Y \equiv (Y_0, \dot{Y}, \Lambda_Y)$  of the set  $\mathbb{B}$ , define*

$$\mathbb{M}(X, Y)(t) := X(t)Y(t) - X_0Y_0 - \int_0^t \{X(\tau)\dot{Y}(\tau) + \dot{X}(\tau)Y(\tau) + \Lambda_X'(\tau)\Lambda_Y(\tau)\} d\tau, \quad t \in [0, T].$$

*Then  $\{(\mathbb{M}(X, Y)(t), \mathcal{F}_t), t \in [0, T]\}$  is a continuous martingale with  $\mathbb{M}(X, Y)(0) = 0$ .*

**Proposition 5.7.** *Assume Conditions 2.1, 4.1 and 5.1. Then the functions  $\Phi$  and  $\Psi$  given by (5.21) and (5.28) are well-defined, with values in  $(-\infty, \infty]$  for each  $X \in \mathbb{B}$ ,  $Y \in \mathbb{B}$ , and*

$$(5.29) \quad \Phi(X) + \Psi(Y) \geq 0, \quad (X, Y) \in \mathbb{B} \times \mathbb{B}.$$

Moreover, for arbitrary  $\bar{X} \equiv (\bar{X}_0, \dot{\bar{X}}, \Lambda_{\bar{X}}) \in \mathbb{B}$  and  $\bar{Y} \equiv (\bar{Y}_0, \dot{\bar{Y}}, \Lambda_{\bar{Y}}) \in \mathbb{B}$ , we have the equality  $\Phi(\bar{X}) + \Psi(\bar{Y}) = 0$  if and only if each of the following conditions hold:

$$(5.30) \quad \begin{cases} (1) & l_0(\bar{X}_0) + m_0(\bar{Y}_0) = \bar{X}_0 \bar{Y}_0, \\ (2) & l_T(\bar{X}(T)) + m_T(\bar{Y}(T)) = -\bar{X}(T)\bar{Y}(T) \quad a.s. \\ (3) & L(t, \bar{X}(t), \dot{\bar{X}}(t), \Lambda_{\bar{X}}(t)) + M(t, \bar{Y}(t), \dot{\bar{Y}}(t), \Lambda_{\bar{Y}}(t)) \\ & = \bar{X}(t)\dot{\bar{Y}}(t) + \dot{\bar{X}}(t)\bar{Y}(t) + \Lambda'_{\bar{X}}(t)\Lambda_{\bar{Y}}(t) \quad a.e. \end{cases}$$

*Proof.* Fix  $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{B}$  and  $Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{B}$ . To establish (5.29), observe from the convex conjugates in (5.24) that, for each  $(\omega, t) \in \Omega \times [0, T]$ ,

$$(5.31) \quad \begin{aligned} l_0(X_0) + m_0(Y_0) &\geq X_0 Y_0 \\ l_T(X(T)) + m_T(Y(T)) &\geq -X(T)Y(T) \\ L(t, X(t), \dot{X}(t), \Lambda_X(t)) + M(t, Y(t), \dot{Y}(t), \Lambda_Y(t)) &\geq X(t)\dot{Y}(t) + \dot{X}(t)Y(t) + \Lambda'_X(t)\Lambda_Y(t). \end{aligned}$$

By (5.21), (5.28), (5.31), and the definition of  $\mathbb{M}(X, Y)$  (see statement of Proposition 5.6),

$$(5.32) \quad \begin{aligned} \Phi(X) + \Psi(Y) &= l_0(X_0) + m_0(Y_0) + \mathbb{E}[l_T(X(T)) + m_T(Y(T))] \\ &+ \mathbb{E} \int_0^T \{L(t, X(t), \dot{X}(t), \Lambda_X(t)) + M(t, Y(t), \dot{Y}(t), \Lambda_Y(t))\} dt \\ &\geq X_0 Y_0 + \mathbb{E}[-X(T)Y(T)] + \mathbb{E} \int_0^T \{X(t)\dot{Y}(t) + \dot{X}(t)Y(t) + \Lambda'_X(t)\Lambda_Y(t)\} dt \\ &= \mathbb{E}[-\mathbb{M}(X, Y)(T)]. \end{aligned}$$

Moreover,  $\mathbb{E}[-\mathbb{M}(X, Y)(T)] = 0$  (from Proposition 5.6), which establishes (5.29). Next, for some  $(\bar{X}, \bar{Y}) \in \mathbb{B} \times \mathbb{B}$ , the equivalence between  $\Phi(\bar{X}) + \Psi(\bar{Y}) = 0$  and (5.30)(1) - (3) follows at once from (5.32) and (5.31), and the fact that  $\mathbb{E}[-\mathbb{M}(X, Y)(T)] = 0$ .  $\square$

We next refine Proposition 5.7 to obtain the following Proposition 5.8. This gives a set of optimality relations which will be essential in constructing an optimal portfolio. Put

$$(5.33) \quad \Theta_Y(t) := -\sigma(t) [\theta(t)Y(t) + \Lambda_Y(t)], \quad \text{for each } Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{B}.$$

**Proposition 5.8.** *Assume Conditions 2.1, 4.1 and 5.1. Then, for arbitrary  $(\bar{X}, \bar{Y}) \in \mathbb{B} \times \mathbb{B}$ , we have*

$$(5.34) \quad \Phi(\bar{X}) = \vartheta_{c,q} = \sup_{Y \in \mathbb{B}} [-\Psi(Y)] = -\Psi(\bar{Y}),$$

if and only if

$$(5.35) \quad \begin{cases} (1') & \bar{X}_0 = x_0 \\ (2') & \bar{X}(T) = -\frac{(\bar{Y}(T) + c)}{a} \quad a.s. \\ (3') & \dot{\bar{Y}}(t) + r(t)\bar{Y}(t) = 0 \quad a.e. \\ (4') & \bar{\pi} \in \mathcal{U}(\bar{X}) \quad \text{and} \quad \delta(\Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) = 0 \quad a.e. \\ & \text{for } \bar{\pi}(t) := [\sigma'(t)]^{-1}\Lambda_{\bar{X}}(t). \end{cases}$$

*Proof.* From (5.18) and (5.26), for arbitrary  $(\omega, t) \in \Omega \times [0, T]$ , and  $(x, v, \xi), (y, s, \gamma) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ , we have the equivalence

$$(5.36) \quad \begin{aligned} L(\omega, t, x, v, \xi) + M(\omega, t, y, s, \gamma) &= xs + vy + \xi' \gamma \\ \iff v &= r(t)x + \xi' \theta(t), \quad [\sigma'(t)]^{-1} \xi \in K, \quad s + r(t)y = 0 \\ \text{and } \delta(-\sigma(t) [\theta(t)y + \gamma]) - \xi' \sigma^{-1}(t) \sigma(t) [\theta(t)y + \gamma] &= 0. \end{aligned}$$

Moreover, from (5.13), (5.20), (5.22), and (5.25), for arbitrary  $x, y \in \mathbb{R}$ ,  $\omega \in \Omega$ , we find that  $l_0(x) + m_0(y) = xy$  if and only if  $x = x_0$ , and  $l_T(\omega, x) + m_T(\omega, y) = -xy$  if and only if  $x = -(y + c(\omega))/a(\omega)$ . From these equivalences, with (5.36), (5.15), and (5.17), we obtain the following: for arbitrary  $(\bar{X}, \bar{Y}) \in \mathbb{B} \times \mathbb{B}$ , (5.35)(1') – (4') hold if and only if (5.30)(1) – (3) hold. But, in view of (5.23) and the universal inequality (5.29), we see that the statement  $\Phi(\bar{X}) + \Psi(\bar{Y}) = 0$  is equivalent to (5.34), hence (5.34) is equivalent to items (5.30)(1) – (3). The result follows from this equivalence, together with the equivalence of (5.35)(1') – (4') and (5.30)(1) – (3), which we have already noted.  $\square$

**Remark 5.9.** It follows from Proposition 5.8 that the solution of problem  $(\mathcal{P}_{c,q})$  in (5.14) reduces to construction of a pair  $(\bar{X}, \bar{Y}) \in \mathbb{B} \times \mathbb{B}$  which satisfies the optimality relations (5.35)(1')–(4'), for then the optimal portfolio  $\bar{\pi}$  is defined by (5.35)(4'). Motivated by the third equality of (5.34), in the remainder of this subsection we show that there exists a solution to the problem of minimizing  $\Psi(\cdot)$  on  $\mathbb{B}$ , henceforth referred to as the *dual problem*. Define

$$(5.37) \quad \mathbb{B}_1 := \{ Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{B} \mid \dot{Y}(t) = -r(t)Y(t) \text{ a.e.} \},$$

and observe from (5.28) and (5.26), that  $\Psi$  necessarily takes the value  $+\infty$  on  $\mathbb{B} - \mathbb{B}_1$ . Then

$$(5.38) \quad \inf_{Y \in \mathbb{B}} \Psi(Y) = \inf_{Y \in \mathbb{B}_1} \Psi(Y),$$

so that the dual problem reduces to minimization of  $\Psi(\cdot)$  over  $\mathbb{B}_1$ . For each  $t \in [0, T]$  put

$$(5.39) \quad \beta(t) := \exp \left[ - \int_0^t r(\tau) d\tau \right], \quad \mathfrak{J}(\gamma)(t) := \int_0^t \beta^{-1}(\tau) \gamma'(\tau) dW(\tau), \quad \gamma \in L_{22},$$

$$(5.40) \quad \Xi(y, \gamma)(t) := \beta(t)[y + \mathfrak{J}(\gamma)(t)], \quad t \in [0, T], \quad (y, \gamma) \in \mathbb{R} \times L_{22}.$$

**Remark 5.10.** From Section 3, we see that  $Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{B}_1$  satisfies the relation

$$(5.41) \quad Y(t) = Y_0 - \int_0^t r(\tau)Y(\tau) d\tau + \int_0^t \Lambda_Y'(\tau) dW(\tau).$$

Then it follows from Itô's formula and Doob's  $L^2$ -inequality that  $\Xi(\cdot) : \mathbb{R} \times L_{22} \rightarrow \mathbb{B}_1$  is a *linear bijection*, and, when  $Y := \Xi(y, \gamma)$  for some  $(y, \gamma) \in \mathbb{R} \times L_{22}$ , then (recalling (5.33))

$$(5.42) \quad Y_0 = y, \quad \Lambda_Y(t) = \gamma(t), \quad \Theta_Y(t) = -\sigma(t)[\theta(t)Y(t) + \gamma(t)], \text{ a.e.}$$

We then obtain

$$(5.43) \quad \inf_{(y,\gamma) \in \mathbb{R} \times L_{22}} \tilde{\Psi}(y, \gamma) = \inf_{Y \in \mathbb{B}_1} \Psi(Y), \quad \text{for } \tilde{\Psi}(y, \gamma) := \Psi(\Xi(y, \gamma)), \quad (y, \gamma) \in \mathbb{R} \times L_{22}.$$

Moreover,  $M(t, Y(t), \dot{Y}(t), \Lambda_Y(t)) = \delta(\Theta_Y(t))$  a.e. for each  $Y \in \mathbb{B}_1$  (see (5.37) and (5.26)), thus from (5.25) and (5.28), for each  $(y, \gamma) \in \mathbb{R} \times L_{22}$  (with  $Y := \Xi(y, \gamma)$ ) we get

$$(5.44) \quad \tilde{\Psi}(y, \gamma) = x_0 y + \mathbb{E} \left[ \frac{(Y(T) + c)^2}{2a} \right] + \mathbb{E} \int_0^T \delta(\Theta_Y(t)) dt - q.$$

**Remark 5.11.** Define norm  $\|\gamma\|_{L_{22}}$  on the real vector space  $L_{22}$  by  $\|\gamma\|_{L_{22}}^2 := \mathbb{E} \int_0^T \|\gamma(t)\|^2 dt$ , and define the norm  $\|(y, \gamma)\|$  on the real vector space  $\mathbb{R} \times L_{22}$  by  $\|(y, \gamma)\|^2 := |y|^2 + \|\gamma\|_{L_{22}}^2$ . With this norm  $\mathbb{R} \times L_{22}$  is a *reflexive* Banach space.

**Proposition 5.12.** *Suppose Conditions 2.1, 4.1, and 5.1. Then*

$$(5.45) \quad \inf_{(y,\gamma) \in \mathbb{R} \times L_{22}} \tilde{\Psi}(y, \gamma) = \tilde{\Psi}(\bar{y}, \bar{\gamma}) \in \mathbb{R}, \quad \text{for some } (\bar{y}, \bar{\gamma}) \in \mathbb{R} \times L_{22}.$$

*Proof.* It is immediate from (5.44), (5.42) and (5.27), that  $\tilde{\Psi}$  is convex on  $\mathbb{R} \times L_{22}$ . From Conditions 4.1 and 5.1 we get  $\tilde{\Psi}(y, \gamma) \geq x_0 y - q > -\infty$  for each  $(y, \gamma) \in \mathbb{R} \times L_{22}$  as well as  $\tilde{\Psi}(0, 0) = \mathbb{E}[c^2/(2a)] - q < \infty$ , hence  $\tilde{\Psi}$  is proper. A routine argument using Fatou's lemma, with the nonnegativity and lower-semicontinuity of  $\delta(\cdot)$ , proves that  $\tilde{\Psi}$  is lower-semicontinuous on  $\mathbb{R} \times L_{22}$  (with respect to the norm  $\|(y, \gamma)\|$  in Remark 5.11). We next show that  $\tilde{\Psi}$  is coercive (i.e.  $\tilde{\Psi}(y, \gamma) \rightarrow \infty$  when  $\|(y, \gamma)\| \rightarrow \infty$ ). From Conditions 2.1 and 5.1 we know that  $\beta^{-1}(T)c$  is  $\mathcal{F}_T$ -measurable square-integrable, thus  $\beta^{-1}(T)c = \tilde{y} + \int_0^T \eta'(\tau) dW(\tau)$ , for  $\tilde{y} = \mathbb{E}[\beta^{-1}(T)c]$  and some  $\eta \in L_{22}$  (see Theorem 3.4.15 of [10]); thus, from (5.39) and (5.40), we obtain  $\Xi(y, \gamma)(T) + c = \Xi(y + \tilde{y}, \gamma + \beta\eta)(T)$ ,  $(y, \gamma) \in \mathbb{R} \times L_{22}$ . Thus, for showing coercivity, with no loss of generality we can and shall take  $c \equiv 0$  in (5.44). In view of the nonrandom and strictly positive uniform lower bounds on  $\beta(T)$  and  $1/(2a)$  (see Conditions 2.1 and 4.1), and the Itô isometry, we find  $\mathbb{E}[(\Xi(y, \gamma)(T))^2/(2a)] \rightarrow \infty$  as  $\|(y, \gamma)\| \rightarrow \infty$ . Coercivity of  $\tilde{\Psi}$  follows from this, together with (5.44) and the non-negativity of  $\delta(\cdot)$ . Existence of a pair  $(\bar{y}, \bar{\gamma}) \in \mathbb{R} \times L_{22}$  which satisfies (5.45) follows from this, together with Remark 5.11 and Proposition II-1.2 of Ekeland and Témam ([7], p.35).  $\square$

**Remark 5.13.** Define  $\bar{Y} := \Xi(\bar{y}, \bar{\gamma})$ , for  $(\bar{y}, \bar{\gamma}) \in \mathbb{R} \times L_{22}$  given by Proposition 5.12. From Remark 5.10 we have  $\bar{Y} \in \mathbb{B}_1 \subset \mathbb{B}$ . Upon combining (5.45), (5.43) and (5.38), we get  $\Psi(\bar{Y}) = \inf_{Y \in \mathbb{B}} \Psi(Y)$ , thus  $\bar{Y}$  solves the dual problem of Remark 5.9.

## 5.2 Construction of the Optimal Portfolio

In the present subsection we shall construct some  $\bar{X} \in \mathbb{B}$  such that the pair  $(\bar{X}, \bar{Y})$ , with  $\bar{Y}$  given by Remark 5.13, satisfies (5.35). To this end, consider the *state price density process* given by (recall (5.39) for  $\beta$ )

$$(5.46) \quad H(t) := \beta(t) \mathcal{E}(-\theta' \bullet W)(t).$$

**Remark 5.14.** In (5.46) the notation  $\mathcal{E}(M)(t) := \exp[M(t) - (1/2)\langle M \rangle(t)]$  indicates the exponential of a continuous local martingale  $M$ , while  $\bullet$  denotes stochastic integration.

**Remark 5.15.** Fix an arbitrary  $p \in \mathbb{R}$ ; since  $\theta$  is uniformly-bounded (recall Remark 2.3), it follows that  $\{\mathcal{E}(-p\theta' \bullet W)(t)\}$  is a continuous  $\{\mathcal{F}_t\}$ -martingale (by the Novikov criterion, Corollary 3.5.14 of [10], p.199), and then it easily follows from the uniform-boundedness of  $\theta$  and  $r$  (Condition 2.1) and Doob's maximal  $L^2$ -inequality that  $\mathbb{E}[\sup_{t \in [0, T]} |H(t)|^p] < \infty$ , for each  $p \in \mathbb{R}$ . Thus  $H$  defined by (5.46) is a member of  $\mathbb{B}$  (take  $p = 2$ ).

Now  $\{(H(t)X^\pi(t), \mathcal{F}_t), t \in [0, T]\}$  is a martingale for each  $\pi \in L_{22}$  (as follows from (5.46), (2.4), Proposition 5.6, Remark 5.15, and Proposition 3.1). This, together with (5.35)(2'), motivates the following definition of  $\bar{X}$  in terms of  $\bar{Y}$  defined by Remark 5.13:

$$(5.47) \quad \bar{X}(t) := -\frac{1}{H(t)} \mathbb{E} \left[ \left( \frac{\bar{Y}(T) + c}{a} \right) H(T) \middle| \mathcal{F}_t \right].$$

**Remark 5.16.** The square-integrability of  $\bar{Y}(T)$  (recall  $\bar{Y} \in \mathbb{B}$ ) and  $c$  (Condition 5.1), and the strictly positive lower-bound on  $a$  (Condition 4.1), ensure that  $(\bar{Y}(T) + c)/a$  is square integrable. Together with Remark 5.15, this certainly establishes the existence of the conditional expectation in (5.47).

Observe that  $\bar{X}$  defined by (5.47) satisfies the ‘‘dynamical part’’ of (2.4), namely

$$(5.48) \quad d\bar{X}(t) = \{r(t)\bar{X}(t) + \bar{\pi}'(t)\sigma(t)\theta(t)\} dt + \bar{\pi}'(t)\sigma(t) dW(t),$$

for some  $\mathbb{R}^N$ -valued  $\bar{\pi} \in \mathcal{F}^*$  such that  $\int_0^T \|\bar{\pi}(t)\|^2 dt < \infty$  a.s. Indeed, from (5.47) and the martingale representation theorem (see 3.4.16 of Karatzas and Shreve [10], p.184), there exists some  $\mathbb{R}^N$ -valued and a.e.-unique  $\psi \in \mathcal{F}^*$ , with  $\int_0^T \|\psi(t)\|^2 dt < \infty$  a.s., such that

$$(5.49) \quad \bar{X}(t)H(t) = \bar{X}(0) + \int_0^t \psi'(\tau) dW(\tau) := \xi_0(t).$$

Expanding the quotient  $\bar{X}(t) := \xi_0(t)/H(t)$  by Itô's formula, we get (5.48) for

$$(5.50) \quad \bar{\pi}(t) := [\sigma'(t)]^{-1} \left[ \frac{\psi(t)}{H(t)} + \bar{X}(t)\theta(t) \right],$$

(since  $\bar{X}$  defined by (5.47) is continuous, Remark 2.2 shows that  $\int_0^T \|\bar{\pi}(t)\|^2 dt < \infty$  a.s.).

**Remark 5.17.** From Remark 5.13 we have seen that  $\bar{Y} \in \mathbb{B}_1$ , thus (5.35)(3') holds (recall (5.37)), and of course (5.35)(2') is immediate from (5.47). In the remainder of this section we shall establish that  $\bar{X} \in \mathbb{B}$  (in which case we see from (5.48) that  $\bar{\pi}$  is also given by  $\bar{\pi} = [\sigma']^{-1}\Lambda_{\bar{X}}$ ), and that (1') and (4') of (5.35) hold. We shall then have verified all items of (5.35), and can conclude (5.34) (from Proposition 5.8), which, together with (5.23), implies  $\Phi(\bar{X}) = \vartheta_{c,q}$ . Moreover, from (5.35)(4'), we obtain  $\bar{\pi} \in \mathcal{A}$  (recall (5.15)), while the dynamical relation (5.48) together with (5.35)(1'), establishes that  $\bar{X} = X^{\bar{\pi}}$  a.e. (for  $X^\pi$  defined by the wealth equation (2.4)). But, in light of (5.35)(1')(4'), (5.20), and (5.19), the first and third terms on the right side of (5.21) are zero when  $X := \bar{X}$ , and then (from (5.22))  $\Phi(\bar{X}) = \mathbb{E}[J(\bar{X}(T))]$ . Thus  $\vartheta_{c,q} = \mathbb{E}[J(X^{\bar{\pi}}(T))]$ , hence  $\bar{\pi} \in \mathcal{A}$  solves problem  $(\mathcal{P}_{c,q})$ .

**Lemma 5.18.** *Assume Conditions 2.1, 4.1, and 5.1. Then  $E \left[ \sup_{t \in [0, T]} |\bar{X}(t)|^2 \right] < \infty$  (for  $\bar{X}$  defined in (5.47)).*

*Proof.* Put  $D := (\bar{Y}(T) + c)/a$ . Then  $E|D|^2 < +\infty$  (Remark 5.16), and it follows from the integrability of  $H(t)$  indicated in Remark 5.15, together with Hölder's inequality, that  $DH(T)H^{-1}(t)$  is integrable. Thus, from (5.47), we have  $\bar{X}(t) = -E[DH(T)H^{-1}(t) | \mathcal{F}_t]$ . Now fix some  $q \in (1, 2)$ , and let  $p \in (2, \infty)$  be the conjugate constant given by  $p^{-1} + q^{-1} = 1$ . Then Hölder's inequality for conditional expectations (see Chow and Teicher, [3], Theorem 7.2.4, p.219) gives

$$(5.51) \quad |\bar{X}(t)| \leq \left\{ E \left[ \left( \frac{H(T)}{H(t)} \right)^p \middle| \mathcal{F}_t \right] \right\}^{1/p} \{E[|D|^q | \mathcal{F}_t]\}^{1/q}, \quad \text{a.s.}$$

for each  $t \in [0, T]$ . From (5.46) and (5.39), along with the uniform bounds on  $r$  and  $\theta$  (Condition 2.1 and Remark 2.3), there is a constant  $k \in (0, \infty)$  such that

$$(5.52) \quad \left( \frac{H(T)}{H(t)} \right)^p \leq k \frac{\mathcal{E}(-p\theta' \bullet W)(T)}{\mathcal{E}(-p\theta' \bullet W)(t)}, \quad \text{a.s.}$$

As noted in Remark 5.15,  $\{\mathcal{E}(-p\theta' \bullet W)(t)\}$  is a  $\{\mathcal{F}_t\}$ -martingale, thus it follows from (5.52) that the first conditional expectation on the right-hand side of (5.51) is upper-bounded a.s. by the constant  $k$ , and therefore

$$(5.53) \quad |\bar{X}(t)|^q \leq k^{q/p} E[|D|^q | \mathcal{F}_t], \quad \text{a.s.}$$

for each  $t \in [0, T]$ . Since  $E|D|^2 < \infty$  and  $q \in (1, 2)$ , we have  $E[|D|^q] < \infty$ . Thus, defining  $N(t) := E[|D|^q | \mathcal{F}_t]$ , we find that  $\{N(t)\}$  is a  $\{\mathcal{F}_t\}$ -martingale. Put  $p_1 := 2/q > 1$ , where the strict inequality follows since  $q \in (1, 2)$ . Thus, from Jensen's inequality, we see that  $E[|N(t)|^{p_1}] \leq E|D|^2 < \infty$ , for each  $t \in [0, T]$ , and consequently

$$(5.54) \quad E \left[ \sup_{t \in [0, T]} |N(t)|^{p_1} \right] \leq \left( \frac{p_1}{p_1 - 1} \right)^{p_1} E|N(T)|^{p_1} < \infty,$$

(from Doob's  $L^{p_1}$ -inequality). From (5.53) and the definition of  $N(t)$ , we have  $|\bar{X}(t)|^2 \leq k^{2/p} |N(t)|^{p_1}$ , and the result follows from (5.54).  $\square$

**Lemma 5.19.** *Suppose Conditions 2.1, 4.1, and 5.1. For  $\bar{X}$  and  $\bar{\pi}$  defined by (5.47), (5.49) and (5.50), we have  $\bar{X} \in \mathbb{B}$  and  $\bar{\pi} \in L_{22}$ .*

*Proof.* For each  $n = 1, 2, \dots$  put  $\tau_n := \inf\{t \in [0, T] \mid \int_0^t \|\bar{\pi}(s)\|^2 ds \geq n\} \wedge T$ . Then  $\tau_n$  is a  $\mathcal{F}_t$ -stopping time (recall (2.1)), and  $\tau_n \uparrow T$  a.s. (since  $\int_0^T \|\bar{\pi}(s)\|^2 ds < \infty$  a.s., as noted following (5.50)). Now we have seen that  $\bar{X}$  and  $\bar{\pi}$  are related by (5.48); using this relation to expand  $t \rightarrow \bar{X}^2(t)$  by Itô's formula, and evaluating at  $t \wedge \tau_n$ , we obtain

$$(5.55) \quad \begin{aligned} \bar{X}^2(t \wedge \tau_n) = \bar{X}^2(0) &+ \int_0^{t \wedge \tau_n} \{2\bar{X}(s)[r(s)\bar{X}(s) + \bar{\pi}'(s)\sigma(s)\theta(s)] + \|\sigma'(s)\bar{\pi}(s)\|^2\} ds \\ &+ 2 \int_0^{t \wedge \tau_n} \bar{X}(s)\bar{\pi}'(s)\sigma(s) dW(s), \quad t \in [0, T]. \end{aligned}$$

Now it follows from Lemma 5.18 and the definition of  $\tau_n$  that the last term on the right side of (5.55) defines a  $\mathcal{F}_t$ -martingale null at  $t = 0$ , and hence has zero expectation for all  $t$ ; thus upon taking expectations on each side of (5.55) at  $t := T$ , and using the non-negativity of  $r$  (Condition 2.1), we obtain

$$(5.56) \quad \mathbb{E}[\bar{X}^2(\tau_n)] + \mathbb{E} \int_0^{\tau_n} [-2\bar{X}(s)\bar{\pi}'(s)\sigma(s)\theta(s)] ds \geq \mathbb{E} \int_0^{\tau_n} \|\sigma'(s)\bar{\pi}(s)\|^2 ds.$$

For arbitrary  $v_1, v_2 \in \mathbb{R}^N$  we have  $v_1'v_2 \leq (1/2)[\|v_1\|^2 + \|v_2\|^2]$ , thus we get the inequality  $-2\bar{X}(s)\theta'(s)\sigma'(s)\bar{\pi}(s) \leq (1/2)[4\bar{X}^2(s)\|\theta(s)\|^2 + \|\sigma'(s)\bar{\pi}(s)\|^2]$ . Substituting this inequality into (5.56) and simplifying then gives

$$(5.57) \quad \frac{1}{2}\mathbb{E} \int_0^{\tau_n} \|\sigma'(s)\bar{\pi}(s)\|^2 ds \leq (1 + Tk_1)\mathbb{E} \left[ \sup_{t \in [0, T]} |\bar{X}(t)|^2 \right], \quad n = 1, 2, \dots$$

for some constant  $k_1 \in [0, \infty)$  depending only on the uniform bound on  $\theta$  (recall Remark 2.3). Now we have seen  $\tau_n \uparrow T$  a.s., thus we get  $\bar{\pi} \in L_{22}$  from Lemma 5.18 and Remark 2.2 upon taking  $n \rightarrow \infty$  in (5.57). Finally, from  $\bar{\pi} \in L_{22}$ , (5.48), and an argument identical to that for Proposition 3.1, we get  $\bar{X} \in \mathbb{B}$  (this proof is suggested by the argument for establishing existence of solutions for backwards SDE's - see e.g. ([16], p.352)).  $\square$

Recalling Remark 5.17, it remains to verify (1') and (4') of (5.35). To this end we need

**Lemma 5.20.** *Assume Conditions 2.1, 4.1, and 5.1. For arbitrary  $(\alpha, \eta) \in \mathbb{R} \times L_{22}$  and  $R := \Xi(\alpha, \eta)$  (recall (5.40)), we have*

$$(5.58) \quad 0 \leq \alpha(x_0 - \bar{X}(0)) + \lim_{\epsilon \searrow 0} \mathbb{E} \int_0^T \left\{ \frac{\delta(\Theta_{\bar{Y}}(t) + \epsilon \Theta_R(t)) - \delta(\Theta_{\bar{Y}}(t))}{\epsilon} + \bar{\pi}'(t)\Theta_R(t) \right\} dt.$$

**Remark 5.21.** From (5.45), Remark 5.13, and (5.44) we have  $\mathbb{E} \int_0^T \delta(\Theta_{\bar{Y}}(t)) dt < \infty$ , thus the expectation in (5.58) exists in  $(-\infty, \infty]$ . Since  $\delta(\cdot)$  is convex, it follows from Ekeland and Témam ([7], p.23) that the limit on the right of (5.58) exists (in the extended reals).

*Proof.* (of Lemma 5.20): For arbitrary  $\epsilon \in (0, \infty)$  define  $(y^\epsilon, \gamma^\epsilon) \in \mathbb{R} \times L_{22}$  by  $y^\epsilon := \bar{y} + \epsilon\alpha$  and  $\gamma^\epsilon := \bar{\gamma} + \epsilon\eta$ . From (5.45) we have  $\epsilon^{-1}[\tilde{\Psi}(y^\epsilon, \gamma^\epsilon) - \tilde{\Psi}(\bar{y}, \bar{\gamma})] \geq 0$  for each  $\epsilon \in (0, \infty)$ . Using (5.35)(2') (which holds in view of (5.47)) and (5.44) to calculate the quantity  $\lim_{\epsilon \rightarrow 0} \epsilon^{-1}[\tilde{\Psi}(y^\epsilon, \gamma^\epsilon) - \tilde{\Psi}(\bar{y}, \bar{\gamma})]$ , we easily obtain

$$(5.59) \quad 0 \leq \alpha x_0 - \mathbb{E} [\bar{X}(T)R(T)] + \lim_{\epsilon \searrow 0} \mathbb{E} \int_0^T \frac{\delta(\Theta_{\bar{Y}}(t) + \epsilon \Theta_R(t)) - \delta(\Theta_{\bar{Y}}(t))}{\epsilon} dt.$$

Now we have shown (5.48) and  $\bar{X} \in \mathbb{B}$  (see Lemma 5.19) hence  $\dot{\bar{X}}(t) = r(t)\bar{X}(t) + \bar{\pi}'(t)\sigma(t)\theta(t)$  and  $\Lambda_{\bar{X}}(t) = \sigma'(t)\bar{\pi}(t)$ , a.e. In view of these observations and Remark 5.10 (applied to  $R = \Xi(\alpha, \eta)$ ), it follows from Proposition 5.6 that  $\mathbb{M}(\bar{X}, R)(t) = \bar{X}(t)R(t) - \alpha\bar{X}(0) + \int_0^t \bar{\pi}'(\tau)\Theta_R(\tau) d\tau$ , is a continuous  $\{\mathcal{F}_t\}$ -martingale null at the origin, hence  $\mathbb{E}[\mathbb{M}(\bar{X}, R)(t)] = \mathbb{E}[\bar{X}(t)R(t)] - \alpha\bar{X}(0) + \mathbb{E} \int_0^t \bar{\pi}'(\tau)\Theta_R(\tau) d\tau = 0$ . Combining this with (5.59) gives (5.58).  $\square$

**Lemma 5.22.** *Assume Condition 2.1. For each  $\rho \in L_{22}$  there is a unique  $\xi \in L_{22}$  such that  $\rho(t) = \xi(t) + \theta(t) \int_0^t \xi'(\tau) dW(\tau)$  a.e.*

The proof of the preceding result is omitted since it is just a simple modification of the usual argument for existence and uniqueness in linear integral equations: With Lemmas 5.20 and 5.22 at hand, we can complete the program outlined in Remark 5.17:

**Proposition 5.23.** *Assume Conditions 2.1, 4.1, and 5.1. Then (1') and (4') of (5.35) hold for  $\bar{Y}$  defined by Remark 5.13, and  $\bar{X}$  defined by (5.47).*

*Proof.* We first establish (5.35)(1'). Fix an arbitrary  $\alpha \in \mathbb{R}$ . Since  $\theta \in L_{22}$  (being uniformly bounded by Remark 2.3) and  $\beta$  is uniformly bounded ((5.39) and Condition 2.1), upon taking  $\rho(t) := -\alpha\theta(t)$  in Lemma 5.22 we easily see that there is some  $\eta \in L_{22}$  such that  $-\alpha\theta(t)\beta(t) = \eta(t) + \theta(t)\beta(t) \int_0^t \beta^{-1}(\tau)\eta'(\tau) dW(\tau)$  a.e. From this, together with (5.40) and (5.39), we find  $\eta(t) + \theta(t)\Xi(\alpha, \eta)(t) = 0$  a.e. Upon defining  $R := \Xi(\alpha, \eta)$ , we get from Remark 5.10 that  $\Theta_R(t) = 0$  a.e., thus Lemma 5.20 gives  $0 \leq \alpha(x_0 - \bar{X}(0))$ . Now (5.35)(1') follows since  $\alpha \in \mathbb{R}$  is arbitrary.

It remains to establish (5.35)(4'): Since (5.48) holds and  $\bar{\pi} \in L_{22}$  (see Lemma 5.19), it is enough to show that  $\bar{\pi}(t) \in K$  a.e. to conclude  $\bar{\pi} \in \mathcal{U}(\bar{X})$  (recall (5.15), (4.8)). Since  $\delta(\cdot)$  is subadditive and positively homogeneous (see [9], p.206) we have  $\delta(\Theta_{\bar{Y}}(t) + \epsilon\Theta_R(t)) \leq \delta(\Theta_{\bar{Y}}(t)) + \epsilon\delta(\Theta_R(t))$  for arbitrary  $\epsilon \in (0, \infty)$  and  $R \in \mathbb{B}$ . Then, since we have shown  $\bar{X}(0) = x_0$ , it follows from Lemma 5.20 that, for each  $(\alpha, \eta) \in \mathbb{R} \times L_{22}$ , we have

$$(5.60) \quad 0 \leq \mathbb{E} \int_0^T \{\delta(\Theta_R(t)) + \bar{\pi}'(t)\Theta_R(t)\} dt, \quad \text{for } R := \Xi(\alpha, \eta).$$

Put  $B := \{(\omega, t) \in \Omega \times [0, T] \mid \bar{\pi}(\omega, t) \in K\}$ . By Lemma 5.4.2 of (Karatzas and Shreve, [9], p.207), there exists some  $\mathcal{F}^*$ -measurable mapping  $\tilde{\nu} : \Omega \times [0, T] \rightarrow \mathbb{R}^N$  such that  $\|\tilde{\nu}(t)\| \leq 1$  and  $|\delta(\tilde{\nu}(t))| \leq 1$  a.e., and

$$(5.61) \quad \delta(\tilde{\nu}(t)) + \bar{\pi}'(t)\tilde{\nu}(t) = 0, \quad \text{a.e. on } B, \quad \delta(\tilde{\nu}(t)) + \bar{\pi}'(t)\tilde{\nu}(t) < 0, \quad \text{a.e. on } B^c.$$

Now suppose that  $(P \otimes \lambda)\{(\Omega \times [0, T]) - B\} > 0$ . Then, by (5.61),

$$(5.62) \quad 0 > \mathbb{E} \int_0^T \{\delta(\tilde{\nu}(t)) + \bar{\pi}'(t)\tilde{\nu}(t)\} dt.$$

Put  $\rho(t) := -\beta^{-1}(t)\sigma^{-1}(t)\tilde{\nu}(t)$ . Since  $\|\tilde{\nu}(t)\|$  is essentially bounded on  $\Omega \times [0, T]$ , it follows from the boundedness of  $\beta^{-1}$  and  $\sigma^{-1}$  (Remark 2.2) that  $\rho \in L_{22}$ . Then, from Lemma 5.22, there exists some  $\tilde{\xi} \in L_{22}$  such that  $-\beta^{-1}(t)\sigma^{-1}(t)\tilde{\nu}(t) = \tilde{\xi}(t) + \theta(t) \int_0^t \tilde{\xi}'(\tau) dW(\tau)$  a.e. Now multiply each side by  $\beta(t)\sigma(t)$ , and define  $\tilde{\eta}(t) := \beta(t)\tilde{\xi}(t) \in L_{22}$  and  $R := \Xi(0, \tilde{\eta}) = \beta(t)\mathfrak{J}(\tilde{\eta})(t)$  (see (5.40)). Then, from (5.39) and Remark 5.10, we get  $\tilde{\nu}(t) = -\sigma(t)[\tilde{\eta}(t) + \theta(t)R(t)] = \Theta_R(t)$  a.e. From this and (5.62), we get  $\mathbb{E} \int_0^T \{\delta(\Theta_R(t)) + \bar{\pi}'(t)\Theta_R(t)\} dt < 0$ , (for  $R := \Xi(0, \tilde{\eta})$ ). Since  $\tilde{\eta} \in L_{22}$ , this last inequality contradicts (5.60) and so  $(P \otimes \lambda)\{(\Omega \times [0, T]) - B\} = 0$ , that is  $\bar{\pi}(t) \in K$  a.e., as required to establish  $\bar{\pi} \in \mathcal{U}(\bar{X})$ . We next show

$$(5.63) \quad \delta(\Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) = 0 \quad \text{a.e.}$$

To this end, put  $R := \Xi(-\bar{y}, -\bar{\gamma})$ . Then, in light of the linearity of  $\Xi(\cdot)$ , and since  $\bar{Y} = \Xi(\bar{y}, \bar{\gamma})$  (see Remark 5.13), we have  $R = -\bar{Y}$ . Thus  $\Theta_R = -\Theta_{\bar{Y}}$  (see (5.33)). Since  $\delta(\cdot)$  is positively homogeneous, for each  $\epsilon \in (0, 1)$  we get  $\delta(\Theta_{\bar{Y}}(t) + \epsilon\Theta_R(t)) = \delta((1-\epsilon)\Theta_{\bar{Y}}(t)) = (1-\epsilon)\delta(\Theta_{\bar{Y}}(t))$  a.e. From this, together with  $\bar{X}(0) = x_0$  (which we have already shown), and Lemma 5.20, we obtain the inequality  $0 \geq E \int_0^T \{\delta(\Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t)\} dt$ . Now we have already seen that  $\bar{\pi}(t) \in K$  a.e., thus (see (5.27))  $\delta(\Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) \geq 0$  a.e. This, together with the inequality just noted, establishes (5.63). Finally, we see from (5.48) that  $\bar{\pi}$  at (5.50) is also given by  $\bar{\pi}(t) := [\sigma'(t)]^{-1}\Lambda_{\bar{X}}(t)$ . This establishes (5.35)(4').  $\square$

For easy reference we summarize the main result of the present section as follows:

**Proposition 5.24.** *Suppose Conditions 2.1, 4.1, and 5.1. Then there exists a pair  $(\bar{y}, \bar{\gamma}) \in \mathbb{R} \times L_{22}$  minimizing the proper convex functional  $\tilde{\Psi}(\cdot, \cdot)$  (see (5.44)) over  $\mathbb{R} \times L_{22}$ . Define  $\bar{Y} := \Xi(\bar{y}, \bar{\gamma})$  (with  $\Xi$  given by (5.40), (5.39)), and  $H$  by (5.46). Put*

$$\bar{X}(t) := -\frac{1}{H(t)} E \left[ \left( \frac{\bar{Y}(T) + c}{a} \right) H(T) \middle| \mathcal{F}_t \right], \quad \bar{\pi}(t) := [\sigma'(t)]^{-1} \left[ \frac{\psi(t)}{H(t)} + \bar{X}(t)\theta(t) \right],$$

(here  $\psi \in \mathcal{F}^*$  is the  $\mathbb{R}^N$ -valued a.e. unique process on  $\Omega \times [0, T]$  such that  $\int_0^T \|\psi(t)\|^2 dt < \infty$  a.s. and  $\bar{X}(t)H(t) = \bar{X}(0) + \int_0^t \psi'(\tau) dW(\tau)$ , given by the martingale representation theorem). Then we have  $\bar{\pi} \in \mathcal{A}$  and  $\bar{X}(t) = X^{\bar{\pi}}(t)$  a.e. (for  $X^\pi$  defined by (2.5)), and

$$\inf_{\pi \in \mathcal{A}} E[J(X^\pi(T))] = E[J(X^{\bar{\pi}}(T))] = - \inf_{(y, \gamma) \in \mathbb{R} \times L_{22}} \tilde{\Psi}(y, \gamma) = -\tilde{\Psi}(\bar{y}, \bar{\gamma}) \in \mathbb{R}.$$

In particular,  $\bar{\pi}$  solves the partially constrained problem  $(\mathcal{P}_{c,q})$  at (5.14).

## 6 Fully Constrained Optimization Problem

In the present section we return to the main goal of this paper, namely the fully constrained problem (4.12). Our approach relies on Proposition 5.24, together with results from Lagrange duality for convex optimization, as set forth in Aubin ([1], Chapter 2, Section 6).

Throughout this section we postulate Conditions 2.1, 4.1, 4.2, and 4.3. Then we have

- (i)  $\mathcal{A}$  is a convex subset of  $L_{22}$  (as follows from convexity of  $K$  in Condition 4.1);
- (ii)  $G$  is an affine functional on  $L_{22}$  (as follows from (2.5) and (4.10));
- (iii)  $\pi \mapsto E[\hat{J}(X^\pi(T))]$  defines an  $\mathbb{R}$ -valued convex mapping on  $\mathcal{A}$  (as follows from Proposition 3.1 and Conditions 4.1 and 4.2).

Now define the *Lagrangian function* for the optimization problem  $(\hat{\mathcal{P}})$  at (4.12):

$$(6.64) \quad \mathcal{L}(\mu; \pi) := E[\hat{J}(X^\pi(T))] + \mu G(\pi), \quad \pi \in L_{22}, \quad \mu \in \mathbb{R}.$$

Then, from Proposition 2.6.1 of ([1], p.36), Condition 4.3, and Theorem 2.6.1 of ([1], p.37), there exists some ‘‘Lagrange multiplier’’  $\bar{\mu} \in \mathbb{R}$  such that (recalling (4.11))

$$(6.65) \quad \hat{\vartheta} = \sup_{\mu \in \mathbb{R}} \inf_{\pi \in \mathcal{A}} \mathcal{L}(\mu; \pi) = \inf_{\pi \in \mathcal{A}} \mathcal{L}(\bar{\mu}; \pi).$$

For each  $(\mu, \omega, x) \in \mathbb{R} \times \Omega \times \mathbb{R}$ , put

$$(6.66) \quad J_1(\mu; \omega, x) := \frac{1}{2} [a(\omega)x^2 + 2c_\mu(\omega)x] - \mu d, \quad c_\mu(\omega) := c_0(\omega) + \mu c_1(\omega),$$

(for  $a, c_0, c_1$  given by Conditions 4.1 and 4.2) and observe, from (4.9), (4.10) and (6.64),

$$(6.67) \quad \mathcal{L}(\mu; \pi) = \mathbb{E}[J_1(\mu; X^\pi(T))], \quad \pi \in L_{22}, \quad \mu \in \mathbb{R}.$$

**Remark 6.1.** For each fixed  $\mu \in \mathbb{R}$ , the function  $J_1(\mu; \cdot, \cdot)$  is identical to the function  $J(\cdot, \cdot)$  in (5.13), with  $c_\mu$  in place of  $c$  and  $-\mu d$  in place of  $q$ . In view of Condition 4.2, we see that  $\mathbb{E}[c_\mu^2] < +\infty$ ; that is Condition 5.1 holds with  $c_\mu$  replacing  $c$  for each  $\mu \in \mathbb{R}$ , and therefore the infima in (6.65) correspond to optimization problems  $(\mathcal{P}_{c,q})$  of the form (5.14) (with  $c := c_\mu$  and  $q := -\mu d$ ), which are addressed by Proposition 5.24.

Motivated by (5.44) and Remark 6.1, for each  $(\mu, y, \gamma) \in \mathbb{R} \times \mathbb{R} \times L_{22}$  and  $Y := \Xi(y, \gamma)$ , put

$$(6.68) \quad \tilde{\Psi}_1(\mu; y, \gamma) := x_0 y + \mathbb{E} \left[ \frac{(Y(T) + c_\mu)^2}{2a} \right] + \mathbb{E} \int_0^T \delta(\Theta_Y(t)) dt + \mu d.$$

**Remark 6.2.** Proposition 5.24 asserts existence of a minimizer  $(\bar{y}(\mu), \bar{\gamma}(\mu)) \in \mathbb{R} \times L_{22}$  of  $\tilde{\Psi}_1(\mu; \cdot)$  over  $\mathbb{R} \times L_{22}$  for each  $\mu \in \mathbb{R}$ ; motivated by Proposition 5.24, define

$$(6.69) \quad \begin{cases} (1) \bar{Y}(\mu; t) := \Xi(\bar{y}(\mu), \bar{\gamma}(\mu))(t), \\ (2) \bar{X}(\mu; t) := -\frac{1}{H(t)} \mathbb{E} \left[ \left( \frac{\bar{Y}(\mu; T) + c_\mu}{a} \right) H(T) \mid \mathcal{F}_t \right], \\ (3) \bar{\pi}(\mu; t) := [\sigma'(t)]^{-1} \left[ \frac{\psi(\mu; t)}{H(t)} + \bar{X}(\mu; t)\theta(t) \right]. \end{cases}$$

for each  $\mu \in \mathbb{R}$ . From (6.69)(2) and the martingale representation theorem there exists some a.e.-unique  $\mathbb{R}^N$ -valued  $\mathcal{F}_t$ -progressively measurable process  $\psi(\mu; \cdot)$  on  $\Omega \times [0, T]$ , such that  $\int_0^T \|\psi(\mu; t)\|^2 dt < \infty$  a.s. and  $\bar{X}(\mu; t)H(t) = \bar{X}(\mu; 0) + \int_0^t \psi'(\mu; \tau) dW(\tau)$  for all  $t \in [0, T]$ ; it is this process which appears on the right side of (6.69)(3). Finally, note from Proposition 5.24 that  $\bar{\pi}(\mu) \in \mathcal{A}$  and  $\bar{X}(\mu; t) = X^{\bar{\pi}(\mu)}(t)$  a.e. for each  $\mu \in \mathbb{R}$  (with  $X^{\bar{\pi}(\mu)}$  given by (2.5)).

It remains to show that  $\bar{\pi}(\bar{\mu}; \cdot)$  solves problem  $(\hat{\mathcal{P}})$  at (4.12). From Proposition 5.24 and (6.67), for each  $\mu \in \mathbb{R}$  we have

$$(6.70) \quad \inf_{\pi \in \mathcal{A}} \mathcal{L}(\mu; \pi) = \mathcal{L}(\mu, \bar{\pi}(\mu)) = - \inf_{(y, \gamma) \in \mathbb{R} \times L_{22}} \tilde{\Psi}_1(\mu; y, \gamma) = -\tilde{\Psi}_1(\mu; \bar{y}(\mu), \bar{\gamma}(\mu)) \in \mathbb{R}.$$

Since  $\bar{\pi}(\bar{\mu}) \in \mathcal{A}$  (by Remark 6.2), it is enough to show that

$$(6.71) \quad G(\bar{\pi}(\bar{\mu})) = 0,$$

for then Proposition 2 of Aubin ([1], p.37), together with the second equality of (6.65) and the first equality of (6.70) establishes that  $\bar{\pi}(\bar{\mu})$  solves the problem (4.12). To obtain (6.71) we use a variational analysis on the optimality of  $\bar{\mu}$ ; from (6.65) and (6.70), we find that

$$(6.72) \quad -\hat{\vartheta} = \inf_{(\mu, y, \gamma) \in \mathbb{R} \times \mathbb{R} \times L_{22}} \tilde{\Psi}_1(\mu; y, \gamma) = \tilde{\Psi}_1(\bar{\mu}; \bar{y}(\bar{\mu}), \bar{\gamma}(\bar{\mu})).$$

Now put  $\mu^\epsilon := \bar{\mu} + \epsilon\rho$  for  $\rho \in \mathbb{R}$  and  $\epsilon \in (0, \infty)$ . Then, from (6.72), we have

$$(6.73) \quad 0 \leq \frac{\tilde{\Psi}_1(\mu^\epsilon; \bar{y}(\bar{\mu}), \bar{\gamma}(\bar{\mu})) - \tilde{\Psi}_1(\bar{\mu}; \bar{y}(\bar{\mu}), \bar{\gamma}(\bar{\mu}))}{\epsilon}, \quad \epsilon \in (0, \infty).$$

From the definition of  $c_\mu$  in (6.66), we have  $c_{\mu^\epsilon} = c_{\bar{\mu}} + \epsilon\rho c_1$ , hence, from (6.73) and (6.68), we obtain  $0 \leq \rho \mathbb{E}[\{(\bar{Y}(\bar{\mu}; T) + c_{\bar{\mu}}) c_1\}/a] + \epsilon\rho^2 \mathbb{E}[c_1^2/(2a)] + \rho d$ , for all  $\epsilon \in (0, \infty)$ . Taking  $\epsilon \rightarrow 0$ , and using the arbitrary choice of  $\rho \in \mathbb{R}$ , then gives  $\mathbb{E}[\{(\bar{Y}(\bar{\mu}; T) + c_{\bar{\mu}}) c_1\}/a] + d = 0$ , which, in view of (6.69)(2), establishes that  $\mathbb{E}[c_1 \bar{X}(\bar{\mu}; T)] = d$ . Now (6.71) follows from this, together with  $\bar{X}(\bar{\mu}; T) = X^{\bar{\pi}(\bar{\mu})}(T)$  (recall Remark 6.2) and (4.10).

**Remark 6.3.** Now we can assemble the preceding and state the main result of the present section. Define  $h(\mu) := \inf_{(y, \gamma) \in \mathbb{R} \times L_{22}} \tilde{\Psi}_1(\mu; y, \gamma)$ ,  $\mu \in \mathbb{R}$ , and note, from (6.70), that the second equality of (6.65) gives  $\inf_{\mu \in \mathbb{R}} h(\mu) = h(\bar{\mu})$ .

**Proposition 6.4.** *Suppose Conditions 2.1, 4.1, 4.2, and 4.3. For each  $\mu \in \mathbb{R}$ , there exists a pair  $(\bar{y}(\mu), \bar{\gamma}(\mu)) \in \mathbb{R} \times L_{22}$  which minimizes the functional  $\tilde{\Psi}_1(\mu; \cdot)$  over  $\mathbb{R} \times L_{22}$  (recall (6.68)), and hence satisfies  $h(\mu) = \tilde{\Psi}_1(\mu; \bar{y}(\mu), \bar{\gamma}(\mu))$ . Moreover, there exists some  $\bar{\mu} \in \mathbb{R}$  which minimizes  $h(\cdot)$  on  $\mathbb{R}$ , and  $\hat{\pi} := \bar{\pi}(\bar{\mu})$  (given by Remark 6.2 with  $\mu := \bar{\mu}$ ) is the optimal portfolio for the problem  $(\hat{\mathcal{P}})$  at (4.12).*

**Example 6.5.** Take  $K := \mathbb{R}^N$  in Condition 4.1 for the *unconstrained case*. From (5.27) we see that  $\delta(0) = 0$ , and  $\delta(z) = +\infty$  when  $z \neq 0$ . Then we need minimize  $\tilde{\Psi}_1(\mu; \cdot)$  at (6.68) only over pairs  $(y, \gamma) \in \mathbb{R} \times L_{22}$  such that  $\Theta_Y(t) = 0$  a.e. (for  $Y := \Xi(y, \gamma)$ ). From Remark 5.10 and nonsingularity of  $\sigma(t)$  (Condition 2.1) we obtain  $Y_0 = y$  and  $\Lambda_Y(t) = -Y(t)\theta(t)$  a.e. Inserting these in (5.41) then shows that  $Y(t) = yH(t)$  a.e. and  $\gamma \in L_{22}$  necessarily has the form  $\gamma(t) = -yH(t)\theta(t)$  a.e. for some  $y \in \mathbb{R}$  (recall (5.46)). Determination of the optimal portfolio reduces to the following: (i) for each  $\mu \in \mathbb{R}$  locate the minimizer  $\bar{y}(\mu) \in \mathbb{R}$  of the functional  $y \mapsto \tilde{\Psi}_2(\mu; y) := \tilde{\Psi}_1(\mu; y, -yH\theta)$  (which is quadratic); (ii) use  $\bar{y}(\mu)$  to minimize the functional  $h(\mu) := \tilde{\Psi}_2(\mu; \bar{y}(\mu))$ ,  $\mu \in \mathbb{R}$  (which is also quadratic). In the special case of the *mean-variance problem* of Remark 4.5, where  $a = 2$ ,  $c_0 = 0$ ,  $c_1 = 1$ , we have  $c_\mu = \mu$  (by (6.66)), and (i) and (ii) lead to the (unique) minimizers  $\bar{y}(\mu) = -\{2x_0 + \mu\mathbb{E}[H(T)]\}/\mathbb{E}[H^2(T)]$  and  $\bar{\mu} = 2\{x_0\mathbb{E}[H(T)] - d\mathbb{E}[H^2(T)]\}/\text{Var}(H(T))$ . Then  $\bar{Y}(\bar{\mu}; t) = \bar{y}(\bar{\mu})H(t)$  and  $\bar{X}(\bar{\mu}; T) = -(1/2)[\bar{Y}(\bar{\mu}; T) + \bar{\mu}]$  (by (6.69)(2)), the optimal portfolio  $\bar{\pi}(\bar{\mu})$  is given by (6.69)(3) with  $\mu := \bar{\mu}$  (by Proposition 6.4), and the least variance (or *efficient frontier*) is given by

$$\inf_{\substack{\pi \in L_{22} \\ \mathbb{E}[X^\pi(T)] = d}} \text{Var}(X^\pi(T)) = \text{Var}(\bar{X}(\bar{\mu}; T)) = \frac{1}{4} \text{Var}(\bar{Y}(\bar{\mu}; T)) = \frac{(x_0 - d\mathbb{E}[H(T)])^2}{\text{Var}(H(T))}.$$

**Example 6.6.** We suppose that  $K \subset \mathbb{R}^N$  in Condition 4.1 is a *closed convex cone*, the market coefficients  $r$ ,  $b$  and  $\sigma$  in Condition 2.1 are *nonrandom* continuous functions on  $[0, T]$ , and  $c_0$ ,  $c_1$  and  $a$  in Conditions 4.1 and 4.2 are also *nonrandom*. In this case the dual problem of minimizing  $\tilde{\Psi}_1(\mu; y, \gamma)$  over the pairs  $(y, \gamma) \in L_{22}$  (recall (6.68)) is particularly well-suited to the application of *dynamic programming* and leads to an essentially explicit formula for the

optimal portfolio  $\bar{\pi}(\mu)$  at (6.69)(3). Since  $K$  is a convex cone, from (5.27) we have  $\delta \equiv 0$  on  $\tilde{K} := \{z \mid \delta(z) < \infty\}$  (the “barrier cone” of  $-K$ ). Thus the third term on the right of (6.68) takes values in the two-point set  $\{0, \infty\}$  according to whether or not  $\Theta_Y(t) \in \tilde{K}$  a.e. We can therefore regard  $u(t) := \Theta_Y(t)$  (rather than  $\gamma$ ) as the “control” in the dual problem, and it then follows from (5.41) and (5.42) that the dual process  $Y$  is subject to

$$(6.74) \quad dY(t) = -r(t)Y(t) dt - [\theta(t)Y(t) + \sigma^{-1}(t)u(t)]' dW(t),$$

with  $u(t) \in \tilde{K}$  a.e. For each  $(y, u) \in \mathbb{R} \times L_{22}$ , let  $\{\tilde{\Xi}(y, u)(t), t \in [0, T]\}$  denote the process  $Y$  given by (6.74) with the initial condition  $Y(0) = y$ . Then, for arbitrary  $\mu \in \mathbb{R}$ , the dual problem of minimizing  $\tilde{\Psi}_1(\mu; y, \gamma)$  at (6.68) over pairs  $(y, \gamma) \in \mathbb{R} \times L_{22}$  is equivalent to the minimization of

$$(6.75) \quad \tilde{\Psi}_3(\mu; y, u) := x_0 y + \mathbb{E}[(\tilde{\Xi}(y, u)(T) + c_\mu)^2 / (2a)] + \mu d,$$

over  $(y, u) \in \mathbb{R} \times L_{22}$  with  $u(t) \in \tilde{K}$  a.e. (a straightforward application of Gronwall’s inequality yields  $\tilde{\Xi}(y, u) \in \mathbb{B}$  for each  $(y, u) \in \mathbb{R} \times L_{22}$ ). We now minimize the second term of (6.75) over  $u \in L_{22}$  for arbitrary  $y \in \mathbb{R}$ . Keeping  $\mu \in \mathbb{R}$  fixed, define the value function

$$(6.76) \quad V(\mu; y) := \inf_{\substack{u \in L_{22} \\ u(t) \in \tilde{K} \text{ a.e.}}} \mathbb{E} \left[ \frac{(\tilde{\Xi}(y, u)(T) + c_\mu)^2}{2a} \right], \quad y \in \mathbb{R},$$

and consider the Bellman equation associated with (6.74) and (6.76), namely

$$(6.77) \quad \begin{cases} (1) \tilde{v}_s(s, y) - r(s)y\tilde{v}_y(s, y) + (1/2) \inf_{\eta \in \tilde{K}} \left\{ \|\sigma^{-1}(s)\eta + \theta(s)y\|^2 \tilde{v}_{yy}(s, y) \right\} = 0; \\ (2) \tilde{v}(T, y) = (y + c_\mu)^2 / (2a), \end{cases}$$

for each  $(s, y) \in [0, T] \times \mathbb{R}$ . This is a particularly tractable equation because the infimum in (6.77)(1) is easily expressed in terms of  $y \in \mathbb{R}$ . Indeed, for  $s \in [0, T]$  and  $i = 1, 2$ , put

$$(6.78) \quad \zeta_i(s) := \arg \min_{\eta \in \tilde{K}} \|\sigma^{-1}(s)\eta - (-1)^i \theta(s)\|^2 = \sigma(s) \text{proj}((-1)^i \theta(s) \mid \sigma^{-1}(s)\tilde{K}),$$

where  $\text{proj}(z \mid C)$  is the (uniquely determined) projection of a vector  $z \in \mathbb{R}^N$  on a closed convex set  $C \subset \mathbb{R}^N$ . Then, for each  $s \in [0, T]$ , it follows that

$$(6.79) \quad \hat{u}(s, y) := \arg \min_{\eta \in \tilde{K}} \|\sigma^{-1}(s)\eta + \theta(s)y\|^2 = \begin{cases} y\zeta_1(s) & \text{if } y \geq 0; \\ -y\zeta_2(s) & \text{if } y < 0. \end{cases}$$

In the light of (6.79), we can easily write down an explicit solution of (6.77). To this end, for  $(s, y) \in [0, T] \times \mathbb{R}$  and  $i = 1, 2$ , (recall  $\beta$  at (5.39) and  $c_\mu$  at (6.66)), define

$$(6.80) \quad A_i(s) := \exp \left\{ \int_s^T \|\theta(\tau) - (-1)^i \sigma^{-1}(\tau)\zeta_i(\tau)\|^2 d\tau \right\},$$

$$P_i(s) := \frac{1}{a} \left( \frac{\beta(T)}{\beta(s)} \right)^2 A_i(s), \quad \chi(\mu; s) := \frac{c_\mu}{a} \frac{\beta(T)}{\beta(s)}, \quad \alpha(\mu) := \frac{c_\mu^2}{2a},$$

$$(6.81) \quad \tilde{v}(\mu; s, y) := \begin{cases} P_1(s)y^2/2 + \chi(\mu; s)y + \alpha(\mu) & \text{if } (s, y) \in [0, T] \times [0, \infty); \\ P_2(s)y^2/2 + \chi(\mu; s)y + \alpha(\mu) & \text{if } (s, y) \in [0, T] \times (-\infty, 0). \end{cases}$$

Then  $\tilde{v}(\mu; \cdot)$  is of class  $C^{1,1}$  over  $[0, T] \times \mathbb{R}$  and of class  $C^{1,2}$  over  $[0, T] \times (\mathbb{R} - \{0\})$ , and a simple direct verification establishes that it satisfies the Bellman equation (6.77)(1) in the classical sense for each  $(s, y) \in [0, T] \times (\mathbb{R} - \{0\})$ , as well as the boundary condition (6.77)(2). Moreover, the second-order parabolic sub/superdifferentials of  $\tilde{v}(\mu; s, y)$  at  $(s, y) \in [0, T] \times \{0\}$  are easily computed to show that  $\tilde{v}(\mu; \cdot)$  defines a viscosity solution of (6.77) on  $[0, T] \times \mathbb{R}$ . It now follows from the verification theorem for dynamic programming ([16], Theorem 5.3, p.270) that  $\hat{u}$  at (6.79) is the optimal feedback control for the problem (6.76) with arbitrary  $y \in \mathbb{R}$ , and  $V(\mu; y) = \tilde{v}(\mu; 0, y)$  for all  $y \in \mathbb{R}$ . In particular the function  $y \mapsto V(\mu; y) = \tilde{v}(\mu; 0, y)$  is the ‘‘asymmetric quadratic’’ given by (6.81). Substituting  $\hat{u}(t, Y(t))$  for  $u(t)$  in (6.74), it follows that the resulting SDE has pathwise-uniqueness (since  $\hat{u}(t, \cdot)$  given by (6.79) is globally Lipschitz continuous on  $\mathbb{R}$ ) with solution (for the initial condition  $Y(0) = y \in \mathbb{R}$ ) given by (recall Remark 5.14)

$$(6.82) \quad \hat{Y}(y; t) := \begin{cases} y\beta(t) \mathcal{E}(-[\theta + \sigma^{-1}\zeta_1]' \bullet W)(t) & \text{if } y \geq 0; \\ y\beta(t) \mathcal{E}(-[\theta - \sigma^{-1}\zeta_2]' \bullet W)(t) & \text{if } y < 0. \end{cases}$$

We are now able to minimize  $\tilde{\Psi}_3(\mu; \cdot)$  at (6.75) (still keeping  $\mu \in \mathbb{R}$  fixed). Let  $\bar{y}(\mu) \in \mathbb{R}$  be the (unique) minimizer (with respect to  $y \in \mathbb{R}$ ) of the ‘‘asymmetric quadratic’’

$$(6.83) \quad \tilde{\Psi}_4(\mu; y) := x_0y + V(\mu; y) + \mu d = x_0y + \tilde{v}(\mu; 0, y) + \mu d, \quad y \in \mathbb{R},$$

given by (6.81), and put  $\bar{u}(\mu; t) := \hat{u}(t, \hat{Y}(\bar{y}(\mu); t))$ ,  $t \in [0, T]$ . Then  $\bar{u}(\mu; t) \in \tilde{K}$  a.e. (see (6.79)), and the pair  $(\bar{y}(\mu), \bar{u}(\mu)) \in \mathbb{R} \times L_{22}$  is the minimizer of the dual cost functional  $\tilde{\Psi}_3(\mu; \cdot)$  defined at (6.75). Comparison of (6.74) with the relations (5.41) and (5.42) then shows that, for  $\bar{\gamma}(\mu) \in L_{22}$  defined by  $\bar{\gamma}(\mu; t) := -[\theta(t) \tilde{\Xi}(\bar{y}(\mu), \bar{u}(\mu))(t) + \sigma^{-1}(t)\bar{u}(\mu; t)]$ , the pair  $(\bar{y}(\mu), \bar{\gamma}(\mu))$  minimizes the functional  $\tilde{\Psi}_1(\mu; \cdot)$  (see (6.68)) over  $\mathbb{R} \times L_{22}$ , and (see (6.69)) the corresponding optimal dual process  $\bar{Y}(\mu)$  is given by  $\bar{Y}(\mu; t) = \hat{Y}(\bar{y}(\mu); t)$ ,  $t \in [0, T]$ . Using this representation for  $\bar{Y}(\mu)$  it is easy to get explicit formulae for the portfolio  $\bar{\pi}(\mu)$  and corresponding wealth  $\bar{X}(\mu)$  (see (6.69)(2)(3)). Indeed, upon substituting  $\hat{Y}(\bar{y}(\mu); T)$  (given by (6.82)) for  $\bar{Y}(\mu; T)$  in (6.69)(2), and using (5.46), the fact that the coefficients  $r$ ,  $b$  and  $\sigma$  are deterministic, and the independent increments of  $W$ , we obtain

$$(6.84) \quad -\bar{X}(\mu; t) = \frac{\hat{Y}(\bar{y}(\mu); t)}{a} \exp \left\{ \int_t^T [-2r(\tau) + \theta'(\tau)[\theta(\tau) + \sigma^{-1}(\tau)\zeta_1(\tau)]] d\tau \right\} + \frac{c_\mu\beta(T)}{a\beta(t)},$$

when  $\bar{y}(\mu) \geq 0$  (just replace  $\zeta_1$  by  $-\zeta_2$  in (6.84) to get  $\bar{X}(\mu)$  when  $\bar{y}(\mu) < 0$ ). Finally, using (6.82) and Itô’s product formula to expand the right side of (6.84) (and its analogue for  $\bar{y}(\mu) < 0$ ), and comparing the result with (2.4), we obtain  $\bar{\pi}(\mu)$  such that  $\bar{X}(\mu) = X^{\bar{\pi}(\mu)}$  as the following feedback policy on the wealth  $\bar{X}(\mu)$ :

$$(6.85) \quad \bar{\pi}(\mu; t) := -[\bar{X}(\mu; t) + a^{-1}\beta^{-1}(t)c_\mu\beta(T)] (\sigma'(t))^{-1} [\theta(t) + \sigma^{-1}(t)\zeta_1(t)] \quad \text{if } \bar{y}(\mu) > 0,$$

$\bar{\pi}(\mu, t)$  is given by (6.85) with  $-\zeta_2(t)$  replacing  $\zeta_1(t)$  when  $\bar{y}(\mu) < 0$ , and  $\bar{\pi}(\mu, t) = 0$  when  $\bar{y}(\mu) = 0$ . We now determine the optimal portfolio and minimum variance in the special

case of Remark 4.5, for which  $a = 2$ ,  $c_0 = 0$ ,  $c_1 = 1$ . To this end we first characterize the set  $\mathcal{R}$  of Remark 4.6. Define the set  $F := \{t \in [0, T] \mid \|\theta(t) + \sigma^{-1}(t)\zeta_1(t)\| > 0\} = \{t \in [0, T] \mid -\sigma(t)\theta(t) \notin \tilde{K}\}$ , where the equality follows from (6.78). Since  $\tilde{K} = \{z \in \mathbb{R}^N \mid -\pi'z \leq 0, \text{ all } \pi \in K\}$  we then have that  $F = \{t \in [0, T] \mid \Gamma(t) \neq \emptyset\}$ , where  $\Gamma(t) := \{\pi \in K \mid \pi'\sigma(t)\theta(t) > 0\}$ . Now suppose  $A_1(0) > 1$ ; then  $\lambda(F) > 0$  (by (6.80)), and by the Aumann selection theorem ([15], Theorem 2.3.12, p.71) there is a measurable selection  $\pi_1(\cdot)$  of  $\Gamma(\cdot)$  on  $F$ . Put  $\pi_2(t) := 0$ ,  $t \notin F$ , and  $\pi_2(t) := \pi_1(t)/\|\pi_1(t)\|$ ,  $t \in F$ . Then  $\pi_2 \in \mathcal{A}$  (since  $K$  is a cone) and  $\pi_2'(t)\sigma(t)\theta(t) > 0$ ,  $t \in F$ , hence from (2.5) we have  $E[X^{\pi_2}(T)] > x_0S_0(T)$ , which establishes that  $[x_0S_0(T), \infty) \subset \mathcal{R}$  when  $A_1(0) > 1$ . Now when  $A_1(0) = 1$  then  $\lambda(F) = 0$  (by (6.80)), thus for each  $\pi \in \mathcal{A}$  we have  $\pi'(t)\sigma(t)\theta(t) \leq 0$  a.e., hence  $E[X^\pi(T)] \leq x_0S_0(T)$  (by (2.5)), hence  $\mathcal{R} \subset (-\infty, x_0S_0(T)]$ . In this latter case the market model is not “interesting” (in the sense of Remark 4.6), hence we shall suppose that  $A_1(0) > 1$  and fix some  $d > x_0S_0(T)$ . From Remark 6.3, (6.75), (6.76), (6.83) we have  $h(\mu) = \tilde{\Psi}_4(\mu, \bar{y}(\mu))$ ,  $\mu \in \mathbb{R}$ . Using (6.81), it is then easy (although tedious) to calculate that  $h(\cdot)$  has the unique minimizer  $\bar{\mu}$  given by  $\bar{\mu} = 2\beta^{-1}(T)[A_1(0) - 1]^{-1}[x_0 - \beta(T)A_1(0)d]$ , and that  $\bar{y}(\bar{\mu}) = 2\beta^{-2}(T)[A_1(0) - 1]^{-1}[\beta(T)d - x_0] > 0$ . From this, together with (6.82) and the fact that  $\text{Var}(\bar{X}(\bar{\mu}; T)) = \text{Var}(\hat{Y}(\bar{y}(\bar{\mu}); T))/4$  (see (6.84) with  $a = 2$ ), we compute the minimum variance or *efficient frontier*, namely

$$\inf_{\substack{\pi \in \mathcal{A} \\ E[X^\pi(T)] = d}} \text{Var}(X^\pi(T)) = \text{Var}(\bar{X}(\bar{\mu}; T)) = \frac{1}{4} \text{Var}(\hat{Y}(\bar{y}(\bar{\mu}); T)) = \frac{[x_0 - \beta(T)d]^2}{[A_1(0) - 1]\beta^2(T)}.$$

The optimal feedback policy is given by (6.85) with  $\mu := \bar{\mu}$  (since we have seen that  $\bar{y}(\bar{\mu}) > 0$ ), and is easy to implement, since only  $\zeta_1(\cdot)$  given by (6.78) need be “precalculated off-line” using the known deterministic coefficients  $r$ ,  $b$  and  $\sigma$ . The simplicity with which dynamic programming applies to the dual problem (for general conical constraints on the portfolio) should be contrasted with the technical complexity involved in applying dynamic programming directly to the primal problem, as in [13], for which the resulting Bellman equation is substantially more involved. As a consequence the analysis in [13] is very specific to the no-shorting constraint (where  $K$  is the positive orthant) and relies on the restriction  $b_n(t) > r(t)$ ,  $t \in [0, T]$ ,  $n = 1, \dots, N$  (see line following (2.2) in [13]). This restriction excludes the very natural possibility that interest rates may increase at some point in the investment interval, exceeding - one hopes only temporarily - the mean rates of return on some stocks, and also excludes the all-too-real possibility that some stocks might temporarily underperform over part of the investment horizon (in the sense that the mean return rate  $b_n(t)$  is less than the interest rate  $r(t)$  for some values of  $t$ ) but perform well in the remainder of the trading interval. The preceding duality analysis removes these restrictions and works for completely general conical constraints.

## 7 Utility Maximization

In this section we put aside the problem of mean-variance minimization of the previous sections and turn attention to problems of *utility maximization* with convex portfolio constraints.

Our goal is to demonstrate that the approach used for mean-variance minimization applies equally well to utility maximization, and thus constitutes a unified method for dealing with both of these preferences. The basic ingredients are the following:

**Condition 7.1.** Given (i) a market with information filtration (2.1), a bond with price  $\{S_0(t)\}$ , and  $N$  stocks with prices  $\{S_n(t)\}$ ,  $n = 1, 2, \dots, N$ , modelled as in Section 2 by the relations (2.2) and (2.3), and subject to Condition 2.1; (ii) a closed convex *portfolio constraint set*  $K \subset \mathbb{R}^N$  with  $0 \in K$ ; (iii) an initial fortune  $x_0 \in (0, \infty)$ ; (iv) a *utility function*  $U : (0, \infty) \rightarrow \mathbb{R}$ , which is of class  $C^1$ , strictly increasing, strictly concave, and (a)  $\lim_{x \rightarrow \infty} U(x) = \infty$ , (b)  $\lim_{x \rightarrow \infty} U^{(1)}(x) = 0$ , (c)  $\lim_{x \downarrow 0} U^{(1)}(x) = \infty$ , (d)  $\lim_{x \downarrow 0} U(x) > -\infty$ .

For an  $\{\mathcal{F}_t\}$ -progressively measurable  $\pi : \Omega \times [0, T] \rightarrow \mathbb{R}^N$  such that  $\int_0^T \|\pi(t)\|^2 dt < \infty$  a.s., let  $\{X^\pi(t), t \in [0, T]\}$  be the unique  $\mathbb{R}$ -valued, continuous,  $\{\mathcal{F}_t\}$ -progressively measurable process determined by

$$(7.86) \quad dX^\pi(t) = X^\pi(t) \{r(t) + \pi'(t)\sigma(t)\theta(t)\} dt + X^\pi(t)\pi'(t)\sigma(t) dW(t), \quad X^\pi(0) = x_0.$$

Then  $X^\pi$  is  $P$ -strictly positive, namely  $\inf_{t \in [0, T]} X^\pi(t) > 0$  a.s., and if  $\pi_n(t)$ , the  $n$ -th entry of  $\pi(t)$ , is interpreted as the *fraction* of a small investor's total wealth put into the stock with price  $S_n(t)$  (as is customary in problems of constrained utility maximization), then  $X^\pi(t)$  gives the investor's total wealth at instant  $t$ , provided that the investor follows a self-funded strategy (see [5], p.770). Define the set of *admissible portfolios*

$$(7.87) \quad \mathcal{A}' := \left\{ \pi : \Omega \times [0, T] \rightarrow \mathbb{R}^N \mid \pi \in \mathcal{F}^*, \pi(t) \in K \text{ a.e.}, \int_0^T \|\pi(t)\|^2 dt < \infty \text{ a.s.} \right\},$$

(recall  $\mathcal{F}^*$  defined in Section 3) and the *value of the portfolio optimization problem*

$$(7.88) \quad \vartheta := \sup_{\pi \in \mathcal{A}'} \mathbb{E}[U(X^\pi(T))].$$

To avoid trivialities, assume  $\vartheta \in \mathbb{R}$ . The utility maximization problem is:

$$(7.89) \quad \text{establish existence of some } \bar{\pi} \in \mathcal{A}' \text{ such that } \vartheta = \mathbb{E}[U(X^{\bar{\pi}}(T))].$$

**Remark 7.2.** In contrast to the problem of mean-variance minimization of the preceding sections, the optimal wealth processes  $X^\pi$  for the utility maximization problem is generally not square-integrable when  $\pi \in \mathcal{A}'$ , hence the set  $\mathbb{B}$  of Section 3 is not the appropriate one in which to embed the utility maximization problem. Instead, we introduce the set  $\mathbb{I}$  of all  $\{\mathcal{F}_t\}$ -Itô processes  $\{X(t), t \in [0, T]\}$  with the form (3.6) for some (a.e.-unique)  $\{\mathcal{F}_t\}$ -progressively measurable mappings  $\dot{X} : \Omega \times [0, T] \rightarrow \mathbb{R}$ , and  $\Lambda_X : \Omega \times [0, T] \rightarrow \mathbb{R}^N$ , such that  $\int_0^T |\dot{X}(t)| dt < +\infty$  and  $\int_0^T \|\Lambda_X(t)\|^2 dt < +\infty$  a.s. and write  $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{I}$  to indicate that  $X_0 \in \mathbb{R}$ ,  $\dot{X}$  and  $\Lambda_X$  satisfy these a.s. bounds, and (3.6) holds.

By analogy with (5.15), and recalling (7.86) and (7.87), put

$$(7.90) \quad \mathcal{C}(X) := \left\{ \pi \in \mathcal{A}' \mid \begin{array}{l} \dot{X}(t) = X(t) \{r(t) + \pi'(t)\sigma(t)\theta(t)\} \\ \text{and } \Lambda_X(t) = X(t)\sigma'(t)\pi(t) \text{ a.e.} \end{array} \right\},$$

for each  $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{I}$ . Then, by an argument identical to that giving (5.16), we have

$$(7.91) \quad \vartheta = \sup_{\substack{X \in \mathbb{I} \\ X_0 = x_0 \\ \mathcal{C}(X) \neq \emptyset}} \mathbb{E}[U(X(T))].$$

We now introduce penalty functions for the constraints in (7.91) (much as we did for (5.16)). From Remark 2.2, (7.87) and (7.90) we see that, for each  $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{I}$  with  $X_0 > 0$ ,

$$(7.92) \quad \mathcal{C}(X) \neq \emptyset \iff \begin{cases} \inf_{t \in [0, T]} X(t) > 0 \text{ a.s.,} & \dot{X}(t) = r(t)X(t) + \Lambda'_X(t)\theta(t) \\ \text{and } X^{-1}(t)[\sigma'(t)]^{-1}\Lambda_X(t) \in K & \text{a.e.} \end{cases}$$

**Remark 7.3.** If  $\mathcal{C}(X) \neq \emptyset$  for  $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{I}$  with  $X_0 > 0$ , then  $X^{-1}[\sigma']^{-1}\Lambda_X \in \mathcal{C}(X)$ .

Motivated by (7.92), define the mapping  $L : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \{0, \infty\}$  by

$$(7.93) \quad L(\omega, t, x, v, \xi) := \begin{cases} 0 & \text{if } x > 0, \quad v = r(\omega, t)x + \xi'\theta(\omega, t), \quad x^{-1}[\sigma'(\omega, t)]^{-1}\xi \in K; \\ \infty & \text{otherwise,} \end{cases}$$

(compare (5.18)). Next, define  $l_0(x)$  as at (5.20), put

$$(7.94) \quad l_T(x) := \begin{cases} -U(x) & \text{if } x \in (0, \infty); \\ \infty & \text{otherwise,} \end{cases}$$

and define the mappings  $m_0(\cdot)$ ,  $m_T(\cdot)$ , and  $M(\cdot)$  as at (5.24) (with  $l_0(\cdot)$ ,  $l_T(\cdot)$  and  $L(\cdot)$  given by (5.20), (7.94) and (7.93), suppressing  $\omega$  in the second relation of (5.24)). Then

$$(7.95) \quad m_0(y) = x_0 y, \quad m_T(y) = \tilde{U}(y) := \sup_{x > 0} [U(x) - xy], \quad y \in \mathbb{R},$$

and an easy calculation based on (7.93), (5.27), and (5.24), shows that

$$(7.96) \quad M(\omega, t, y, s, \gamma) := \begin{cases} 0 & \text{when } s + r(\omega, t)y + \delta(-\sigma(\omega, t)[\theta(\omega, t)y + \gamma]) \leq 0; \\ \infty & \text{otherwise,} \end{cases}$$

for each  $(\omega, t, y, s, \gamma) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ .

**Remark 7.4.** For  $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{I}$  and  $Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{I}$  we shall continue to use the notation  $\mathbb{M}(X, Y)(t)$  declared in the statement of Proposition 5.6, as well as the notation  $\Theta_Y(t)$  at (5.33). In this case, it follows at once from Itô's formula that  $\{(\mathbb{M}(X, Y)(t), \mathcal{F}_t), t \in [0, T]\}$  is a continuous *local* martingale with  $\mathbb{M}(X, Y)(0) = 0$ , but it is not necessarily a genuine martingale. It follows that we can no longer avail ourselves of Proposition 5.6 in constructing the dual problem (as we did in Proposition 5.7). In order to deal with this we define

$$(7.97) \quad \mathbb{I}_1 := \{X \in \mathbb{I} \mid X_0 = x_0, \mathcal{C}(X) \neq \emptyset\},$$

$$(7.98) \quad \mathbb{I}_2 := \{Y \in \mathbb{I} \mid \dot{Y}(t) + r(t)Y(t) + \delta(\Theta_Y(t)) \leq 0, Y(t) \geq 0 \text{ a.e.}\};$$

from (5.20), (7.92), (7.93) and (7.96), for each  $(X, Y) \in \mathbb{I}_1 \times \mathbb{I}_2$  we then have

$$(7.99) \quad l_0(X_0) = 0, \quad L(t, X(t), \dot{X}(t), \Lambda_X(t)) = 0 \quad \text{and} \quad M(t, Y(t), \dot{Y}(t), \Lambda_Y(t)) = 0 \quad \text{a.e.}$$

This, together with the third relation of (5.24), ensures that for each  $(X, Y) \in \mathbb{I}_1 \times \mathbb{I}_2$

$$(7.100) \quad \begin{aligned} 0 &= L(t, X(t), \dot{X}(t), \Lambda_X(t)) + M(t, Y(t), \dot{Y}(t), \Lambda_Y(t)) \\ &\geq X(t)\dot{Y}(t) + \dot{X}(t)Y(t) + \Lambda'_X(t)\Lambda_Y(t), \end{aligned} \quad \text{a.e.,}$$

and thus, we find  $\mathbb{M}(X, Y)(t) \geq X(t)Y(t) - X_0Y_0$  a.e., for each  $(X, Y) \in \mathbb{I}_1 \times \mathbb{I}_2$ . Since  $X(t)Y(t) \geq 0$  a.e. for each  $(X, Y) \in \mathbb{I}_1 \times \mathbb{I}_2$  (by (7.92), (7.97), (7.98)), it follows that  $\mathbb{M}(X, Y)(t) \geq -X_0Y_0$  a.e., and thus, from Remark 7.4 and Fatou's lemma, for  $(X, Y) \in \mathbb{I}_1 \times \mathbb{I}_2$

$$(7.101) \quad \{(\mathbb{M}(X, Y)(t), \mathcal{F}_t), t \in [0, T]\} \quad \text{is a supermartingale with } \mathbb{M}(X, Y)(0) = 0.$$

**Remark 7.5.** From now on we define  $\Phi(X)$  for each  $X \in \mathbb{I}_1$ , and  $\Psi(Y)$  for each  $Y \in \mathbb{I}_2$ , by (5.21) and (5.28), with  $l_0$ ,  $l_T$  and  $L$  given by (5.20), (7.94) and (7.93) respectively, and with  $m_0$ ,  $m_T$  and  $M$  given by (7.95) and (7.96) respectively. Then, from (7.99), we have

$$(7.102) \quad \Phi(X) = -\mathbb{E}[U(X(T))], \quad \Psi(Y) = x_0Y_0 + \mathbb{E}[\tilde{U}(Y(T))], \quad (X, Y) \in \mathbb{I}_1 \times \mathbb{I}_2.$$

**Proposition 7.6.** *Assume Condition 7.1 and  $\vartheta \in \mathbb{R}$  (see (7.88)). Then  $\Phi(X) > -\infty$  and  $\Psi(Y) > -\infty$  for each  $(X, Y) \in \mathbb{I}_1 \times \mathbb{I}_2$  (recall Remark 7.5), and*

$$(7.103) \quad \Phi(X) + \Psi(Y) \geq 0, \quad (X, Y) \in \mathbb{I}_1 \times \mathbb{I}_2.$$

Moreover, for arbitrary  $\bar{X} \equiv (\bar{X}_0, \dot{\bar{X}}, \Lambda_{\bar{X}}) \in \mathbb{I}_1$  and  $\bar{Y} \equiv (\bar{Y}_0, \dot{\bar{Y}}, \Lambda_{\bar{Y}}) \in \mathbb{I}_2$ , we have the equality  $\Phi(\bar{X}) + \Psi(\bar{Y}) = 0$  if and only if each of the following conditions hold:

$$(7.104) \quad \left\{ \begin{array}{l} (1) \quad l_0(\bar{X}_0) + m_0(\bar{Y}_0) = \bar{X}_0\bar{Y}_0, \\ (2) \quad l_T(\bar{X}(T)) + m_T(\bar{Y}(T)) = -\bar{X}(T)\bar{Y}(T) \quad \text{a.s.} \\ (3) \quad L(t, \bar{X}(t), \dot{\bar{X}}(t), \Lambda_{\bar{X}}(t)) + M(t, \bar{Y}(t), \dot{\bar{Y}}(t), \Lambda_{\bar{Y}}(t)) \\ \quad = \bar{X}(t)\dot{\bar{Y}}(t) + \dot{\bar{X}}(t)\bar{Y}(t) + \Lambda'_{\bar{X}}(t)\Lambda_{\bar{Y}}(t) \quad \text{a.e.} \\ (4) \quad \{\bar{X}(t)\bar{Y}(t), t \in [0, T]\} \quad \text{is a } \mathcal{F}_t\text{-martingale.} \end{array} \right.$$

*Proof.* Fix  $(X, Y) \in \mathbb{I}_1 \times \mathbb{I}_2$ . That  $\Phi(X) > -\infty$  and  $\Psi(Y) > -\infty$  is an immediate consequence of Condition 7.1(iv) and  $\vartheta \in \mathbb{R}$ . From the definitions of  $\Phi(X)$  and  $\Psi(Y)$  at Remark 7.5, we see that the chain of equalities and inequalities at (5.32) continues to hold. Moreover  $\mathbb{E}[-\mathbb{M}(X, Y)(T)] \geq 0$  (from (7.101)), and (7.103) follows from this and (5.32). Next, suppose  $\Phi(\bar{X}) + \Psi(\bar{Y}) = 0$  for some  $(\bar{X}, \bar{Y}) \in \mathbb{I}_1 \times \mathbb{I}_2$ . Then, since  $\mathbb{E}[-\mathbb{M}(\bar{X}, \bar{Y})(T)] \geq 0$  (from (7.101)), the inequality in (5.32) must be an equality (with  $(\bar{X}, \bar{Y})$  in place of  $(X, Y)$ ) and  $\mathbb{E}[\mathbb{M}(\bar{X}, \bar{Y})(T)] = 0$ . Now (7.104)(1) - (3) follows from this and the general relations (5.31), and it follows from (7.101) that  $\{\mathbb{M}(\bar{X}, \bar{Y})(t)\}$  is actually a  $\{\mathcal{F}_t\}$ -martingale (being a supermartingale with constant expectation). But (7.104)(3), together with (7.99), gives  $\mathbb{M}(\bar{X}, \bar{Y})(t) = \bar{X}(t)\bar{Y}(t) - \bar{X}_0\bar{Y}_0$ , which establishes (7.104)(4). The converse, that (7.104)(1) - (4) implies  $\Phi(\bar{X}) + \Psi(\bar{Y}) = 0$ , is immediate from (7.99) and (5.32).  $\square$

**Remark 7.7.** Condition 7.1 ensures that  $\tilde{U}(\cdot)$  is smooth, and the derivative  $U^{(1)}(\cdot)$  has a continuous strictly decreasing inverse  $I : (0, \infty) \rightarrow (0, \infty)$  with  $I(y) = -\tilde{U}^{(1)}(y)$ ,  $y \in (0, \infty)$  (see [9], Lemma 3.4.3, p.96). From (7.94) and (7.95), for each  $x, y \in \mathbb{R}$ , we have the equivalence  $l_T(x) + m_T(y) = -xy \iff y \in (0, \infty)$  and  $x = I(y) \in (0, \infty)$ . Similarly, from (5.20) and (7.95), for each  $x, y \in \mathbb{R}$  we have  $l_0(x) + m_0(y) = xy \iff x = x_0$ .

**Proposition 7.8.** *Suppose the conditions of Proposition 7.6. Then, for arbitrary  $(\bar{X}, \bar{Y}) \in \mathbb{I}_1 \times \mathbb{I}_2$ , we have (recalling  $\Theta_Y(\cdot)$ ,  $\vartheta$ , and  $\delta(\cdot)$  defined at Remark 7.4, (7.88), and (5.27))*

$$(7.105) \quad E[U(\bar{X}(T))] = \vartheta = \inf_{Y \in \mathbb{I}_2} \Psi(Y) = \Psi(\bar{Y}),$$

if and only if

$$(7.106) \quad \left\{ \begin{array}{l} (1') \quad \bar{X}_0 = x_0; \\ (2') \quad \bar{Y}(T) > 0 \quad \text{and} \quad \bar{X}(T) = I(\bar{Y}(T)) > 0 \quad \text{a.s.}; \\ (3') \quad \dot{\bar{Y}}(t) + r(t)\bar{Y}(t) + \delta(\Theta_{\bar{Y}}(t)) = 0 \quad \text{a.e.}; \\ (4') \quad \bar{\pi} \in \mathcal{C}(\bar{X}) \quad \text{and} \quad \delta(\Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) = 0 \quad \text{a.e.} \\ \quad \quad \quad \text{for } \bar{\pi}(t) := \bar{X}^{-1}(t)[\sigma'(t)]^{-1}\Lambda_{\bar{X}}(t); \\ (5') \quad \{\bar{X}(t)\bar{Y}(t), t \in [0, T]\} \text{ is a } \{\mathcal{F}_t\} \text{-martingale.} \end{array} \right.$$

*Proof.* In view of (5.27), (7.93), and (7.96), we have the following equivalence: for arbitrary  $(\omega, t) \in \Omega \times [0, T]$ ,  $x, y, v, s \in \mathbb{R}$ ,  $\xi, \gamma \in \mathbb{R}^N$ ,

$$(7.107) \quad \begin{aligned} L(\omega, t, x, v, \xi) + M(\omega, t, y, s, \gamma) &= xs + yv + \xi'\gamma \\ \iff x > 0, \quad v = r(t)x + \xi'\theta(t), \quad x^{-1}[\sigma'(t)]^{-1}\xi &\in K, \\ s + yr(t) + \delta(-\sigma(t)[\theta(t)y + \gamma]) &= 0, \\ \text{and } \delta(-\sigma(t)[\theta(t)y + \gamma]) + x^{-1}\xi'\sigma^{-1}(t)(-\sigma(t)[\theta(t)y + \gamma]) &= 0. \end{aligned}$$

Fix arbitrary  $(\bar{X}, \bar{Y}) \in \mathbb{I}_1 \times \mathbb{I}_2$ . In view of (7.107), (7.97), (7.92), Remark 7.7 and Remark 7.3, we find that (7.104)(1) - (4) is equivalent to (7.106)(1') - (5'). Moreover, from (7.91), (7.97), and Remark 7.5, we have  $\vartheta = \sup_{X \in \mathbb{I}_1} \{-\Phi(X)\}$ , and thus the condition  $\Phi(\bar{X}) + \Psi(\bar{Y}) = 0$  is equivalent to (7.105), as follows from the weak duality (7.103). The equivalence of (7.105) and (7.106) now follows from Proposition 7.6.  $\square$

**Remark 7.9.** It follows from Proposition 7.8 that solving problem (7.89) requires constructing a pair  $(\bar{X}, \bar{Y}) \in \mathbb{I}_1 \times \mathbb{I}_2$  which satisfies the relations (7.106), for then the optimal portfolio  $\bar{\pi}$  is given in terms of  $\bar{X}$  by (7.106)(4'). To this end we observe the following:

(a) If  $(\bar{X}, \bar{Y}) \in \mathbb{I}_1 \times \mathbb{I}_2$  is a pair satisfying (7.106), then  $\bar{Y}$  is necessarily  $P$ -strictly positive (since (7.106)(2') ensures  $\bar{X}(T)\bar{Y}(T) > 0$  a.s., hence it follows from (7.106)(5') and ([10], 1.3.29, p.21) that  $\bar{X}\bar{Y}$  is a  $P$ -strictly positive process, while (7.106)(4') and (7.92) ensure that the process  $\bar{X}$  is  $P$ -strictly positive). This fact, together with (7.106)(3'), says that we need minimize  $\Psi(\cdot)$ , not over all of  $\mathbb{I}_2$ , but instead over the smaller set  $\mathbb{I}_3 \subset \mathbb{I}_2$  defined by

$$(7.108) \quad \mathbb{I}_3 := \{Y \in \mathbb{I} \mid \inf_{t \in [0, T]} Y(t) > 0 \quad \text{a.s.} \quad \text{and} \quad \dot{Y}(t) + r(t)Y(t) + \delta(\Theta_Y(t)) = 0 \quad \text{a.e.}\}.$$

The advantage of minimizing over  $\mathbb{I}_3$  is that  $Y \in \mathbb{I}_3$  are exponential semimartingales. In fact, with  $\mathcal{Q} := \{\nu : \Omega \times [0, T] \rightarrow \mathbb{R}^N \mid \nu \in \mathcal{F}^* \text{ and } \int_0^T [\|\nu(t)\|^2 + \delta(\nu(t))] dt < \infty \text{ a.s.}\}$ , and

$$(7.109) \quad H_\nu(t) := \exp \left[ - \int_0^t \{r(\tau) + \delta(\nu(\tau))\} d\tau \right] \mathcal{E}(-[\theta + \sigma^{-1}\nu]' \bullet W)(t), \quad \nu \in \mathcal{Q},$$

(see Remark 5.14), it follows easily from Itô's formula that  $\mathbb{I}_3 = \{yH_\nu \mid y \in (0, \infty), \nu \in \mathcal{Q}\}$ .

(b) Now minimization of  $\Psi(\cdot)$  over  $\mathbb{I}_3$  is still difficult because the set  $\mathcal{Q}$  is "very large". Accordingly, we restrict attention to  $\nu \in \mathcal{D} := \{\nu \in \mathcal{Q} \mid \mathbb{E} \int_0^T \|\nu(t)\|^2 dt < \infty\}$  (i.e. *square integrable* members of  $\mathcal{Q}$ ), and minimize  $\Psi(\cdot)$  over  $\mathbb{I}_4 := \{yH_\nu \mid y \in (0, \infty), \nu \in \mathcal{D}\} \subset \mathbb{I}_3$ . That is, we shall establish  $\Psi(\bar{Y}) = \inf_{Y \in \mathbb{I}_4} \Psi(Y)$  for some  $\bar{Y} \in \mathbb{I}_4$ . To this end, in addition to Condition 7.1 we must assume:

- $x \mapsto xU^{(1)}(x)$  is nondecreasing on  $(0, \infty)$ ;
- there exists  $\gamma \in (1, \infty)$ ,  $\alpha \in (0, 1)$  such that  $\alpha U^{(1)}(x) \geq U^{(1)}(\gamma x)$  for all  $x \in (0, \infty)$ ;
- for each  $y \in (0, \infty)$  there is some  $\nu \in \mathcal{D}$  such that  $\mathbb{E}[\tilde{U}(yH_\nu(T))] < \infty$ .

Using these conditions and Proposition II.1.2 of ([7], p.35) - which relies on the square integrability of  $\nu \in \mathcal{D}$  - together with a trivial variant of the proof of ([5], Proposition 13.2, p.795) we see that there exists  $(\bar{y}, \bar{\nu}) \in (0, \infty) \times \mathcal{D}$  such that  $\inf_{Y \in \mathbb{I}_4} \Psi(Y) = \Psi(\bar{Y})$  for  $\bar{Y} := \bar{y}H_{\bar{\nu}} \in \mathbb{I}_4$  (this is analogous to Proposition 5.12). It remains to construct an  $\bar{X} \in \mathbb{I}_1$  in terms of this  $\bar{Y}$  such that (7.106) holds. Motivated by (7.106)(2')(5'), define

$$(7.110) \quad \bar{X}(t) := \bar{Y}^{-1}(t) \mathbb{E}[\bar{Y}(T)I(\bar{Y}(T)) \mid \mathcal{F}_t] = H_{\bar{\nu}}^{-1}(t) \mathbb{E}[H_{\bar{\nu}}(T)I(\bar{y}H_{\bar{\nu}}(T)) \mid \mathcal{F}_t].$$

From (7.110) we have  $\bar{X}(t)H_{\bar{\nu}}(t) = \bar{X}(0) + \int_0^t \bar{\psi}'(\tau) dW(\tau) := \xi_0(t)$ , for some  $\mathbb{R}^N$ -valued and a.e.-unique  $\bar{\psi} \in \mathcal{F}^*$ , with  $\int_0^T \|\bar{\psi}(t)\|^2 dt < \infty$  a.s. (by the martingale representation theorem). Using this, together with (7.109), to expand the quotient  $\bar{X}(t) = \xi_0(t)/H_{\bar{\nu}}(t)$  by Itô's formula then gives  $\bar{X} \in \mathbb{I}$  (recall Remark 7.2) with  $\{\dot{\bar{X}}(t)\}$  and  $\{\Lambda_{\bar{X}}(t)\}$  given by

$$(7.111) \quad d\bar{X}(t) = \bar{X}(t)\{r(t) + \bar{\pi}'(t)\sigma(t)\theta(t) + \delta(\bar{\nu}(t)) + \bar{\pi}'(t)\bar{\nu}(t)\} dt + \bar{X}(t)\bar{\pi}'(t)\sigma(t) dW(t),$$

for  $\bar{\pi}(t) := [\sigma'(t)]^{-1}[H_{\bar{\nu}}^{-1}(t)\bar{X}^{-1}(t)\bar{\psi}(t) + \{\theta(t) + \sigma^{-1}(t)\bar{\nu}(t)\}]$  (compare (5.48), (5.49), (5.50)).

(c) From (7.110) and (7.108) we see that the pair  $(\bar{X}, \bar{Y}) \in \mathbb{I} \times \mathbb{I}_4$  satisfies (7.106)(2')(3')(5'). It remains to show that  $\bar{X} \in \mathbb{I}_1$  and that (7.106)(1')(4') hold. To this end we use *necessary conditions* resulting from the optimality of  $\bar{Y} \equiv \bar{y}H_{\bar{\nu}}$  established in (b). From this optimality, together with (7.102), we find that  $x_0y + \mathbb{E}[\tilde{U}(yH_\nu(T))] \geq x_0\bar{y} + \mathbb{E}[\tilde{U}(\bar{y}H_{\bar{\nu}}(T))]$  for all  $(y, \nu) \in (0, \infty) \times \mathcal{D}$ . In particular,  $\bar{y}$  minimizes the function  $y \rightarrow x_0y + \mathbb{E}[\tilde{U}(yH_{\bar{\nu}}(T))]$ , and upon taking the derivative in  $y$  and using the identity  $\tilde{U}^{(1)}(y) = -I(y)$  (see Remark 7.7), we obtain  $x_0 = \mathbb{E}[H_{\bar{\nu}}(T)I(\bar{y}H_{\bar{\nu}}(T))] = \bar{X}(0)$ , the second equality following from (7.110); this verifies (7.106)(1'). Again, from the preceding optimality, we obtain  $\mathbb{E}[\tilde{U}(\bar{y}H_\nu(T))] \geq \mathbb{E}[\tilde{U}(\bar{y}H_{\bar{\nu}}(T))]$  for all  $\nu \in \mathcal{D}$ , that is  $(1/\epsilon)\mathbb{E}[\tilde{U}(\bar{y}H_{\bar{\nu}+\epsilon\eta}(T)) - \tilde{U}(\bar{y}H_{\bar{\nu}}(T))] \geq 0$  for each  $\epsilon \in (0, 1)$  and  $\eta \in \mathcal{D}$ . Evaluating the limit as  $\epsilon \rightarrow 0$  for suitable choices of  $\eta \in \mathcal{D}$  (by a calculation essentially identical to that in ([5], pp.781 - 783)) we obtain  $\bar{\pi}(t) \in K$  and  $\delta(\bar{\nu}(t)) + \bar{\pi}'(t)\bar{\nu}(t) = 0$  a.e. It follows from this, together with (7.111) and (7.90), that  $\bar{\pi} \in \mathcal{C}(\bar{X})$ , thus  $\bar{X} \in \mathbb{I}_1$ . Moreover, it is clear that  $\Theta_{\bar{Y}}(t) = \bar{Y}(t)\bar{\nu}(t)$  for  $\bar{Y} = \bar{y}H_{\bar{\nu}}$ , thus  $\delta(\Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) = 0$  a.e. (since  $\bar{Y}(t) > 0$  and  $\delta(\cdot)$  is positively homogeneous), as required to verify (7.106)(4').

**Remark 7.10.** To focus on just the essential ideas we have considered maximization of utility from terminal wealth only, without utility from consumption. A straightforward modification of the the preceding approach suffices to include intertemporal consumption (see [11]).

**Remark 7.11.** The approach used in the present section for problems of constrained utility-maximization contrasts with that of ([4], [5], [8]) which rely on the *a-priori* introduction of a complete *fictitious market*, in which the money-market rate and mean return-rate on stocks are such that *unconstrained* utility maximization in the fictitious market amounts to *constrained* utility maximization in the given market. The approach of the present section avoids fictitious markets (the formulation of which is not at all simple), and proceeds algorithmically by relying on elementary convex analysis to synthesize optimality relations (see (7.106)), the solution of which yields the optimal portfolio. The same approach establishes existence of optimal portfolios for problems of mean-variance minimization. In cases of genuine practical interest this optimal portfolio is explicitly computable (see Example 6.6). Finally, while methods based on the introduction of a complete fictitious market are undoubtedly effective for problems of constrained utility maximization, provided that one can find the “correct” fictitious market (always a significant challenge), this approach does not appear to adapt easily to constrained mean-variance minimization. On the other hand, methods of stochastic Riccati equations and stochastic LQ control, which are the preferred mathematical technology for problems of mean-variance minimization, are unlikely to be appropriate for preferences based on utility maximization, relying as they do in an essential way on the quadratic form of the loss function. In contrast, the approach of the present work applies with equal facility to both of the two main preference structures of utility maximization and mean-variance reduction, and deals easily with portfolio constraints.

**Example 7.12.** We add an induced constraint on the terminal wealth to the problem (7.89). Fix some  $\alpha \in [0, 1)$  and define  $\zeta := \alpha x_0 S_0(T)$ . Put  $\mathcal{A}^* := \{\pi \in \mathcal{A}' \mid X^\pi(T) \geq \zeta \text{ a.s.}\}$  (recall (7.87)) and  $\vartheta_1 := \sup_{\pi \in \mathcal{A}^*} \mathbb{E}[U(X^\pi(T))]$ . The problem of utility maximization that we study is to establish existence of some  $\bar{\pi} \in \mathcal{A}^*$  such that  $\vartheta_1 = \mathbb{E}[U(X^{\bar{\pi}}(T))]$ . This problem could represent the preference of a cautious investor, whose goal is to maximize expected utility from terminal wealth subject to the usual portfolio constraints, while also insisting that the terminal wealth not be less than the fortune  $\zeta$  that would have been obtained by just investing some fraction  $\alpha$  of the initial wealth  $x_0$  in a money-market account. While it is difficult to use the method of fictitious markets to establish optimality relations for this problem, we shall see that the approach of the present section applies quite easily. Define a modified utility by  $U_1(\omega, x) := U(x)$  when  $x \geq \zeta(\omega)$ , and  $U_1(\omega, x) := -\infty$  otherwise, and put  $\tilde{U}_1(\omega, y) := \sup_{x > 0} [U_1(\omega, x) - xy]$  and  $I_1(\omega, y) := -\tilde{U}_1^{(1)}(\omega, y)$  for all  $(\omega, y) \in \Omega \times (0, \infty)$ . Then, exactly as at (7.91), we get  $\vartheta_1 = \sup\{\mathbb{E}[U_1(X(T))] : X \in \mathbb{I}, X_0 = x_0, \mathcal{C}(X) \neq \emptyset\}$ . Finally, by analogy with (7.102), put  $\Phi_1(X) := -\mathbb{E}[U_1(X(T))]$  for each  $X \in \mathbb{I}_1$ , and  $\Psi_1(Y) := x_0 Y_0 + \mathbb{E}[\tilde{U}_1(Y(T))]$  for each  $Y \in \mathbb{I}_{2,1} := \{Y \in \mathbb{I}_2 \mid \mathbb{E}[Y(T)] < \infty\}$ . One can now repeat the analysis which led to Proposition 7.8, but using the utility function  $U_1$  in place of  $U$ , to obtain the following: for arbitrary  $(\bar{X}, \bar{Y}) \in \mathbb{I}_1 \times \mathbb{I}_{2,1}$  we have  $\mathbb{E}[U_1(\bar{X}(T))] = \vartheta_1 = \inf_{Y \in \mathbb{I}_{2,1}} \Psi_1(Y) = \Psi_1(\bar{Y})$  if and only if the Euler-Lagrange and transversality relations (7.106)(1')(3')(4')(5') and

$$(7.112) \quad \bar{Y}(T) > 0 \quad \text{and} \quad \bar{X}(T) = I_1(\omega, \bar{Y}(T)) \quad \text{a.s.}$$

hold (that is, we have (7.112) in place of the transversality relation (7.106)(2')). It is now necessary to resolve these relations, exactly as in Remark 7.9. For the sake of simplicity we look at the special case of  $K = \mathbb{R}^N$ . Then, just as for the Example 6.5, the dual problem reduces to minimization of the functional  $y \rightarrow \Psi_1(yH) : (0, \infty) \rightarrow \mathbb{R}$  (recall (5.46)). It is clear from the definition of  $\tilde{U}$  that  $\lim_{y \downarrow 0} \Psi_1(yH) = +\infty$ , and, since  $\alpha < 1$ , it is easily verified that  $\lim_{y \rightarrow \infty} \Psi_1(yH) = +\infty$ , so the existence of a minimizing  $\bar{y} \in (0, \infty)$  follows. Now define  $\bar{Y}(t) := \bar{y}H(T)$  and  $\bar{X}(t) := H^{-1}(t)E[H(T)I_1(\bar{y}H(T)) | \mathcal{F}_t]$ . Then it is clear that  $(\bar{X}, \bar{Y}) \in \mathbb{I}_1 \times \mathbb{I}_{2,1}$ , and, just as in Remark 7.9, it can be established that this pair verifies the relations (7.106)(1')(3')(4')(5') and (7.112). Now it follows that  $\bar{\pi}$  defined in terms of  $\bar{X}$  by (7.106)(4') is the optimal portfolio. When the constraint on the terminal wealth binds, one can use the fact that  $\alpha < 1$  to relax the dual problem (by essentially following the approach of Dubovitskii and Mil'yutin [6]) in order to establish existence of a Lagrange multiplier in  $L^\infty(\Omega, \mathcal{F}, P)^*$  (the topological dual of  $L^\infty(\Omega, \mathcal{F}, P)$ ) which enforces the terminal wealth constraint.

## References

- [1] J-P. AUBIN, *Applied Functional Analysis*, Wiley, New York, (1978).
- [2] J.M. BISMUT, Conjugate convex functions in optimal stochastic control, *J. Math. Analysis Appl.*, pp. 384–404, v.44, (1973).
- [3] Y.S. CHOW AND H. TEICHER, *Probability Theory: Independence, Interchangeability, Martingales*, 2nd Ed., Springer-Verlag, New York (1988).
- [4] D. CUOCO AND H. LIU, A martingale characterization of consumption choices and hedging costs with margin requirements, *Mathematical Finance*, pp. 355–385, v.10, (2000).
- [5] J. CVITANIĆ AND I. KARATZAS, Convex duality in constrained portfolio optimization, *Annals Appl. Probability*, pp. 767–818, v.2, (1992).
- [6] A. YA. DUBOVITSKII AND A. A. MIL'YUTIN, Necessary conditions for a weak extremum in problems of optimal control with mixed inequality constraints, *Zhur. Vychislitel. Mat. i Mat. Fys.*, pp. 725–779, v.8 (1968).
- [7] I. EKELAND AND R. TÉMAM, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, (1976) (reprinted by SIAM as *Classics in Applied Mathematics*, no. 88).
- [8] I. KARATZAS, J.P. LEHOCZKY, S.E. SHREVE, AND G.L. XU, Martingale and duality methods for utility maximization in an incomplete market, *SIAM J. Control and Optimization*, pp. 702–730, v.29, (1991).
- [9] I. KARATZAS AND S.E. SHREVE, *Methods of Mathematical Finance*, Springer-Verlag, New York, (1998).
- [10] I. KARATZAS AND S.E. SHREVE, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York, (1988).

- [11] C. LABBÉ, Contributions to the theory of constrained portfolio optimization, PhD thesis, Department of Statistics and Actuarial Sciences, University of Waterloo (2004).
- [12] A.E.B. LIM AND X.Y. ZHOU, Mean-variance portfolio selection with random parameters in a complete market, *Math. Operations Research*, pp. 101–120, v.27, (2002).
- [13] X. LI, X.Y. ZHOU AND A.E.B. LIM, Dynamic mean-variance portfolio selection with no-shorting constraints, *SIAM J. Control and Optimization*, pp. 1540–1555, v.40, (2002).
- [14] L.C.G. ROGERS, Duality in constrained optimal investment and consumption problems: a synthesis, (Paris-Princeton Lectures on Mathematical Finance 2002), *Springer-Verlag Lecture Notes in Mathematics*, pp. 95–131, no. 1814, (2003).
- [15] R.B. VINTER, *Optimal Control*, Birkhauser, Boston (2000).
- [16] J. YONG AND X.Y. ZHOU, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York (1999).