

CONVEX DUALITY IN CONSTRAINED MEAN-VARIANCE PORTFOLIO OPTIMIZATION

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Summary

We apply conjugate duality to establish existence of optimal portfolios in an asset-allocation problem, with the goal of minimizing the variance of the final wealth which results from trading over a fixed finite horizon in a continuous-time complete market, subject to the constraints that the expected final wealth equal a specified target value, and the portfolio of the investor, defined by the dollar amount invested in each stock, takes values in a given closed convex set. The asset prices are modelled by Itô processes, for which the market parameters are random processes adapted to the information filtration available to the investor. We synthesize a dual optimization problem and establish a set of optimality relations, similar to the Euler-Lagrange and transversality relations of calculus of variations, giving necessary and sufficient conditions for the given optimization problem and its dual to each have a solution, with zero duality gap. We then resolve these relations to establish existence of an optimal portfolio.

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1 Introduction

In this work we study an asset-allocation problem, the goal of which is to minimize the variance of the final wealth which results from trading on a fixed finite horizon in a continuous-time complete market with random market parameters, subject to the constraints that the expected final wealth equal a specified target value and the portfolio of the investor (defined by the dollar amount in each stock) always takes values in a given closed convex set; this constraint is general enough to model a prohibition on short-selling of stock, incomplete markets, limits on the dollar amount allocated to each stock, and other trading restrictions. Our goal is to establish existence of an optimal portfolio and characterize it.

Problems of this kind belong to the general area of *mean-variance portfolio selection*, and their financial relevance, as compared with the more common objective of maximizing expected utility, has been discussed by Lim and Zhou [9] and Li, Zhou and Lim [10]. In fact [9] addresses this problem, but for *unconstrained* portfolios, using the methods of stochastic LQ control. The follow-up work [10] deals with a similar problem, but includes a no-short selling constraint; it is postulated that the market coefficients are *nonrandom*, and viscosity solutions of the (correspondingly nonrandom) Bellman equation are used to characterize the constrained optimal portfolio. The problem of interest here involves a combination of both random market parameters *and* general portfolio constraints. This rules out application of stochastic LQ theory, as in [9] (which relies on the absence of portfolio constraints), as well as the approach of [10] (for which the market parameters must be nonrandom).

In light of the preceding we turn to the use of *conjugate duality*. The goal is to formulate an associated “dual” optimization problem for which it is (hopefully) easy to directly establish existence of a solution, and then to construct an optimal portfolio in terms of the solution of the dual problem. Our approach is motivated by a recent work of Rogers [11], in which the central idea is to regard the dynamical relation satisfied by the wealth and the portfolio as itself defining a constraint, a point of view which then provides the key for synthesizing a dual optimization problem. We cannot in fact directly apply the method of Rogers [11], since this work does not address the problem of existence of optimal portfolios, but nevertheless the fundamental viewpoint of [11], namely that the wealth equation is a constraint, is essential to us. We shall account for this constraint in a way which is suggested by a work of Bismut [2] on stochastic convex control problems. The basic idea is to remove the portfolio “variable” to obtain a Bolza problem in the (stochastic) calculus of variations which amounts to minimization of a convex functional over a set of Itô processes large enough to include all of the possible wealth processes. Bismut [2] establishes a powerful duality theory for dealing with such stochastic Bolza problems, which we shall use to construct a dual optimization problem, together with optimality relations (similar to the Euler-Lagrange and transversality relations of calculus of variations) which are equivalent to the primal and dual problems being solvable with zero duality gap. We then use these relations to establish existence of an optimal portfolio and corresponding wealth process.

In Sections 2 to 4 we introduce the market model and formulate the problem of constrained mean-variance portfolio selection, and, in Sections 5 and 6, we use conjugate duality to construct the optimal portfolio and wealth process.

2 Market Model

Throughout the sequel $T \in (0, \infty)$ is a given constant, and $\{W(t), t \in [0, T]\}$ is a given \mathbb{R}^N -valued standard Brownian motion on the complete probability space (Ω, \mathcal{F}, P) . Put

$$(2.1) \quad \mathcal{F}_t := \sigma\{W(\tau), \tau \in [0, t]\} \vee \mathcal{N}(P),$$

in which $\mathcal{N}(P)$ denotes the collection of all P -null events in (Ω, \mathcal{F}, P) . We consider a market comprising $N + 1$ assets traded continuously on the interval $[0, T]$, namely a *bond* with price $\{S_0(t)\}$ given by

$$(2.2) \quad dS_0(t) = r(t)S_0(t) dt, \quad 0 \leq t \leq T, \quad S_0(0) = 1,$$

and N *stocks* with prices $\{S_n(t)\}$, $n = 1, 2, \dots, N$, given by

$$(2.3) \quad dS_n(t) = S_n(t) \left[b_n(t) dt + \sum_{m=1}^N \sigma_{nm}(t) dW_m(t) \right], \quad 0 \leq t \leq T,$$

the initial values $S_n(0)$ being given strictly positive constants. We shall always postulate

Condition 2.1. In (2.2) and (2.3) the *interest rate* $\{r(t)\}$, the entries $\{b_n(t)\}$ of the \mathbb{R}^N -valued process $\{b(t)\}$ of *mean rates of return on stocks*, and the entries $\{\sigma_{nm}(t)\}$ of the $N \times N$ matrix-valued *volatility process* $\{\sigma(t)\}$ are *uniformly bounded* and $\{\mathcal{F}_t\}$ -progressively measurable scalar processes on $\Omega \times [0, T]$, and $\{r(t)\}$ is non-negative. There is a constant $\kappa \in (0, \infty)$ such that $z' \sigma(\omega, t) \sigma'(\omega, t) z \geq \kappa \|z\|^2$ for all $(z, \omega, t) \in \mathbb{R}^N \times \Omega \times [0, T]$.

Remark 2.2. In view of Condition 2.1 and Karatzas and Shreve ([8], 5.8.1, p.372), there exists a constant $\kappa_1 \in (0, \infty)$ such that $\max\{\|(\sigma(\omega, t))^{-1}z\|, \|(\sigma'(\omega, t))^{-1}z\|\} \leq \kappa_1 \|z\|$ for all $(z, \omega, t) \in \mathbb{R}^N \times \Omega \times [0, T]$. This bound will often be used.

Remark 2.3. Define the usual *market price of risk* $\theta(t) := (\sigma(t))^{-1}[b(t) - r(t)\mathbf{1}]$, in which $\mathbf{1} \in \mathbb{R}^N$ has all unit entries. From Condition 2.1 and Remark 2.2, we see that $\{\theta(t)\}$ is uniformly bounded on $\Omega \times [0, T]$.

Given some $x_0 \in \mathbb{R}$, and some $\{\mathcal{F}_t\}$ -progressively measurable process $\pi : \Omega \times [0, T] \rightarrow \mathbb{R}^N$ satisfying $\int_0^T \|\pi(t)\|^2 dt < \infty$ a.s., it follows that there exists a scalar-valued, continuous, and $\{\mathcal{F}_t\}$ -progressively measurable process $\{X^\pi(t), t \in [0, T]\}$ such that

$$(2.4) \quad dX^\pi(t) = \{r(t)X^\pi(t) + \pi'(t)\sigma(t)\theta(t)\} dt + \pi'(t)\sigma(t) dW(t), \quad X^\pi(0) = x_0,$$

which is unique (to within indistinguishability) and given by

$$(2.5) \quad X^\pi(t) = S_0(t) \left\{ x_0 + \int_0^t S_0^{-1}(\tau)\pi'(\tau)\sigma(\tau)\theta(\tau) d\tau + \int_0^t S_0^{-1}(\tau)\pi'(\tau)\sigma(\tau) dW(\tau) \right\}.$$

From now on we consider a small investor who trades in the market following a *self-funded strategy* from a given initial wealth $x_0 \in (0, \infty)$. If $\pi_n(t)$, the n -th entry of the \mathbb{R}^N -valued vector $\pi(t)$, is interpreted as the *dollar amount* invested in the stock with price $S_n(t)$, $n = 1, 2, \dots, N$, then it follows from (2.2), (2.3) and Remark 1.3.3 of Karatzas and Shreve ([7], p.10), that $X^\pi(t)$ gives the investor's wealth at instant $t \in [0, T]$.

3 A Class of Square-Integrable Itô Processes

We formulate the optimization problem in the following section, but must first define a class of square-integrable Itô processes which will be essential in all later developments.

Write \mathcal{F}^* for the $\{\mathcal{F}_t\}$ -progressively measurable σ -algebra on $\Omega \times [0, T]$. The measure space $(\Omega \times [0, T], \mathcal{F}^*, (P \otimes \lambda))$, where λ stands for the Lebesgue measure (on the Borel σ -algebra on $[0, T]$), is used extensively, and the qualifier “a.e.” always refers to the measure $(P \otimes \lambda)$ on $\Omega \times [0, T]$. For example, if π is an \mathbb{R}^N -valued \mathcal{F}^* -measurable mapping on $\Omega \times [0, T]$ and $K \subset \mathbb{R}^N$, then $\pi(t) \in K$ a.e. means that $\pi(\omega, t) \in K$ for $(P \otimes \lambda)$ -almost all (ω, t) . Similarly, the qualifier “a.s.” is always with reference to the probability P on \mathcal{F} . For any mapping ξ on $\Omega \times [0, T]$ with values in some Euclidean space (the dimensionality of which will be clear from the context) we write $\xi \in \mathcal{F}^*$ to indicate that ξ is \mathcal{F}^* -measurable. Motivated by Bismut ([2], p.386, p.390), put

$$\begin{aligned} L_{21} &:= \left\{ v : \Omega \times [0, T] \rightarrow \mathbb{R} \mid v \in \mathcal{F}^* \text{ and } E \left(\int_0^T |v(t)| dt \right)^2 < \infty \right\}, \\ L_{22} &:= \left\{ \xi : \Omega \times [0, T] \rightarrow \mathbb{R}^N \mid \xi \in \mathcal{F}^* \text{ and } E \int_0^T \|\xi(t)\|^2 dt < \infty \right\}, \\ \mathbb{B} &:= \mathbb{R} \times L_{21} \times L_{22}, \end{aligned}$$

in which $\|\xi\|$ denotes the usual Euclidean length of $\xi \in \mathbb{R}^N$. Write $X \in \mathbb{B}$ to indicate that $\{(X(t), \mathcal{F}_t), t \in [0, T]\}$ is a continuous semimartingale of the form

$$(3.6) \quad X(t) = X_0 + \int_0^t \dot{X}(\tau) d\tau + \int_0^t \Lambda'_X(\tau) dW(\tau),$$

for some $(X_0, \dot{X}, \Lambda_X) \in \mathbb{B}$, and write $X \equiv (X_0, \dot{X}, \Lambda_X)$ to indicate that (3.6) holds. In the expansion (3.6) it is clear that the integrands \dot{X} and Λ_X are uniquely determined a.e. on $\Omega \times [0, T]$. The set \mathbb{B} is essentially the collection of all *square-integrable* Itô processes with respect to the Brownian motion $\{W(t)\}$. From Doob's L^2 -inequality we immediately have

$$(3.7) \quad E \left[\sup_{0 \leq t \leq T} |X(t)|^2 \right] < \infty, \quad \text{for each } X \in \mathbb{B}.$$

Note from (2.4) that X^π , for any given \mathbb{R}^N -valued $\pi \in \mathcal{F}^*$ for which the stochastic integration is defined, is an Itô process with respect to the Brownian motion $\{W(t)\}$. The next result gives conditions on π for membership of X^π in \mathbb{B} . The proof is elementary and is omitted.

Proposition 3.1. *Assume Condition 2.1 and suppose that $\pi : \Omega \times [0, T] \rightarrow \mathbb{R}^N$ is \mathcal{F}^* -measurable and $\int_0^T \|\pi(t)\|^2 dt < \infty$ a.s. Then $X^\pi \in \mathbb{B}$ if and only if $\pi \in L_{22}$.*

4 The Optimization Problem

In order to formulate the optimization problem we postulate the following basic ingredients:

Condition 4.1. We are given a closed convex set $K \subset \mathbb{R}^N$ with $0 \in K$, and an \mathcal{F}_T -measurable random variable a on (Ω, \mathcal{F}, P) such that $0 < \inf_{\omega \in \Omega} a(\omega) \leq \sup_{\omega \in \Omega} a(\omega) < \infty$.

Condition 4.2. We are given a number $d \in \mathbb{R}$, together with \mathcal{F}_T -measurable square-integrable random variables c_0 and c_1 on (Ω, \mathcal{F}, P) .

Put

$$(4.8) \quad \mathcal{A} := \{\pi \in L_{22} \mid \pi(t) \in K \text{ a.e.}\},$$

$$(4.9) \quad \hat{J}(\omega, x) := \frac{1}{2}[a(\omega)x^2 + 2c_0(\omega)x], \quad (\omega, x) \in \Omega \times \mathbb{R},$$

$$(4.10) \quad G(\pi) := \mathbb{E}[c_1 X^\pi(T)] - d, \quad \pi \in L_{22},$$

$$(4.11) \quad \hat{\vartheta} := \inf_{\substack{\pi \in \mathcal{A} \\ G(\pi)=0}} \mathbb{E}[\hat{J}(X^\pi(T))].$$

We regard \mathcal{A} as the set of *admissible portfolios*, while $G(\pi) = 0$ represents a constraint on the terminal wealth. The problem of interest, which we denote by $(\hat{\mathcal{P}})$, is

$$(4.12) \quad \text{determine some } \hat{\pi} \in \mathcal{A} \text{ such that } G(\hat{\pi}) = 0 \text{ and } \hat{\vartheta} = \mathbb{E}[\hat{J}(X^{\hat{\pi}}(T))],$$

in the sense of demonstrating existence of $\hat{\pi}$ and characterizing its dependence on the market parameters $\{r(t)\}$, $\{b(t)\}$, $\{\sigma(t)\}$ and the information filtration $\{\mathcal{F}_t\}$ available to the investor. We must also postulate $0 \in \{G(\pi) \mid \pi \in \mathcal{A}\}$, for otherwise the constraints on π in (4.11) are mutually contradictory and we will have $\hat{\vartheta} = +\infty$, rendering the problem (4.12) meaningless. In fact, we impose the following *constraint qualification*:

Condition 4.3. The constant d , the set K and the random variable c_1 are such that the set $\{G(\pi) \mid \pi \in \mathcal{A}\} \subset \mathbb{R}$ has a nonempty interior which includes 0 (see Remark 4.6).

Example 4.4. $K = \mathbb{R}^N$ in (4.8) corresponds to the case of no constraints on the portfolio. On the other hand, $K = [0, \infty)^N$ represents a short-selling prohibition on stocks, while the constraint set $K = \{\pi \in \mathbb{R}^N \mid \pi_{n+1} = \dots = \pi_N = 0\}$ enforces an incomplete market in which the dimension N of the Brownian motion $\{W(t)\}$ exceeds the number of stocks n available to the investor. Other examples can be similarly formulated.

Remark 4.5. The most important case of problem (4.12) occurs when $a = 2$, $c_0 = 0$, and $c_1 = 1$, for then $\mathbb{E}[\hat{J}(X^\pi(T))] - d^2 = \text{Var}(X^\pi(T))$ (the *variance* of the terminal wealth) when $G(\pi) = 0$. Now problem (4.12) amounts to minimizing this variance subject to the terminal wealth constraint $\mathbb{E}[X^\pi(T)] = d$, together with the portfolio constraint $\pi \in \mathcal{A}$. This is the problem of *constrained mean-variance portfolio selection*.

Remark 4.6. We show that Condition 4.3 holds in the case where $c_1 \equiv 1$ in (4.10), the market model is “interesting” in the sense that $\mathbb{E}[X^{\hat{\pi}}(T)] > \mathbb{E}[x_0 S_0(T)]$ for some portfolio $\hat{\pi} \in \mathcal{A}$ (the problem (4.12) is pointless otherwise, since the best expected terminal wealth would be attained by just investing the entire fortune risk-free in the money-market), and

the expected terminal wealth d in the constraint (4.10) is “reasonable” in a sense to be specified. From (4.8), convexity of $K \subset \mathbb{R}^N$ (see Condition 4.1), and (2.5), it follows that $\mathcal{R} := \{E[X^\pi(T)] \mid \pi \in \mathcal{A}\} \subset \mathbb{R}$ is convex, hence an interval. Thus, the interior of \mathcal{R} is identical to $\mathcal{I} := (\inf_{\pi \in \mathcal{A}} E[X^\pi(T)], \sup_{\pi \in \mathcal{A}} E[X^\pi(T)])$, which is non-empty since $0 \in K$ and the market model is “interesting”. Now it follows from (4.10) that Condition 4.3 holds provided that d is specified in the “reasonable range” $d \in \mathcal{I}$.

5 Partially Constrained Problem

Here we establish duality relations for a *partially constrained* optimization problem in which the terminal wealth condition $G(\pi) = 0$ of (4.12) is discarded. In Section 6 these relations will then be used to establish existence for the fully constrained problem (4.12). We postulate

Condition 5.1. We are given a constant $q \in \mathbb{R}$, along with an \mathcal{F}_T -measurable square-integrable random variable c on (Ω, \mathcal{F}, P) .

Recalling the random variable a and convex set K in Condition 4.1, the set \mathcal{A} in (4.8), put

$$(5.13) \quad J(\omega, x) = \frac{1}{2} [a(\omega)x^2 + 2c(\omega)x] + q, \quad (\omega, x) \in \Omega \times \mathbb{R},$$

$$(5.14) \quad \vartheta_{c,q} := \inf_{\pi \in \mathcal{A}} E[J(X^\pi(T))].$$

The *partially constrained optimization problem*, which we denote by $(\mathcal{P}_{c,q})$, is:

$$(5.15) \quad \text{determine some } \bar{\pi} \in \mathcal{A} \text{ such that } \vartheta_{c,q} = E[J(X^{\bar{\pi}}(T))].$$

Remark 5.2. We distinguish between the coefficients c_0 and c in the linear terms of \hat{J} and J respectively (recall (4.9) and (5.13)), because, in Section 6, these coefficients will play somewhat different roles. Although the parameter q in (5.13) is of course immaterial for the problem $(\mathcal{P}_{c,q})$, we introduce it here because it will be essential in Section 6 to account for the Lagrange multiplier associated with the terminal wealth constraint in $(\hat{\mathcal{P}})$.

Remark 5.3. It follows at once from the quadratic form of $x \mapsto J(\omega, x)$ in (5.13), Conditions 4.1 and 5.1, and Proposition 3.1, that $-\infty < \vartheta_{c,q} < +\infty$.

5.1 Synthesis of a Dual Problem and Optimality Relations

Remark 5.4. Our goal in this subsection is to eliminate the portfolio π from the problem (5.15) to get a “primal” optimization problem over the set \mathbb{B} of Section 3 (see **Step I** and (5.24) to follow). This is a Bolza problem in the stochastic calculus of variations for which we synthesize a corresponding “dual” optimization problem by following an algorithmic approach motivated by Bismut [2] (see **Step II** and Remark 5.10 to follow). We also establish optimality relations (of Euler-Lagrange type) which are equivalent to the primal and dual problems each having a solution with zero “duality gap” (Proposition 5.9), and show that a solution of the dual problem exists (Proposition 5.13).

Step I: From Proposition 3.1 we know that $X^\pi \in \mathbb{B}$ for each admissible $\pi \in \mathcal{A}$. We therefore express the value (5.14) as the infimum over the set \mathbb{B} of some appropriate mapping $\Phi : \mathbb{B} \rightarrow (-\infty, \infty]$, by introducing “penalty terms” on \mathbb{B} , which account for the initial-wealth constraint $X(0) = x_0$, the portfolio constraint $\pi(t) \in K$ a.e., and the “dynamical constraint” implicit in (2.4); these will be defined to give zero penalty when the constraints hold and “infinite” penalty otherwise. For each $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{B}$ (recall Section 3), put

$$(5.16) \quad \mathcal{U}(X) := \{ \pi \in \mathcal{A} \mid \dot{X}(t) = r(t)X(t) + \pi'(t)\sigma(t)\theta(t) \text{ and } \Lambda_X(t) = \sigma'(t)\pi(t) \text{ a.e.} \}.$$

We then see the following: for each $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{B}$,

$$X(t) = X^\pi(t) \text{ a.e. for some } \pi \in \mathcal{A} \iff X_0 = x_0 \text{ and } \mathcal{U}(X) \neq \emptyset,$$

and, from this equivalence together with (5.14), we obtain

$$(5.17) \quad \vartheta_{c,q} = \inf_{\substack{X \in \mathbb{B} \\ X_0 = x_0 \\ \mathcal{U}(X) \neq \emptyset}} \mathbb{E}[J(X(T))].$$

Now define a penalty function on \mathbb{B} giving zero penalty when the constraint $\mathcal{U}(X) \neq \emptyset$ is satisfied, and infinite penalty otherwise. From Remark 2.2, for each $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{B}$

$$(5.18) \quad \mathcal{U}(X) \neq \emptyset \iff \dot{X}(t) = r(t)X(t) + \Lambda'_X(t)\theta(t) \text{ and } [\sigma'(t)]^{-1}\Lambda_X(t) \in K \text{ a.e.}$$

Motivated by (5.18), define the mapping $L : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \{0, \infty\}$ by

$$(5.19) \quad L(\omega, t, x, v, \xi) = \begin{cases} 0 & \text{if } v = r(\omega, t)x + \xi'\theta(\omega, t) \text{ and } [\sigma'(\omega, t)]^{-1}\xi \in K; \\ \infty & \text{otherwise.} \end{cases}$$

It is clear that $L(t, X(t), \dot{X}(t), \Lambda_X(t))$ is \mathcal{F}^* -measurable, and, in view of (5.18) and (5.19),

$$(5.20) \quad \mathbb{E} \int_0^T L(t, X(t), \dot{X}(t), \Lambda_X(t)) dt = \begin{cases} 0 & \text{if } \mathcal{U}(X) \neq \emptyset; \\ \infty & \text{otherwise,} \end{cases}$$

for each $X \in \mathbb{B}$. We see that (5.20) establishes a penalty for the constraint $\mathcal{U}(X) \neq \emptyset$ in (5.17). As for the initial-wealth constraint $X_0 = x_0$ in (5.17), put

$$(5.21) \quad l_0(x) := \begin{cases} 0 & \text{if } x = x_0; \\ \infty & \text{otherwise,} \end{cases}$$

for each $x \in \mathbb{R}$. Now define

$$(5.22) \quad \Phi(X) := l_0(X_0) + \mathbb{E}[l_T(X(T))] + \mathbb{E} \int_0^T L(t, X(t), \dot{X}(t), \Lambda_X(t)) dt,$$

for each $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{B}$, where, for consistency of notation, we put

$$(5.23) \quad l_T(\omega, x) := J(\omega, x), \quad (\omega, x) \in \Omega \times \mathbb{R}.$$

Upon combining (5.17), (5.20), (5.21), (5.22), and (5.23), we obtain

$$(5.24) \quad \vartheta_{c,q} = \inf_{X \in \mathbb{B}} \Phi(X).$$

Remark 5.5. From (5.20) and (5.21) it is clear that $\Phi(X)$ exists in $(-\infty, \infty]$ for each $X \in \mathbb{B}$.

Step II: In this step we synthesize a “cost” functional $\Psi : \mathbb{B} \rightarrow (-\infty, \infty]$ for an optimization problem which is dual to the primal problem (5.24). To this end define the convex conjugate functions:

$$(5.25) \quad \begin{aligned} m_0(y) &:= l_0^*(y) := \sup_{x \in \mathbb{R}} \{xy - l_0(x)\} \\ m_T(\omega, y) &:= l_T^*(\omega, -y) := \sup_{x \in \mathbb{R}} \{x(-y) - l_T(\omega, x)\} \\ M(\omega, t, y, s, \gamma) &:= L^*(\omega, t, s, y, \gamma) := \sup_{\substack{x, v \in \mathbb{R} \\ \xi \in \mathbb{R}^N}} \{xs + vy + \xi' \gamma - L(\omega, t, x, v, \xi)\}, \end{aligned}$$

for each $y \in \mathbb{R}$, $s \in \mathbb{R}$, $\gamma \in \mathbb{R}^N$, $\omega \in \Omega$ and $t \in [0, T]$. From the definitions of l_0 and l_T (recall (5.21) and (5.23)), for each $(\omega, y) \in \Omega \times \mathbb{R}$ we obtain

$$(5.26) \quad m_0(y) = x_0 y, \quad m_T(\omega, y) = \frac{(y + c(\omega))^2}{2a(\omega)} - q,$$

and, from (5.19), we easily find

$$(5.27) \quad M(\omega, t, y, s, \gamma) = \begin{cases} \delta(-\sigma(t) [\theta(t)y + \gamma]) & \text{if } s + r(t)y = 0, \\ \infty & \text{otherwise,} \end{cases}$$

where

$$(5.28) \quad \delta(z) := \sup_{\pi \in K} \{-\pi' z\}, \quad z \in \mathbb{R}^N.$$

For each $Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{B}$, define

$$(5.29) \quad \Psi(Y) := m_0(Y_0) + \mathbb{E}[m_T(Y(T))] + \mathbb{E} \int_0^T M(t, Y(t), \dot{Y}(t), \Lambda_Y(t)) dt.$$

Remark 5.6. Since $\delta(\cdot)$ is lower semicontinuous on \mathbb{R}^N , it is easily seen from (5.27) that $M(t, Y(t), \dot{Y}(t), \Lambda_Y(t))$ is \mathcal{F}^* -measurable for each $Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{B}$, and it is clear that $\Psi(Y)$ exists in $(-\infty, \infty]$ for each $Y \in \mathbb{B}$.

Next we require the following result from Bismut ([2], Proposition I-1, p.387):

Proposition 5.7. For members $X \equiv (X_0, \dot{X}, \Lambda_X)$ and $Y \equiv (Y_0, \dot{Y}, \Lambda_Y)$ of the set \mathbb{B} , define

$$\mathbb{M}(X, Y)(t) := X(t)Y(t) - X_0Y_0 - \int_0^t \{X(\tau)\dot{Y}(\tau) + \dot{X}(\tau)Y(\tau) + \Lambda_X'(\tau)\Lambda_Y(\tau)\} d\tau, \quad t \in [0, T].$$

Then $\{(\mathbb{M}(X, Y)(t), \mathcal{F}_t), t \in [0, T]\}$ is a continuous martingale with $\mathbb{M}(X, Y)(0) = 0$.

Proposition 5.8. *Assume Conditions 2.1, 4.1 and 5.1. Then the functions Φ and Ψ given by (5.22) and (5.29) are well-defined, with values in $(-\infty, \infty]$ for each $X \in \mathbb{B}$, $Y \in \mathbb{B}$, and*

$$(5.30) \quad \Phi(X) + \Psi(Y) \geq 0, \quad (X, Y) \in \mathbb{B} \times \mathbb{B}.$$

Moreover, for arbitrary $\bar{X} \equiv (\bar{X}_0, \dot{\bar{X}}, \Lambda_{\bar{X}}) \in \mathbb{B}$ and $\bar{Y} \equiv (\bar{Y}_0, \dot{\bar{Y}}, \Lambda_{\bar{Y}}) \in \mathbb{B}$, we have the equality $\Phi(\bar{X}) + \Psi(\bar{Y}) = 0$ if and only if each of the following conditions hold:

1. $l_0(\bar{X}_0) + m_0(\bar{Y}_0) = \bar{X}_0 \bar{Y}_0$,
2. $l_T(\bar{X}(T)) + m_T(\bar{Y}(T)) = -\bar{X}(T) \bar{Y}(T)$ a.s.,
3. $L(t, \bar{X}(t), \dot{\bar{X}}(t), \Lambda_{\bar{X}}(t)) + M(t, \bar{Y}(t), \dot{\bar{Y}}(t), \Lambda_{\bar{Y}}(t)) = \bar{X}(t) \dot{\bar{Y}}(t) + \dot{\bar{X}}(t) \bar{Y}(t) + \Lambda'_{\bar{X}}(t) \Lambda_{\bar{Y}}(t)$ a.e. on $\Omega \times [0, T]$.

Proof. Fix $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{B}$ and $Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{B}$. To establish (5.30), observe from the convex conjugates in (5.25) that, for each $(\omega, t) \in \Omega \times [0, T]$,

$$(5.31) \quad \begin{aligned} l_0(X_0) + m_0(Y_0) &\geq X_0 Y_0 \\ l_T(X(T)) + m_T(Y(T)) &\geq -X(T) Y(T) \\ L(t, X(t), \dot{X}(t), \Lambda_X(t)) + M(t, Y(t), \dot{Y}(t), \Lambda_Y(t)) &\geq X(t) \dot{Y}(t) + \dot{X}(t) Y(t) + \Lambda'_X(t) \Lambda_Y(t). \end{aligned}$$

By (5.22), (5.29), and (5.31),

$$(5.32) \quad \begin{aligned} \Phi(X) + \Psi(Y) &= l_0(X_0) + m_0(Y_0) + \mathbb{E}[l_T(X(T)) + m_T(Y(T))] \\ &\quad + \mathbb{E} \int_0^T \{L(t, X(t), \dot{X}(t), \Lambda_X(t)) + M(t, Y(t), \dot{Y}(t), \Lambda_Y(t))\} dt \\ &\geq X_0 Y_0 + \mathbb{E}[-X(T) Y(T)] + \mathbb{E} \int_0^T \{X(t) \dot{Y}(t) + \dot{X}(t) Y(t) + \Lambda'_X(t) \Lambda_Y(t)\} dt \\ &= \mathbb{E}[-\mathbb{M}(X, Y)(T)] = 0, \end{aligned}$$

where the last two equalities of (5.32) follow from Proposition 5.7. This establishes (5.30). Next, for some $(\bar{X}, \bar{Y}) \in \mathbb{B} \times \mathbb{B}$, the equivalence between $\Phi(\bar{X}) + \Psi(\bar{Y}) = 0$ and items (1) - (3) follows at once from (5.32) and (5.31). \square

We next refine Proposition 5.8 to obtain the following Proposition 5.9. This gives a set of optimality relations which will be essential in constructing an optimal portfolio. Put

$$(5.33) \quad \Theta_Y(t) := -\sigma(t) [\theta(t) Y(t) + \Lambda_Y(t)], \quad \text{for each } Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{B}.$$

Proposition 5.9. *Assume Conditions 2.1, 4.1 and 5.1. Then, for arbitrary $(\bar{X}, \bar{Y}) \in \mathbb{B} \times \mathbb{B}$, we have*

$$(5.34) \quad \Phi(\bar{X}) = \inf_{X \in \mathbb{B}} \Phi(X) = \sup_{Y \in \mathbb{B}} [-\Psi(Y)] = -\Psi(\bar{Y}),$$

if and only if

$$(5.35) \quad \begin{cases} (1') & \bar{X}_0 = x_0 \\ (2') & \bar{X}(T) = -\frac{(\bar{Y}(T) + c)}{a} \quad a.s. \\ (3') & \dot{\bar{Y}}(t) + r(t)\bar{Y}(t) = 0 \quad a.e. \\ (4') & \bar{\pi} \in \mathcal{U}(\bar{X}) \quad \text{and} \quad \delta(\Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) = 0 \quad a.e. \\ & \text{for } \bar{\pi}(t) := [\sigma'(t)]^{-1}\Lambda_{\bar{X}}(t). \end{cases}$$

Proof. From (5.19) and (5.27), for arbitrary $(\omega, t) \in \Omega \times [0, T]$, and $(x, v, \xi), (y, s, \gamma) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$, we have the equivalences

$$(5.36) \quad \begin{aligned} L(\omega, t, x, v, \xi) + M(\omega, t, y, s, \gamma) &= xs + vy + \xi'\gamma \\ \iff v = r(t)x + \xi'\theta(t), \quad [\sigma'(t)]^{-1}\xi \in K, \quad s + r(t)y = 0 \\ &\text{and } \delta(-\sigma(t)[\theta(t)y + \gamma]) = xs + vy + \xi'\gamma \\ \iff v = r(t)x + \xi'\theta(t), \quad [\sigma'(t)]^{-1}\xi \in K, \quad s + r(t)y = 0 \\ &\text{and } \delta(-\sigma(t)[\theta(t)y + \gamma]) - \xi'\sigma^{-1}(t)\sigma(t)[\theta(t)y + \gamma] = 0. \end{aligned}$$

Moreover, from (5.13), (5.21), (5.23), and (5.26), for arbitrary $x, y \in \mathbb{R}$, $\omega \in \Omega$, we find that $l_0(x) + m_0(y) = xy$ if and only if $x = x_0$, and $l_T(\omega, x) + m_T(\omega, y) = -xy$ if and only if $x = -(y + c(\omega))/a(\omega)$. From these equivalences, with (5.36), (5.16), and (5.18), we obtain the following: for arbitrary $(\bar{X}, \bar{Y}) \in \mathbb{B} \times \mathbb{B}$,

$$(5.37) \quad (1') - (4') \text{ of (5.35) hold} \iff (1) - (3) \text{ of Proposition 5.8 hold.}$$

But, in view of the universal inequality (5.30), we see that the statement $\Phi(\bar{X}) + \Psi(\bar{Y}) = 0$ is equivalent to (5.34), hence (5.34) is equivalent to items (1) to (3) of Proposition 5.8. The result follows from this equivalence, together with (5.37). \square

Remark 5.10. It follows from Proposition 5.9 that the solution of problem $(\mathcal{P}_{c,q})$ in (5.15) reduces to construction of a pair $(\bar{X}, \bar{Y}) \in \mathbb{B} \times \mathbb{B}$ which satisfies the optimality relations (5.35)(1')-(4'), for then the optimal portfolio $\bar{\pi}$ is defined by (5.35)(4') (recall (5.24)). Motivated by the third equality of (5.34), in the remainder of this subsection we show that there exists a solution to the problem of minimizing $\Psi(\cdot)$ on \mathbb{B} , henceforth referred to as the *dual problem*. Define

$$(5.38) \quad \mathbb{B}_1 := \{Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{B} \mid \dot{Y}(t) = -r(t)Y(t) \quad a.e.\},$$

and observe from (5.29) and (5.27), that Ψ necessarily takes the value $+\infty$ on $\mathbb{B} - \mathbb{B}_1$. Then

$$(5.39) \quad \inf_{Y \in \mathbb{B}} \Psi(Y) = \inf_{Y \in \mathbb{B}_1} \Psi(Y),$$

so that the dual problem reduces to minimization of $\Psi(\cdot)$ over \mathbb{B}_1 . For each $t \in [0, T]$ put

$$(5.40) \quad \beta(t) := \exp\left[-\int_0^t r(\tau) d\tau\right], \quad \mathfrak{J}(\gamma)(t) := \int_0^t \beta^{-1}(\tau)\gamma'(\tau) dW(\tau), \quad \gamma \in L_{22},$$

$$(5.41) \quad \Xi(y, \gamma)(t) := \beta(t)[y + \mathfrak{J}(\gamma)(t)], \quad t \in [0, T], \quad (y, \gamma) \in \mathbb{R} \times L_{22}.$$

Remark 5.11. From Section 3, we see that $Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{B}_1$ satisfies the relation

$$(5.42) \quad Y(t) = Y_0 - \int_0^t r(\tau)Y(\tau) \, d\tau + \int_0^t \Lambda'_Y(\tau) \, dW(\tau).$$

Then it follows from Itô's formula and Doob's L^2 -inequality that $\Xi(\cdot) : \mathbb{R} \times L_{22} \rightarrow \mathbb{B}_1$ is a *linear bijection*, and, when $Y := \Xi(y, \gamma)$ for some $(y, \gamma) \in \mathbb{R} \times L_{22}$, then (recalling (5.33))

$$(5.43) \quad Y_0 = y, \quad \Lambda_Y(t) = \gamma(t), \quad \Theta_Y(t) = -\sigma(t)[\theta(t)Y(t) + \gamma(t)], \text{ a.e.}$$

We then obtain

$$(5.44) \quad \inf_{(y, \gamma) \in \mathbb{R} \times L_{22}} \tilde{\Psi}(y, \gamma) = \inf_{Y \in \mathbb{B}_1} \Psi(Y), \quad \text{for } \tilde{\Psi}(y, \gamma) := \Psi(\Xi(y, \gamma)), \quad (y, \gamma) \in \mathbb{R} \times L_{22}.$$

Moreover, $M(t, Y(t), \dot{Y}(t), \Lambda_Y(t)) = \delta(\Theta_Y(t))$ a.e. for each $Y \in \mathbb{B}_1$ (see (5.38) and (5.27)), thus from (5.26) and (5.29), for each $(y, \gamma) \in \mathbb{R} \times L_{22}$ (with $Y := \Xi(y, \gamma)$) we get

$$(5.45) \quad \tilde{\Psi}(y, \gamma) = x_0 y + \mathbb{E} \left[\frac{(Y(T) + c)^2}{2a} \right] + \mathbb{E} \int_0^T \delta(\Theta_Y(t)) \, dt - q.$$

Remark 5.12. Define norm $\|\gamma\|_{L_{22}}$ on the real vector space L_{22} by $\|\gamma\|_{L_{22}}^2 := \mathbb{E} \int_0^T \|\gamma(t)\|^2 \, dt$, and define the norm $\|(y, \gamma)\|$ on the real vector space $\mathbb{R} \times L_{22}$ by $\|(y, \gamma)\|^2 := |y|^2 + \|\gamma\|_{L_{22}}^2$. With this norm $\mathbb{R} \times L_{22}$ is a *reflexive* Banach space.

Proposition 5.13. *Suppose Conditions 2.1, 4.1, and 5.1. Then*

$$(5.46) \quad \inf_{(y, \gamma) \in \mathbb{R} \times L_{22}} \tilde{\Psi}(y, \gamma) = \tilde{\Psi}(\bar{y}, \bar{\gamma}) \in \mathbb{R}, \quad \text{for some } (\bar{y}, \bar{\gamma}) \in \mathbb{R} \times L_{22}.$$

Proof. It is immediate from (5.45), (5.43) and (5.28), that $\tilde{\Psi}$ is convex on $\mathbb{R} \times L_{22}$. From Conditions 4.1 and 5.1 we get $\tilde{\Psi}(y, \gamma) \geq x_0 y - q > -\infty$ for each $(y, \gamma) \in \mathbb{R} \times L_{22}$ as well as $\tilde{\Psi}(0, 0) = \mathbb{E}[c^2/(2a)] - q < \infty$, hence $\tilde{\Psi}$ is proper. A routine argument using Fatou's lemma, with the nonnegativity and lower-semicontinuity of $\delta(\cdot)$, proves that $\tilde{\Psi}$ is lower-semicontinuous on $\mathbb{R} \times L_{22}$ (with respect to the norm $\|(y, \gamma)\|$ in Remark 5.12). We next show that $\tilde{\Psi}$ is coercive (i.e. $\tilde{\Psi}(y, \gamma) \rightarrow \infty$ when $\|(y, \gamma)\| \rightarrow \infty$). From Conditions 2.1 and 5.1 we know that $\beta^{-1}(T)c$ is \mathcal{F}_T -measurable square-integrable, thus $\beta^{-1}(T)c = \tilde{y} + \int_0^T \eta'(\tau) \, dW(\tau)$, for $\tilde{y} = \mathbb{E}[\beta^{-1}(T)c]$ and some $\eta \in L_{22}$ (see Theorem 3.4.15 of [8]); thus, from (5.40) and (5.41), we obtain $\Xi(y, \gamma)(T) + c = \Xi(y + \tilde{y}, \gamma + \beta\eta)(T)$, $(y, \gamma) \in \mathbb{R} \times L_{22}$. Thus, for showing coercivity, with no loss of generality we can and shall take $c \equiv 0$ in (5.45). In view of the nonrandom and strictly positive uniform lower bounds on $\beta(T)$ and $1/(2a)$ (see Conditions 2.1 and 4.1), and the Itô isometry, we find $\mathbb{E}[(\Xi(y, \gamma)(T))^2/(2a)] \rightarrow \infty$ as $\|(y, \gamma)\| \rightarrow \infty$. Coercivity of $\tilde{\Psi}$ follows from this, together with (5.45) and the non-negativity of $\delta(\cdot)$. Existence of a pair $(\bar{y}, \bar{\gamma}) \in \mathbb{R} \times L_{22}$ which satisfies (5.46) follows from this, together with Remark 5.12 and Proposition II-1.2 of Ekeland and Témam ([5], p.35). \square

Remark 5.14. Define $\bar{Y} := \Xi(\bar{y}, \bar{\gamma})$, for $(\bar{y}, \bar{\gamma}) \in \mathbb{R} \times L_{22}$ given by Proposition 5.13. From Remark 5.11 we have $\bar{Y} \in \mathbb{B}_1 \subset \mathbb{B}$. Upon combining (5.46), (5.44) and (5.39), we get $\Psi(\bar{Y}) = \inf_{Y \in \mathbb{B}} \Psi(Y)$, thus \bar{Y} solves the dual problem of Remark 5.10.

5.2 Construction of the Optimal Portfolio

In the present subsection we shall construct some $\bar{X} \in \mathbb{B}$ such that the pair (\bar{X}, \bar{Y}) , with \bar{Y} given by Remark 5.14, satisfies (5.35). To this end, consider the *state price density process* given by (recall (5.40) for β)

$$(5.47) \quad H(t) := \beta(t)\mathcal{E}(-\theta' \bullet W)(t).$$

Remark 5.15. In (5.47) the notation $\mathcal{E}(M)(t) := \exp[M(t) - (1/2)\langle M \rangle(t)]$ indicates the exponential of a continuous local martingale M , while \bullet denotes stochastic integration.

Remark 5.16. Fix an arbitrary $p \in \mathbb{R}$; since θ is uniformly-bounded (recall Remark 2.3), it follows that $\{\mathcal{E}(-p\theta' \bullet W)(t)\}$ is a continuous $\{\mathcal{F}_t\}$ -martingale (by the Novikov criterion, Corollary 3.5.14 of [8], p.199), and then it easily follows from the uniform-boundedness of θ and r (Condition 2.1) and Doob's maximal L^2 -inequality that $\mathbb{E}[\sup_{t \in [0, T]} |H(t)|^p] < \infty$, for each $p \in \mathbb{R}$. Thus H defined by (5.47) is a member of \mathbb{B} (take $p = 2$).

Now $\{(H(t)X^\pi(t), \mathcal{F}_t), t \in [0, T]\}$ is a martingale for each $\pi \in L_{22}$ (as follows from (5.47), (2.4), Proposition 5.7, Remark 5.16, and Proposition 3.1). This, together with (5.35)(2'), motivates the following definition of \bar{X} in terms of \bar{Y} defined by Remark 5.14:

$$(5.48) \quad \bar{X}(t) := -\frac{1}{H(t)} \mathbb{E} \left[\left(\frac{\bar{Y}(T) + c}{a} \right) H(T) \middle| \mathcal{F}_t \right].$$

Remark 5.17. The square-integrability of $\bar{Y}(T)$ (recall $\bar{Y} \in \mathbb{B}$) and c (Condition 5.1), and the strictly positive lower-bound on a (Condition 4.1), ensure that $(\bar{Y}(T) + c)/a$ is square integrable. Together with Remark 5.16, this certainly establishes the existence of the conditional expectation in (5.48).

Observe that \bar{X} defined by (5.48) satisfies the ‘‘dynamical part’’ of (2.4), namely

$$(5.49) \quad d\bar{X}(t) = \{r(t)\bar{X}(t) + \bar{\pi}'(t)\sigma(t)\theta(t)\} dt + \bar{\pi}'(t)\sigma(t) dW(t),$$

for some \mathbb{R}^N -valued $\bar{\pi} \in \mathcal{F}^*$ such that $\int_0^T \|\bar{\pi}(t)\|^2 dt < \infty$ a.s. Indeed, from (5.48) and the martingale representation theorem (see 3.4.16 of Karatzas and Shreve [8], p.184), there exists some \mathbb{R}^N -valued and a.e.-unique $\psi \in \mathcal{F}^*$, with $\int_0^T \|\psi(t)\|^2 dt < \infty$ a.s., such that

$$(5.50) \quad \bar{X}(t)H(t) = \bar{X}(0) + \int_0^t \psi'(\tau) dW(\tau) := \xi_0(t).$$

Expanding the quotient $\bar{X}(t) := \xi_0(t)/H(t)$ by Itô's formula, we get (5.49) for

$$(5.51) \quad \bar{\pi}(t) := [\sigma'(t)]^{-1} \left[\frac{\psi(t)}{H(t)} + \bar{X}(t)\theta(t) \right],$$

(since \bar{X} defined by (5.48) is continuous, Remark 2.2 shows that $\int_0^T \|\bar{\pi}(t)\|^2 dt < \infty$ a.s.).

Remark 5.18. From Remark 5.14 we have seen that $\bar{Y} \in \mathbb{B}_1$, thus (5.35)(3') holds (recall (5.38)), and of course (5.35)(2') is immediate from (5.48). In the remainder of this section we shall establish that $\bar{X} \in \mathbb{B}$ (in which case we see from (5.49) that $\bar{\pi}$ is also given by $\bar{\pi} = [\sigma']^{-1} \Lambda_{\bar{X}}$), and that (1') and (4') of (5.35) hold. We shall then have verified all items of (5.35), and can conclude (5.34) (from Proposition 5.9), which, together with (5.24), implies $\Phi(\bar{X}) = \vartheta_{c,q}$. Moreover, from (5.35)(4'), we obtain $\bar{\pi} \in \mathcal{A}$ (recall (5.16)), while the dynamical relation (5.49) together with (5.35)(1'), establishes that $\bar{X} = X^{\bar{\pi}}$ a.e. (for $X^{\bar{\pi}}$ defined by the wealth equation (2.4)). But, in light of (5.35)(1')(4'), (5.21), and (5.20), the first and third terms on the right side of (5.22) are zero when $X := \bar{X}$, and then (from (5.23)) $\Phi(\bar{X}) = \mathbb{E}[J(\bar{X}(T))]$. Thus $\vartheta_{c,q} = \mathbb{E}[J(X^{\bar{\pi}}(T))]$, hence $\bar{\pi} \in \mathcal{A}$ solves problem $(\mathcal{P}_{c,q})$.

Lemma 5.19. *Assume Conditions 2.1, 4.1, and 5.1. Then $E \left[\sup_{t \in [0, T]} |\bar{X}(t)|^2 \right] < \infty$ (for \bar{X} defined in (5.48)).*

Proof. Put $D := (\bar{Y}(T) + c)/a$. Then $E|D|^2 < +\infty$ (Remark 5.17), and it follows from the integrability of $H(t)$ indicated in Remark 5.16, together with Hölder's inequality, that $DH(T)H^{-1}(t)$ is integrable. Thus, from (5.48), we have $\bar{X}(t) = -\mathbb{E}[DH(T)H^{-1}(t) | \mathcal{F}_t]$. Now fix some $q \in (1, 2)$, and let $p \in (2, \infty)$ be the conjugate constant given by $p^{-1} + q^{-1} = 1$. Then Hölder's inequality for conditional expectations (see Chow and Teicher, [3], Theorem 7.2.4, p.219) gives

$$(5.52) \quad |\bar{X}(t)| \leq \left\{ \mathbb{E} \left[\left(\frac{H(T)}{H(t)} \right)^p \middle| \mathcal{F}_t \right] \right\}^{1/p} \{ \mathbb{E}[|D|^q | \mathcal{F}_t] \}^{1/q}, \quad \text{a.s.}$$

for each $t \in [0, T]$. From (5.47) and (5.40), along with the uniform bounds on r and θ (Condition 2.1 and Remark 2.3), there is a constant $k \in (0, \infty)$ such that

$$(5.53) \quad \left(\frac{H(T)}{H(t)} \right)^p \leq k \frac{\mathcal{E}(-p\theta' \bullet W)(T)}{\mathcal{E}(-p\theta' \bullet W)(t)}, \quad \text{a.s.}$$

As noted in Remark 5.16, $\{\mathcal{E}(-p\theta' \bullet W)(t)\}$ is a $\{\mathcal{F}_t\}$ -martingale, thus it follows from (5.53) that the first conditional expectation on the right-hand side of (5.52) is upper-bounded a.s. by the constant k , and therefore

$$(5.54) \quad |\bar{X}(t)|^q \leq k^{q/p} \mathbb{E}[|D|^q | \mathcal{F}_t], \quad \text{a.s.}$$

for each $t \in [0, T]$. Since $E|D|^2 < \infty$ and $q \in (1, 2)$, we have $\mathbb{E}[|D|^q] < \infty$. Thus, defining

$$(5.55) \quad N(t) := \mathbb{E}[|D|^q | \mathcal{F}_t],$$

we find that $\{N(t)\}$ is a $\{\mathcal{F}_t\}$ -martingale. Put $p_1 := 2/q > 1$, where the strict inequality follows since $q \in (1, 2)$. Thus, from Jensen's inequality, we see that $\mathbb{E}[|N(t)|^{p_1}] \leq \mathbb{E}[|D|^2] < \infty$, for each $t \in [0, T]$, and consequently, from Doob's L^{p_1} -inequality,

$$(5.56) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |N(t)|^{p_1} \right] \leq \left(\frac{p_1}{p_1 - 1} \right)^{p_1} \mathbb{E}[|N(T)|^{p_1}] < \infty.$$

From (5.55) and (5.54) we have $|\bar{X}(t)|^2 \leq k^{2/p} |N(t)|^{p_1}$, and the result follows from (5.56). \square

Lemma 5.20. *Suppose Conditions 2.1, 4.1, and 5.1. For \bar{X} and $\bar{\pi}$ defined by (5.48), (5.50) and (5.51), we have $\bar{\pi} \in L_{22}$ as well as $\bar{X} \in \mathbb{B}$.*

Proof. For each $n = 1, 2, \dots$ put $\tau_n := \inf\{t \in [0, T] \mid \int_0^t \|\bar{\pi}(s)\|^2 ds \geq n\} \wedge T$. Then τ_n is a \mathcal{F}_t -stopping time (recall (2.1)), and $\tau_n \uparrow T$ a.s. (since $\int_0^T \|\bar{\pi}(s)\|^2 ds < \infty$ a.s., as noted following (5.51)). Now we have seen that \bar{X} and $\bar{\pi}$ are related by (5.49); using this relation to expand $t \rightarrow \bar{X}^2(t)$ by Itô's formula, and evaluating at $t \wedge \tau_n$, we obtain

$$(5.57) \quad \begin{aligned} \bar{X}^2(t \wedge \tau_n) = \bar{X}^2(0) &+ \int_0^{t \wedge \tau_n} \{2\bar{X}(s)[r(s)\bar{X}(s) + \bar{\pi}'(s)\sigma(s)\theta(s)] + \|\sigma'(s)\bar{\pi}(s)\|^2\} ds \\ &+ 2 \int_0^{t \wedge \tau_n} \bar{X}(s)\bar{\pi}'(s)\sigma(s) dW(s), \quad t \in [0, T]. \end{aligned}$$

Now it follows from Lemma 5.19 and the definition of τ_n that the last term on the right side of (5.57) defines a \mathcal{F}_t -martingale null at $t = 0$, and hence has zero expectation for all t ; thus upon taking expectations on each side of (5.57) at $t := T$, and using the non-negativity of r (Condition 2.1), we obtain

$$(5.58) \quad \mathbb{E}[\bar{X}^2(\tau_n)] + \mathbb{E} \int_0^{\tau_n} [-2\bar{X}(s)\bar{\pi}'(s)\sigma(s)\theta(s)] ds \geq \mathbb{E} \int_0^{\tau_n} \|\sigma'(s)\bar{\pi}(s)\|^2 ds.$$

For arbitrary $v_1, v_2 \in \mathbb{R}^N$ we have $v_1'v_2 \leq (1/2)[\|v_1\|^2 + \|v_2\|^2]$, thus

$$(5.59) \quad -2\bar{X}(s)\theta'(s)\sigma'(s)\bar{\pi}(s) \leq (1/2)[4\bar{X}^2(s)\|\theta(s)\|^2 + \|\sigma'(s)\bar{\pi}(s)\|^2].$$

Substituting (5.59) into (5.58) and simplifying then gives

$$(5.60) \quad \frac{1}{2}\mathbb{E} \int_0^{\tau_n} \|\sigma'(s)\bar{\pi}(s)\|^2 ds \leq (1 + Tk_1)\mathbb{E} \left[\sup_{t \in [0, T]} |\bar{X}(t)|^2 \right], \quad n = 1, 2, \dots$$

for some constant $k_1 \in [0, \infty)$ depending only on the uniform bound on θ (recall Remark 2.3). Now we have seen $\tau_n \uparrow T$ a.s., thus we get $\bar{\pi} \in L_{22}$ from Lemma 5.19 and Remark 2.2 upon taking $n \rightarrow \infty$ in (5.60). Finally, from $\bar{\pi} \in L_{22}$, (5.49), and an argument identical to that for Proposition 3.1, we get $\bar{X} \in \mathbb{B}$ (this proof is suggested by the argument for establishing existence of solutions for backwards SDE's - see e.g. ([13], p.352)). \square

Recalling Remark 5.18, it remains to verify (1') and (4') of (5.35). To this end we need

Lemma 5.21. *Assume Conditions 2.1, 4.1, and 5.1. For arbitrary $(\alpha, \eta) \in \mathbb{R} \times L_{22}$ and $R := \Xi(\alpha, \eta)$ (recall (5.41)), we have*

$$(5.61) \quad 0 \leq \alpha(x_0 - \bar{X}(0)) + \lim_{\epsilon \searrow 0} \mathbb{E} \int_0^T \left\{ \frac{\delta(\Theta_{\bar{Y}}(t) + \epsilon \Theta_R(t)) - \delta(\Theta_{\bar{Y}}(t))}{\epsilon} + \bar{\pi}'(t)\Theta_R(t) \right\} dt.$$

Remark 5.22. From (5.46), Remark 5.14, and (5.45) we have $\mathbb{E} \int_0^T \delta(\Theta_{\bar{Y}}(t)) dt < \infty$, thus the expectations in (5.61) exist in $(-\infty, \infty]$. Since $\delta(\cdot)$ is convex, it follows from Ekeland and Témam ([5], p.23) that the limit on the right of (5.61) exists (in the extended reals).

Proof. (of Lemma 5.21): For arbitrary $\epsilon \in (0, \infty)$ define $(y^\epsilon, \gamma^\epsilon) \in \mathbb{R} \times L_{22}$ by $y^\epsilon := \bar{y} + \epsilon \alpha$ and $\gamma^\epsilon := \bar{\gamma} + \epsilon \eta$, and define $Y^\epsilon \in \mathbb{B}_1$ by $Y^\epsilon := \Xi(y^\epsilon, \gamma^\epsilon)$. From Remark 5.11, we have $Y^\epsilon = \bar{Y} + \epsilon R$ and $\Theta_{Y^\epsilon} = \Theta_{\bar{Y}} + \epsilon \Theta_R$, thus by (5.45),

$$(5.62) \quad \frac{\tilde{\Psi}(y^\epsilon, \gamma^\epsilon) - \tilde{\Psi}(\bar{y}, \bar{\gamma})}{\epsilon} = \alpha x_0 + \mathbb{E} \left[\left(\frac{\bar{Y}(T) + c}{a} \right) R(T) \right] + \epsilon \mathbb{E} \left[\frac{R^2(T)}{2a} \right] \\ + \mathbb{E} \int_0^T \frac{\delta(\Theta_{\bar{Y}}(t) + \epsilon \Theta_R(t)) - \delta(\Theta_{\bar{Y}}(t))}{\epsilon} dt.$$

From (5.46) the left-hand side of (5.62) is nonnegative. Using (5.35)(2') (which holds in view of (5.48)), and taking limits in (5.62) then gives

$$(5.63) \quad 0 \leq \alpha x_0 - \mathbb{E} [\bar{X}(T)R(T)] + \lim_{\epsilon \searrow 0} \mathbb{E} \int_0^T \frac{\delta(\Theta_{\bar{Y}}(t) + \epsilon \Theta_R(t)) - \delta(\Theta_{\bar{Y}}(t))}{\epsilon} dt.$$

Now we have shown (5.49) and $\bar{X} \in \mathbb{B}$ (see Lemma 5.20) hence $\dot{\bar{X}}(t) = r(t)\bar{X}(t) + \bar{\pi}'(t)\sigma(t)\theta(t)$ and $\Lambda_{\bar{X}}(t) = \sigma'(t)\bar{\pi}(t)$, a.e. In view of these observations and Remark 5.11 (applied to $R = \Xi(\alpha, \eta)$), it follows from Proposition 5.7 that

$$\mathbb{M}(\bar{X}, R)(t) = \bar{X}(t)R(t) - \alpha \bar{X}(0) + \int_0^t \bar{\pi}'(\tau)\Theta_R(\tau) d\tau, \quad t \in [0, T],$$

is a continuous $\{\mathcal{F}_t\}$ -martingale null at the origin, hence $\mathbb{E}[\mathbb{M}(\bar{X}, R)(T)] = 0$. Combining this with (5.63) gives (5.61). \square

We also require the following result, the proof of which will be omitted since it is just a simple modification of the usual argument for existence and uniqueness in linear integral equations:

Lemma 5.23. *Assume Condition 2.1. For each $\rho \in L_{22}$ there is a unique $\xi \in L_{22}$ such that*

$$(5.64) \quad \rho(t) = \xi(t) + \theta(t) \int_0^t \xi'(\tau) dW(\tau) \quad a.e.$$

With Lemmas 5.21 and 5.23 at hand, we can complete the program outlined in Remark 5.18:

Proposition 5.24. *Assume Conditions 2.1, 4.1, and 5.1. Then (1') and (4') of (5.35) hold for \bar{Y} defined by Remark 5.14, and \bar{X} defined by (5.48).*

Proof. We first establish (5.35)(1'). Fix an arbitrary $\alpha \in \mathbb{R}$. Since $\theta \in L_{22}$ (being uniformly bounded by Remark 2.3), Lemma 5.23 gives some $\xi \in L_{22}$ such that (5.64) holds with $\rho(t) := -\alpha\theta(t)$; upon multiplying by $\beta(t)$ we then obtain

$$(5.65) \quad -\alpha\theta(t)\beta(t) = \eta(t) + \theta(t)\beta(t) \int_0^t \beta^{-1}(\tau)\eta'(\tau) dW(\tau) \quad a.e.,$$

in which $\eta(t) := \beta(t)\xi(t)$. Since β is uniformly bounded ((5.40) and Condition 2.1), we have $\eta \in L_{22}$, and from (5.65), (5.41), and (5.40), we find $\eta(t) + \theta(t)\Xi(\alpha, \eta)(t) = 0$ a.e. Upon

defining $R := \Xi(\alpha, \eta)$, we get from Remark 5.11 that $\Theta_R(t) = 0$ a.e., thus Lemma 5.21 gives $0 \leq \alpha(x_0 - \bar{X}(0))$. Now (5.35)(1') follows since $\alpha \in \mathbb{R}$ is arbitrary.

It remains to establish (5.35)(4'): Since (5.49) holds and $\bar{\pi} \in L_{22}$ (see Lemma 5.20), it is enough to show that $\bar{\pi}(t) \in K$ a.e. to conclude $\bar{\pi} \in \mathcal{U}(\bar{X})$ (recall (5.16), (4.8)). Since $\delta(\cdot)$ is subadditive and positively homogeneous (see [7], p.206) we have $\delta(\Theta_{\bar{Y}}(t) + \epsilon\Theta_R(t)) \leq \delta(\Theta_{\bar{Y}}(t)) + \epsilon\delta(\Theta_R(t))$ for arbitrary $\epsilon \in (0, \infty)$ and $R \in \mathbb{B}$. Then, since we have shown $\bar{X}(0) = x_0$, it follows from Lemma 5.21 that, for each $(\alpha, \eta) \in \mathbb{R} \times L_{22}$, we have

$$(5.66) \quad 0 \leq \mathbb{E} \int_0^T \{\delta(\Theta_R(t)) + \bar{\pi}'(t)\Theta_R(t)\} dt, \quad \text{for } R := \Xi(\alpha, \eta).$$

Put $B := \{(\omega, t) \in \Omega \times [0, T] \mid \bar{\pi}(\omega, t) \in K\}$. By Lemma 5.4.2 of (Karatzas and Shreve, [7], p.207), there exists some \mathcal{F}^* -measurable mapping $\tilde{\nu} : \Omega \times [0, T] \rightarrow \mathbb{R}^N$ such that

$$(5.67) \quad \begin{cases} \|\tilde{\nu}(t)\| \leq 1, & |\delta(\tilde{\nu}(t))| \leq 1, & \text{a.e. on } \Omega \times [0, T], \\ \delta(\tilde{\nu}(t)) + \bar{\pi}'(t)\tilde{\nu}(t) = 0, & & \text{a.e. on } B, \\ \delta(\tilde{\nu}(t)) + \bar{\pi}'(t)\tilde{\nu}(t) < 0, & & \text{a.e. on } (\Omega \times [0, T]) - B. \end{cases}$$

Now suppose that $(P \otimes \lambda)\{(\Omega \times [0, T]) - B\} > 0$. Then, by (5.67),

$$(5.68) \quad 0 > \mathbb{E} \int_0^T \{\delta(\tilde{\nu}(t)) + \bar{\pi}'(t)\tilde{\nu}(t)\} dt.$$

Put $\rho(t) := -\beta^{-1}(t)\sigma^{-1}(t)\tilde{\nu}(t)$. Since $\|\tilde{\nu}(t)\|$ is essentially bounded on $\Omega \times [0, T]$, it follows from the boundedness of β^{-1} and σ^{-1} (Remark 2.2) that $\rho \in L_{22}$. Then, from Lemma 5.23, there exists some $\tilde{\xi} \in L_{22}$ such that

$$(5.69) \quad -\beta^{-1}(t)\sigma^{-1}(t)\tilde{\nu}(t) = \tilde{\xi}(t) + \theta(t) \int_0^t \tilde{\xi}'(\tau) dW(\tau) \quad \text{a.e.}$$

Now define $\tilde{\eta}(t) := \beta(t)\tilde{\xi}(t) \in L_{22}$ and $R := \Xi(0, \tilde{\eta})$. From (5.41) we get $R(t) = \beta(t)\mathfrak{J}(\tilde{\eta})(t)$ a.e. Upon combining this with (5.69), (5.40), and Remark 5.11, we obtain

$$(5.70) \quad \tilde{\nu}(t) = -\sigma(t)[\tilde{\eta}(t) + \theta(t)R(t)] = \Theta_R(t) \quad \text{a.e.}$$

From (5.70) and (5.68), we get $\mathbb{E} \int_0^T \{\delta(\Theta_R(t)) + \bar{\pi}'(t)\Theta_R(t)\} dt < 0$, (for $R := \Xi(0, \tilde{\eta})$). Since $\tilde{\eta} \in L_{22}$, this last inequality contradicts (5.66) and so $(P \otimes \lambda)\{(\Omega \times [0, T]) - B\} = 0$, that is $\bar{\pi}(t) \in K$ a.e., as required to establish $\bar{\pi} \in \mathcal{U}(\bar{X})$. We next show

$$(5.71) \quad \delta(\Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) = 0 \quad \text{a.e.}$$

To this end, put $R := \Xi(-\bar{y}, -\bar{\gamma})$. Then, in light of the linearity of $\Xi(\cdot)$, and since $\bar{Y} = \Xi(\bar{y}, \bar{\gamma})$ (see Remark 5.14), we have $R = -\bar{Y}$. Thus $\Theta_R = -\Theta_{\bar{Y}}$ (see (5.33)). Since $\delta(\cdot)$ is positively homogeneous, for each $\epsilon \in (0, 1)$ we get

$$(5.72) \quad \delta(\Theta_{\bar{Y}}(t) + \epsilon\Theta_R(t)) = \delta((1 - \epsilon)\Theta_{\bar{Y}}(t)) = (1 - \epsilon)\delta(\Theta_{\bar{Y}}(t)) \quad \text{a.e.}$$

But we have shown $\bar{X}(0) = x_0$; in view of (5.72) and Lemma 5.21, we find

$$(5.73) \quad 0 \geq \mathbb{E} \int_0^T \{\delta(\Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t)\} dt.$$

Now we have already seen that $\bar{\pi}(t) \in K$ a.e., thus (see (5.28)) $\delta(\Theta_{\bar{Y}}(t)) + \bar{\pi}'(t)\Theta_{\bar{Y}}(t) \geq 0$ a.e. This, together with (5.73), establishes (5.71). Finally, we see from (5.49) that $\bar{\pi}$ at (5.51) is also given by $\bar{\pi}(t) := [\sigma'(t)]^{-1}\Lambda_{\bar{X}}(t)$. This establishes (5.35)(4'). \square

For easy reference we summarize the main result of the present section as follows:

Proposition 5.25. *Suppose Conditions 2.1, 4.1, and 5.1. Then there exists a pair $(\bar{y}, \bar{\gamma}) \in \mathbb{R} \times L_{22}$ minimizing the proper convex functional $\tilde{\Psi}(\cdot, \cdot)$ (see (5.45)) over $\mathbb{R} \times L_{22}$. Define $\bar{Y} := \Xi(\bar{y}, \bar{\gamma})$ (with Ξ given by (5.41), (5.40)), and H by (5.47). Put*

$$\bar{X}(t) := -\frac{1}{H(t)} \mathbb{E} \left[\left(\frac{\bar{Y}(T) + c}{a} \right) H(T) \mid \mathcal{F}_t \right], \quad \bar{\pi}(t) := [\sigma'(t)]^{-1} \left[\frac{\psi(t)}{H(t)} + \bar{X}(t)\theta(t) \right],$$

(here $\psi \in \mathcal{F}^*$ is the \mathbb{R}^N -valued a.e. unique process on $\Omega \times [0, T]$ such that $\int_0^T \|\psi(t)\|^2 dt < \infty$ a.s. and $\bar{X}(t)H(t) = \bar{X}(0) + \int_0^t \psi'(\tau) dW(\tau)$, given by the martingale representation theorem). Then we have $\bar{\pi} \in \mathcal{A}$ and $\bar{X}(t) = X^{\bar{\pi}}(t)$ a.e. (for $X^{\bar{\pi}}$ defined by (2.5)), and

$$\inf_{\pi \in \mathcal{A}} \mathbb{E}[J(X^\pi(T))] = \mathbb{E}[J(X^{\bar{\pi}}(T))] = - \inf_{(y, \gamma) \in \mathbb{R} \times L_{22}} \tilde{\Psi}(y, \gamma) = -\tilde{\Psi}(\bar{y}, \bar{\gamma}) \in \mathbb{R}.$$

In particular, $\bar{\pi}$ solves the partially constrained problem $(\mathcal{P}_{c,q})$ at (5.15).

Remark 5.26. As noted by Lim and Zhou [9] existence of optimal portfolios for the partially constrained problem (5.15) is non-trivial even without portfolio constraints i.e. $K = \mathbb{R}^N$ (in [9] existence for this problem in the unconstrained case is related to solvability of a backward Riccati SDE - itself a nontrivial question). On the other hand, existence for the dual problem is easy to secure even with portfolio constraints (see Proposition 5.13), and then the optimality relations of Proposition 5.9 are used to synthesize a solution $\bar{\pi}$ of problem (5.15) in terms of a solution of the dual problem, thus establishing existence for (5.15). We shall also see (in Example 6.6) that the duality approach offers genuine computational advantages as well.

Remark 5.27. Cvitanic and Karatzas [4] have established a powerful conjugate duality method for showing existence of optimal portfolios in problems of *constrained utility maximization*. The idea of [4] is to construct a family of *unconstrained* utility maximization problems in a “fictitious market” for which the money-market rate and mean return-rate on the stocks are (very cleverly) parametrized by an element ν of a Hilbert space \mathcal{H} (see [4], Section 8), and then establish existence of some $\nu^* \in \mathcal{H}$ such that the optimal portfolio for the *unconstrained problem* in the fictitious market corresponding to ν^* nevertheless satisfies the given portfolio constraints and is in fact the optimal portfolio for the *constrained* optimization problem. It is shown ([4], Sections 10, 12) how such a ν^* can be characterized as the solution of a *dual optimization problem* over the Hilbert space \mathcal{H} . Essential to the approach

of [4] is the *a-priori* definition of the dependence of the fictitious market on the parameter ν (see (8.3)-(8.4) of [4], p.776); as noted in the introductory discussion of [4] this parametric dependence was not at all easy to come by, and is in fact a very subtle generalization of the more concrete (but also far from obvious) fictitious market introduced in an earlier work on the specific case of utility maximization in incomplete markets (see [6], especially Sections 7, 11, 12). Although the approach of [4] and [6] is extremely effective for problems of utility maximization (in which the wealth process is strictly positive and forced as far to the right as the constraints allow), it does not appear to adapt very easily to problems of variance minimization (in which the wealth process is pushed towards the minimum of an associated quadratic function) and we have not been able to obtain an appropriate analog of the unconstrained fictitious markets of [4]. Moreover, the approach used in the present section, which is motivated by the duality syntheses of Bismut [2] and Rogers [11], has the advantage of giving the dual problem by a completely algorithmic method which obviates the need for the very inspired *a-priori* choice of a fictitious market that is central to the approach of [4].

6 Fully Constrained Optimization Problem

In the present section we return to the main goal of this paper, namely the fully constrained problem (4.12). Our approach relies on Proposition 5.25, together with results from Lagrange duality for convex optimization, as set forth in Aubin ([1], Chapter 2, Section 6).

Throughout this section we postulate Conditions 2.1, 4.1, 4.2, and 4.3. Then we have

- (i) \mathcal{A} is a convex subset of L_{22} (as follows from convexity of K in Condition 4.1);
- (ii) G is an affine functional on L_{22} (as follows from (2.5) and (4.10));
- (iii) $\pi \mapsto \mathbb{E}[\hat{J}(X^\pi(T))]$ defines an \mathbb{R} -valued convex mapping on \mathcal{A} (as follows from Proposition 3.1 and Conditions 4.1 and 4.2).

Now define the *Lagrangian function* for the optimization problem ($\hat{\mathcal{P}}$) at (4.12):

$$(6.74) \quad \mathcal{L}(\mu; \pi) := \mathbb{E}[\hat{J}(X^\pi(T))] + \mu G(\pi), \quad \pi \in L_{22}, \quad \mu \in \mathbb{R}.$$

Then, from Proposition 2.6.1 of ([1], p.36), Condition 4.3, and Theorem 2.6.1 of ([1], p.37), there exists some ‘‘Lagrange multiplier’’ $\bar{\mu} \in \mathbb{R}$ such that (recalling (4.11))

$$(6.75) \quad \hat{\vartheta} = \sup_{\mu \in \mathbb{R}} \inf_{\pi \in \mathcal{A}} \mathcal{L}(\mu; \pi) = \inf_{\pi \in \mathcal{A}} \mathcal{L}(\bar{\mu}; \pi).$$

For each $(\mu, \omega, x) \in \mathbb{R} \times \Omega \times \mathbb{R}$, put

$$(6.76) \quad J_1(\mu; \omega, x) := \frac{1}{2} [a(\omega)x^2 + 2c_\mu(\omega)x] - \mu d, \quad c_\mu(\omega) := c_0(\omega) + \mu c_1(\omega),$$

(for a, c_0, c_1 given by Conditions 4.1 and 4.2) and observe, from (4.9), (4.10) and (6.74),

$$(6.77) \quad \mathcal{L}(\mu; \pi) = \mathbb{E}[J_1(\mu; X^\pi(T))], \quad \pi \in L_{22}, \quad \mu \in \mathbb{R}.$$

Remark 6.1. For each fixed $\mu \in \mathbb{R}$, the function $J_1(\mu; \cdot, \cdot)$ is identical to the function $J(\cdot, \cdot)$ in (5.13), with c_μ in place of c and $-\mu d$ in place of q . In view of Condition 4.2, we see that $\mathbb{E}[c_\mu^2] < +\infty$; that is Condition 5.1 holds with c_μ replacing c for each $\mu \in \mathbb{R}$, and therefore the infima in (6.75) correspond to optimization problems $(\mathcal{P}_{c,q})$ of the form (5.15) (with $c := c_\mu$ and $q := -\mu d$), which are addressed by Proposition 5.25.

Motivated by (5.45) and Remark 6.1, for each $(\mu, y, \gamma) \in \mathbb{R} \times \mathbb{R} \times L_{22}$ and $Y := \Xi(y, \gamma)$, put

$$(6.78) \quad \tilde{\Psi}_1(\mu; y, \gamma) := x_0 y + \mathbb{E} \left[\frac{(Y(T) + c_\mu)^2}{2a} \right] + \mathbb{E} \int_0^T \delta(\Theta_Y(t)) dt + \mu d.$$

Remark 6.2. Proposition 5.25 asserts existence of a minimizer $(\bar{y}(\mu), \bar{\gamma}(\mu)) \in \mathbb{R} \times L_{22}$ of $\tilde{\Psi}_1(\mu; \cdot)$ over $\mathbb{R} \times L_{22}$ for each $\mu \in \mathbb{R}$; motivated by Proposition 5.25, define

$$(6.79) \quad \begin{cases} (1) \bar{Y}(\mu; t) := \Xi(\bar{y}(\mu), \bar{\gamma}(\mu))(t), \\ (2) \bar{X}(\mu; t) := -\frac{1}{H(t)} \mathbb{E} \left[\left(\frac{\bar{Y}(\mu; T) + c_\mu}{a} \right) H(T) \middle| \mathcal{F}_t \right], \\ (3) \bar{\pi}(\mu; t) := [\sigma'(t)]^{-1} \left[\frac{\psi(\mu; t)}{H(t)} + \bar{X}(\mu; t)\theta(t) \right]. \end{cases}$$

for each $\mu \in \mathbb{R}$. From (6.79)(2) and the martingale representation theorem there exists some a.e.-unique \mathbb{R}^N -valued \mathcal{F}_t -progressively measurable process $\psi(\mu; \cdot)$ on $\Omega \times [0, T]$, such that $\int_0^T \|\psi(\mu; t)\|^2 dt < \infty$ a.s. and $\bar{X}(\mu; t)H(t) = \bar{X}(\mu; 0) + \int_0^t \psi'(\mu; \tau) dW(\tau)$ for all $t \in [0, T]$; it is this process which appears on the right side of (6.79)(3). Finally, note from Proposition 5.25 that $\bar{\pi}(\mu) \in \mathcal{A}$ and $\bar{X}(\mu; t) = X^{\bar{\pi}(\mu)}(t)$ a.e. for each $\mu \in \mathbb{R}$ (with $X^{\bar{\pi}(\mu)}$ given by (2.5)).

It remains to show that $\bar{\pi}(\bar{\mu}; \cdot)$ solves problem $(\hat{\mathcal{P}})$ at (4.12). From Proposition 5.25 and (6.77), for each $\mu \in \mathbb{R}$ we have

$$(6.80) \quad \inf_{\pi \in \mathcal{A}} \mathcal{L}(\mu; \pi) = \mathcal{L}(\mu, \bar{\pi}(\mu)) = - \inf_{(y, \gamma) \in \mathbb{R} \times L_{22}} \tilde{\Psi}_1(\mu; y, \gamma) = -\tilde{\Psi}_1(\mu; \bar{y}(\mu), \bar{\gamma}(\mu)) \in \mathbb{R}.$$

Since $\bar{\pi}(\bar{\mu}) \in \mathcal{A}$ (by Remark 6.2), it is enough to show that

$$(6.81) \quad G(\bar{\pi}(\bar{\mu})) = 0,$$

for then Proposition 2 of Aubin ([1], p.37), together with the second equality of (6.75) and the first equality of (6.80) establishes that $\bar{\pi}(\bar{\mu})$ solves the problem (4.12). To obtain (6.81) we use a variational analysis on the optimality of $\bar{\mu}$; from (6.75) and (6.80), we find that

$$(6.82) \quad -\hat{\vartheta} = \inf_{(\mu, y, \gamma) \in \mathbb{R} \times \mathbb{R} \times L_{22}} \tilde{\Psi}_1(\mu; y, \gamma) = \tilde{\Psi}_1(\bar{\mu}; \bar{y}(\bar{\mu}), \bar{\gamma}(\bar{\mu})).$$

Now put $\mu^\epsilon := \bar{\mu} + \epsilon \rho$ for $\rho \in \mathbb{R}$ and $\epsilon \in (0, \infty)$. Then, from (6.82), we have

$$(6.83) \quad 0 \leq \frac{\tilde{\Psi}_1(\mu^\epsilon; \bar{y}(\bar{\mu}), \bar{\gamma}(\bar{\mu})) - \tilde{\Psi}_1(\bar{\mu}; \bar{y}(\bar{\mu}), \bar{\gamma}(\bar{\mu}))}{\epsilon}, \quad \epsilon \in (0, \infty).$$

From the definition of c_μ in (6.76), we have $c_{\mu^\epsilon} = c_{\bar{\mu}} + \epsilon\rho c_1$, hence, from (6.83) and (6.78),

$$0 \leq \rho \mathbb{E} \left[\frac{(\bar{Y}(\bar{\mu}; T) + c_{\bar{\mu}}) c_1}{a} \right] + \epsilon \rho^2 \mathbb{E} \left[\frac{c_1^2}{2a} \right] + \rho d, \quad \epsilon \in (0, \infty);$$

taking $\epsilon \rightarrow 0$, and using arbitrary choice of $\rho \in \mathbb{R}$, then gives $\mathbb{E}[(\bar{Y}(\bar{\mu}; T) + c_{\bar{\mu}}) c_1/a] + d = 0$, which, in view of (6.79)(2), establishes that $\mathbb{E}[c_1 \bar{X}(\bar{\mu}; T)] = d$. Now (6.81) follows from this, together with $\bar{X}(\bar{\mu}; T) = X^{\bar{\pi}(\bar{\mu})}(T)$ (recall Remark 6.2) and (4.10).

Remark 6.3. Notice that the variational analysis used to obtain (6.81) makes essential use (at (6.82)) of the dual cost functional $\tilde{\Psi}_1(\cdot)$ for the partially constrained problem (5.15).

Now we can assemble the preceding and state the main result of the present section. Put

$$(6.84) \quad h(\mu) := \inf_{(y, \gamma) \in \mathbb{R} \times L_{22}} \tilde{\Psi}_1(\mu; y, \gamma), \quad \mu \in \mathbb{R},$$

and note, from (6.80), that the second equality of (6.75) gives $\inf_{\mu \in \mathbb{R}} h(\mu) = h(\bar{\mu})$.

Proposition 6.4. *Suppose Conditions 2.1, 4.1, 4.2, and 4.3. For each $\mu \in \mathbb{R}$, there exists a pair $(\bar{y}(\mu), \bar{\gamma}(\mu)) \in \mathbb{R} \times L_{22}$ which minimizes the functional $\tilde{\Psi}_1(\mu; \cdot)$ over $\mathbb{R} \times L_{22}$ (recall (6.78)), and hence satisfies $h(\mu) = \tilde{\Psi}_1(\mu; \bar{y}(\mu), \bar{\gamma}(\mu))$. Moreover, there exists some $\bar{\mu} \in \mathbb{R}$ which minimizes $h(\cdot)$ on \mathbb{R} , and $\hat{\pi} := \bar{\pi}(\bar{\mu})$ (given by Remark 6.2 with $\mu := \bar{\mu}$) is the optimal portfolio for the problem (\hat{P}) at (4.12).*

Example 6.5. Take $K := \mathbb{R}^N$ in Condition 4.1 for the *unconstrained case*. From (5.28) we see that $\delta(0) = 0$, and $\delta(z) = +\infty$ when $z \neq 0$. Then we need minimize $\tilde{\Psi}_1(\mu; \cdot)$ at (6.78) only over pairs $(y, \gamma) \in \mathbb{R} \times L_{22}$ such that $\Theta_Y(t) = 0$ a.e. (for $Y := \Xi(y, \gamma)$). From Remark 5.11 and nonsingularity of $\sigma(t)$ (Condition 2.1) we obtain $Y_0 = y$ and $\Lambda_Y(t) = -Y(t)\theta(t)$ a.e. Inserting these in (5.42) then shows that $Y(t) = yH(t)$ a.e. and $\gamma \in L_{22}$ necessarily has the form $\gamma(t) = -yH(t)\theta(t)$ a.e. for some $y \in \mathbb{R}$ (recall (5.47)). Determination of the optimal portfolio reduces to the following: (i) for each $\mu \in \mathbb{R}$ locate the minimizer $\bar{y}(\mu) \in \mathbb{R}$ of the functional $y \mapsto \tilde{\Psi}_2(\mu; y) := \tilde{\Psi}_1(\mu; y, -yH\theta)$ (which is quadratic); (ii) use $\bar{y}(\mu)$ to minimize the functional $h(\mu) := \tilde{\Psi}_2(\mu; \bar{y}(\mu))$, $\mu \in \mathbb{R}$ (which is also quadratic). In the special case of the *mean-variance problem* of Remark 4.5, where $a = 2$, $c_0 = 0$, $c_1 = 1$, we have $c_\mu = \mu$ (by (6.76)), and (i) and (ii) lead to the (unique) minimizers $\bar{y}(\mu) = -\{2x_0 + \mu\mathbb{E}[H(T)]\}/\mathbb{E}[H^2(T)]$ and $\bar{\mu} = 2\{x_0\mathbb{E}[H(T)] - d\mathbb{E}[H^2(T)]\}/\text{Var}(H(T))$. Then $\bar{Y}(\bar{\mu}; t) = \bar{y}(\bar{\mu})H(t)$ and $\bar{X}(\bar{\mu}; T) = -(1/2)[\bar{Y}(\bar{\mu}; T) + \bar{\mu}]$ (by (6.79)(2)), the optimal portfolio $\bar{\pi}(\bar{\mu})$ is given by (6.79)(3) with $\mu := \bar{\mu}$ (by Proposition 6.4), and the least variance (or *efficient frontier*) is given by

$$\inf_{\substack{\pi \in L_{22} \\ \mathbb{E}[X^\pi(T)] = d}} \text{Var}(X^\pi(T)) = \text{Var}(\bar{X}(\bar{\mu}; T)) = \frac{1}{4} \text{Var}(\bar{Y}(\bar{\mu}; T)) = \frac{(x_0 - d\mathbb{E}[H(T)])^2}{\text{Var}(H(T))}.$$

Example 6.6. We suppose that $K \subset \mathbb{R}^N$ in Condition 4.1 is a *closed convex cone*, the market coefficients r , b and σ in Condition 2.1 are *nonrandom* continuous functions on $[0, T]$, and

c_0 , c_1 and a in Conditions 4.1 and 4.2 are also *nonrandom*. In this case the dual problem of minimizing $\tilde{\Psi}_1(\mu; y, \gamma)$ over the pairs $(y, \gamma) \in L_{22}$ (recall (6.78)) is particularly well-suited to the application of *dynamic programming* and leads to an essentially explicit formula for the optimal portfolio $\bar{\pi}(\mu)$ at (6.79)(3). Since K is a convex cone, from (5.28) we have $\delta \equiv 0$ on $\tilde{K} := \{z \mid \delta(z) < \infty\}$ (the “barrier cone” of $-K$). Thus the third term on the right of (6.78) takes values in the two-point set $\{0, \infty\}$ according to whether or not $\Theta_Y(t) \in \tilde{K}$ a.e. We can therefore regard $u(t) := \Theta_Y(t)$ (rather than γ) as the “control” in the dual problem, and it then follows from (5.42) and (5.43) that the dual process Y is subject to

$$(6.85) \quad dY(t) = -r(t)Y(t) dt - [\theta(t)Y(t) + \sigma^{-1}(t)u(t)]' dW(t),$$

with $u(t) \in \tilde{K}$ a.e. For each $(y, u) \in \mathbb{R} \times L_{22}$, let $\{\tilde{\Xi}(y, u)(t), t \in [0, T]\}$ denote the process Y given by (6.85) with the initial condition $Y(0) = y$. Then, for arbitrary $\mu \in \mathbb{R}$, the dual problem of minimizing $\tilde{\Psi}_1(\mu; y, \gamma)$ at (6.78) over pairs $(y, \gamma) \in \mathbb{R} \times L_{22}$ is equivalent to the minimization of

$$(6.86) \quad \tilde{\Psi}_3(\mu; y, u) := x_0 y + \mathbb{E}[(\tilde{\Xi}(y, u)(T) + c_\mu)^2 / (2a)] + \mu d,$$

over $(y, u) \in \mathbb{R} \times L_{22}$ with $u(t) \in \tilde{K}$ a.e. (a straightforward application of Gronwall’s inequality yields $\tilde{\Xi}(y, u) \in \mathbb{B}$ for each $(y, u) \in \mathbb{R} \times L_{22}$). We now minimize the second term of (6.86) over $u \in L_{22}$ for arbitrary $y \in \mathbb{R}$. Keeping $\mu \in \mathbb{R}$ fixed, define the value function

$$(6.87) \quad V(\mu; y) := \inf_{\substack{u \in L_{22} \\ u(t) \in \tilde{K} \text{ a.e.}}} \mathbb{E} \left[\frac{(\tilde{\Xi}(y, u)(T) + c_\mu)^2}{2a} \right], \quad y \in \mathbb{R},$$

and consider the Bellman equation associated with (6.85) and (6.87), namely

$$(6.88) \quad \begin{cases} (1) \tilde{v}_s(s, y) - r(s)y \tilde{v}_y(s, y) + (1/2) \inf_{\eta \in \tilde{K}} \left\{ \|\sigma^{-1}(s)\eta + \theta(s)y\|^2 \tilde{v}_{yy}(s, y) \right\} = 0; \\ (2) \tilde{v}(T, y) = (y + c_\mu)^2 / (2a), \end{cases}$$

for each $(s, y) \in [0, T] \times \mathbb{R}$. This is a particularly tractable equation because the infimum in (6.88)(1) is easily expressed in terms of $y \in \mathbb{R}$. Indeed, for $s \in [0, T]$ and $i = 1, 2$, put

$$(6.89) \quad \zeta_i(s) := \arg \min_{\eta \in \tilde{K}} \|\sigma^{-1}(s)\eta - (-1)^i \theta(s)\|^2 = \sigma(s) \text{proj}((-1)^i \theta(s) \mid \sigma^{-1}(s)\tilde{K}),$$

where $\text{proj}(z \mid C)$ is the (uniquely determined) projection of a vector $z \in \mathbb{R}^N$ on a closed convex set $C \subset \mathbb{R}^N$. Then, for each $s \in [0, T]$, it follows that

$$(6.90) \quad \hat{u}(s, y) := \arg \min_{\eta \in \tilde{K}} \|\sigma^{-1}(s)\eta + \theta(s)y\|^2 = \begin{cases} y\zeta_1(s) & \text{if } y \geq 0; \\ -y\zeta_2(s) & \text{if } y < 0. \end{cases}$$

In the light of (6.90), we can easily write down an explicit solution of (6.88). To this end, for $(s, y) \in [0, T] \times \mathbb{R}$ and $i = 1, 2$, (recall β at (5.40) and c_μ at (6.76)), define

$$(6.91) \quad A_i(s) := \exp \left\{ \int_s^T \|\theta(\tau) - (-1)^i \sigma^{-1}(\tau) \zeta_i(\tau)\|^2 d\tau \right\},$$

$$P_i(s) := \frac{1}{a} \left(\frac{\beta(T)}{\beta(s)} \right)^2 A_i(s), \quad \chi(\mu; s) := \frac{c_\mu}{a} \frac{\beta(T)}{\beta(s)}, \quad \alpha(\mu) := \frac{c_\mu^2}{2a},$$

$$(6.92) \quad \tilde{v}(\mu; s, y) := \begin{cases} P_1(s)y^2/2 + \chi(\mu; s)y + \alpha(\mu) & \text{if } (s, y) \in [0, T] \times [0, \infty); \\ P_2(s)y^2/2 + \chi(\mu; s)y + \alpha(\mu) & \text{if } (s, y) \in [0, T] \times (-\infty, 0). \end{cases}$$

Then $\tilde{v}(\mu; \cdot)$ is of class $C^{1,1}$ over $[0, T] \times \mathbb{R}$ and of class $C^{1,2}$ over $[0, T] \times (\mathbb{R} - \{0\})$, and a simple direct verification establishes that it satisfies the Bellman equation (6.88)(1) in the classical sense for each $(s, y) \in [0, T] \times (\mathbb{R} - \{0\})$, as well as the boundary condition (6.88)(2). Moreover, the second-order parabolic sub/superdifferentials of $\tilde{v}(\mu; s, y)$ at $(s, y) \in [0, T] \times \{0\}$ are easily computed to show that $\tilde{v}(\mu; \cdot)$ defines a viscosity solution of (6.88) on $[0, T] \times \mathbb{R}$. It now follows from the verification theorem for dynamic programming ([13], Theorem 5.3, p.270) that \hat{u} at (6.90) is the optimal feedback control for the problem (6.87) with arbitrary $y \in \mathbb{R}$, and $V(\mu; y) = \tilde{v}(\mu; 0, y)$ for all $y \in \mathbb{R}$. In particular the function $y \mapsto V(\mu; y) = \tilde{v}(\mu; 0, y)$ is the ‘‘asymmetric quadratic’’ given by (6.92). Substituting $\hat{u}(t, Y(t))$ for $u(t)$ in (6.85), it follows that the resulting SDE has pathwise-uniqueness (since $\hat{u}(t, \cdot)$ given by (6.90) is globally Lipschitz continuous on \mathbb{R}) with solution (for the initial condition $Y(0) = y \in \mathbb{R}$) given by (recall Remark 5.15)

$$(6.93) \quad \hat{Y}(y; t) := \begin{cases} y\beta(t) \mathcal{E}(-[\theta + \sigma^{-1}\zeta_1]' \bullet W)(t) & \text{if } y \geq 0; \\ y\beta(t) \mathcal{E}(-[\theta - \sigma^{-1}\zeta_2]' \bullet W)(t) & \text{if } y < 0. \end{cases}$$

We are now able to minimize $\tilde{\Psi}_3(\mu; \cdot)$ at (6.86) (still keeping $\mu \in \mathbb{R}$ fixed). Let $\bar{y}(\mu) \in \mathbb{R}$ be the (unique) minimizer (with respect to $y \in \mathbb{R}$) of the ‘‘asymmetric quadratic’’

$$(6.94) \quad \tilde{\Psi}_4(\mu; y) := x_0 y + V(\mu; y) + \mu d = x_0 y + \tilde{v}(\mu; 0, y) + \mu d, \quad y \in \mathbb{R},$$

given by (6.92), and put $\bar{u}(\mu; t) := \hat{u}(t, \hat{Y}(\bar{y}(\mu); t))$, $t \in [0, T]$. Then $\bar{u}(\mu; t) \in \tilde{K}$ a.e. (see (6.90)), and the pair $(\bar{y}(\mu), \bar{u}(\mu)) \in \mathbb{R} \times L_{22}$ is the minimizer of the dual cost functional $\tilde{\Psi}_3(\mu; \cdot)$ defined at (6.86). Comparison of (6.85) with the relations (5.42) and (5.43) then shows that, for $\bar{\gamma}(\mu) \in L_{22}$ defined by $\bar{\gamma}(\mu; t) := -[\theta(t) \tilde{\Xi}(\bar{y}(\mu), \bar{u}(\mu))(t) + \sigma^{-1}(t)\bar{u}(\mu; t)]$, the pair $(\bar{y}(\mu), \bar{\gamma}(\mu))$ minimizes the functional $\tilde{\Psi}_1(\mu; \cdot)$ (see (6.78)) over $\mathbb{R} \times L_{22}$, and (see (6.79)) the corresponding optimal dual process $\bar{Y}(\mu)$ is given by $\bar{Y}(\mu; t) = \hat{Y}(\bar{y}(\mu); t)$, $t \in [0, T]$. Using this representation for $\bar{Y}(\mu)$ it is easy to get explicit formulae for the portfolio $\bar{\pi}(\mu)$ and corresponding wealth $\bar{X}(\mu)$ (see (6.79)(2)(3)). Indeed, upon substituting $\hat{Y}(\bar{y}(\mu); T)$ (given by (6.93)) for $\bar{Y}(\mu; T)$ in (6.79)(2), and using (5.47), the fact that the coefficients r , b and σ are deterministic, and the independent increments of W , we obtain

$$(6.95) \quad -\bar{X}(\mu; t) = \frac{\hat{Y}(\bar{y}(\mu); t)}{a} \exp \left\{ \int_t^T [-2r(\tau) + \theta'(\tau)[\theta(\tau) + \sigma^{-1}(\tau)\zeta_1(\tau)]] d\tau \right\} + \frac{c_\mu \beta(T)}{a\beta(t)},$$

when $\bar{y}(\mu) \geq 0$ (just replace ζ_1 by $-\zeta_2$ in (6.95) to get $\bar{X}(\mu)$ when $\bar{y}(\mu) < 0$). Finally, using (6.93) and Itô's product formula to expand the right side of (6.95) (and its analogue for $\bar{y}(\mu) < 0$), and comparing the result with (2.4), we obtain $\bar{\pi}(\mu)$ such that $\bar{X}(\mu) = X^{\bar{\pi}(\mu)}$ as the following feedback policy on the wealth $\bar{X}(\mu)$:

$$(6.96) \quad \bar{\pi}(\mu; t) := \begin{cases} - \left(\bar{X}(\mu; t) + \frac{c_\mu \beta(T)}{a\beta(t)} \right) (\sigma'(t))^{-1} [\theta(t) + \sigma^{-1}(t)\zeta_1(t)] & \text{if } \bar{y}(\mu) > 0; \\ 0 & \text{if } \bar{y}(\mu) = 0; \\ - \left(\bar{X}(\mu; t) + \frac{c_\mu \beta(T)}{a\beta(t)} \right) (\sigma'(t))^{-1} [\theta(t) - \sigma^{-1}(t)\zeta_2(t)] & \text{if } \bar{y}(\mu) < 0. \end{cases}$$

We now determine the optimal portfolio and minimum variance in the special case of Remark 4.5, for which $a = 2$, $c_0 = 0$, $c_1 = 1$. To this end we first characterize the set \mathcal{R} of Remark 4.6. Define the set $F := \{t \in [0, T] \mid \|\theta(t) + \sigma^{-1}(t)\zeta_1(t)\| > 0\} = \{t \in [0, T] \mid -\sigma(t)\theta(t) \notin \tilde{K}\}$, where the equality follows from (6.89). Since $\tilde{K} = \{z \in \mathbb{R}^N \mid -\pi'z \leq 0, \text{ all } \pi \in K\}$ we then have that $F = \{t \in [0, T] \mid \Gamma(t) \neq \emptyset\}$, where $\Gamma(t) := \{\pi \in K \mid \pi'\sigma(t)\theta(t) > 0\}$. Now suppose $A_1(0) > 1$; then $\lambda(F) > 0$ (by (6.91)), and by the Aumann selection theorem ([12], Theorem 2.3.12, p.71) there is a measurable selection $\pi_1(\cdot)$ of $\Gamma(\cdot)$ on F . Put $\pi_2(t) := 0$, $t \notin F$, and $\pi_2(t) := \pi_1(t)/\|\pi_1(t)\|$, $t \in F$. Then $\pi_2 \in \mathcal{A}$ (since K is a cone) and $\pi_2'(t)\sigma(t)\theta(t) > 0$, $t \in F$, hence from (2.5) we have $E[X^{\pi_2}(T)] > x_0 S_0(T)$, which establishes that $[x_0 S_0(T), \infty) \subset \mathcal{R}$ when $A_1(0) > 1$. Now when $A_1(0) = 1$ then $\lambda(F) = 0$ (by (6.91)), thus for each $\pi \in \mathcal{A}$ we have $\pi'(t)\sigma(t)\theta(t) \leq 0$ a.e., hence $E[X^\pi(T)] \leq x_0 S_0(T)$ (by (2.5)), hence $\mathcal{R} \subset (-\infty, x_0 S_0(T)]$. In this latter case the market model is not “interesting” (in the sense of Remark 4.6), hence we shall suppose that $A_1(0) > 1$ and fix some $d > x_0 S_0(T)$. From (6.84), (6.86), (6.87), (6.94) we have $h(\mu) = \tilde{\Psi}_4(\mu, \bar{y}(\mu))$, $\mu \in \mathbb{R}$. Using (6.92), it is then easy (although tedious) to calculate that $h(\cdot)$ has the unique minimizer $\bar{\mu}$ given by

$$(6.97) \quad \bar{\mu} = \frac{2[x_0 - \beta(T)A_1(0)d]}{\beta(T)[A_1(0) - 1]}, \quad \text{and that} \quad \bar{y}(\bar{\mu}) = \frac{2[\beta(T)d - x_0]}{[A_1(0) - 1]\beta^2(T)} > 0.$$

From this, together with (6.93) and the fact that $\text{Var}(\bar{X}(\bar{\mu}; T)) = \text{Var}(\hat{Y}(\bar{y}(\bar{\mu}); T))/4$ (see (6.95) with $a = 2$), we compute the minimum variance or *efficient frontier*, namely

$$\inf_{\substack{\pi \in \mathcal{A} \\ E[X^\pi(T)] = d}} \text{Var}(X^\pi(T)) = \text{Var}(\bar{X}(\bar{\mu}; T)) = \frac{1}{4} \text{Var}(\hat{Y}(\bar{y}(\bar{\mu}); T)) = \frac{[x_0 - \beta(T)d]^2}{[A_1(0) - 1]\beta^2(T)}.$$

The optimal feedback policy is given by (6.96) with $\mu := \bar{\mu}$, and is easy to implement, since only $\zeta_1(\cdot)$ given by (6.89) need be “precalculated off-line” using the known deterministic coefficients r , b and σ . The simplicity with which dynamic programming applies to the dual problem (for general conical constraints on the portfolio) should be contrasted with the technical complexity involved in applying dynamic programming directly to the primal problem, as in [10], for which the resulting Bellman equation is substantially more involved. As a consequence the analysis in [10] is very specific to the no-shorting constraint (where K is the positive orthant) and relies on the restriction $b_n(t) > r(t)$, $t \in [0, T]$, $n = 1, \dots, N$ (see line following (2.2) in [10]). This restriction excludes the very natural possibility that

interest rates may increase at some point in the investment interval, exceeding - one hopes only temporarily - the mean rates of return on some stocks, and also excludes the all-too-real possibility that some stocks might temporarily underperform over part of the investment horizon (in the sense that the mean return rate $b_n(t)$ is less than the interest rate $r(t)$ for some values of t) but perform well in the remainder of the trading interval. The preceding duality analysis removes these restrictions and works for completely general conical constraints.

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