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Stochastic Processes and their Applications 104 (2003) 57–80
stochastic processes and their applications
www.elsevier.com/locate/spa

Strong invariance principle for singular diffusions

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Received 14 October 2001; received in revised form 25 August 2002; accepted 25 August 2002

Abstract

We study a singular diffusion on Euclidean space which is characterized by the solution of a classical Itô stochastic differential equation in which the diffusion coefficient is not necessarily of full rank. Our motivation is in earlier results of Basak (J. Multivariate Anal. 39 (1991) 44) and Basak and Bhattacharya (Ann. Probab. 20 (1992) 312), who establish sufficient conditions for singular diffusions to have a unique invariant probability and obtain a functional central limit theorem and functional law of the iterated logarithm for a large class of real-valued functions of the diffusion. Under similar conditions we establish a strong invariance principle for vector-valued functions of the diffusion, and use this to derive several asymptotic properties of the singular diffusion, including upper/lower-function estimates and a vector form of the functional law of the iterated logarithm.

MSC: primary 60F17; secondary 60J60

Keywords: Stochastic differential equations; Singular diffusions; Invariant probability measures; Multivariate strong invariance principles; Upper/lower-function estimates; Laws of the iterated logarithm

1. Introduction

Consider an $\mathfrak{R}^d$-valued diffusion $\{\xi(t), t \in [0,\infty)\}$ given by the solution of the classical Itô stochastic differential equation

$$\mathrm{d}\xi(t) = b(\xi(t))\,\mathrm{d}t + \sigma(\xi(t))\,\mathrm{d}W(t), \tag{1}$$

in which the coefficients $b(\cdot)$ and $\sigma(\cdot)$ are globally Lipschitz continuous, and $\{W(t), t \in [0,\infty)\}$ is an R^D-valued standard Wiener process. Of particular interest are asymptotic

[1] Supported by NSERC

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0304-4149/03/$ - see front matter
PII: S0304-4149(02)00222-3

properties of the *singular diffusion* $\{\xi(t), t\in[0,\infty)\}$ which is given by (1) when the matrix $a(x) \triangleq \sigma(x)\sigma^{\mathrm{T}}(x)$ fails to be nonsingular for all $x\in\mathfrak{R}^d$. Basak (1991, Theorem 2.1) and Basak and Bhattacharya (1992, Theorem 2.1) establish simple and general sufficient conditions on the coefficients $b(\cdot)$ and $\sigma(\cdot)$ which, even in the singular case, ensure that the Markov process defined by (1) has a unique invariant probability measure (see also Theorem 4.1 on p. 593 of Bhattacharya and Waymire, 1990 for a special case of these results). Moreover, subject to a natural strengthening of these conditions, Basak (1991) establishes a functional central limit theorem and a Strassen functional law of the iterated logarithm for the real-valued process $\{f(\xi(t)), t\in[0,\infty)\}$, for each mapping f belonging to a large class C_r of Hölder-continuous mappings $f:\mathfrak{R}^d\to\mathfrak{R}$ (see (3.15), together with Theorem 3.3(b) and Theorem 3.4 of Basak, 1991).

The particular challenge involved in establishing these results is that, as pointed out in Basak (1991) and Basak and Bhattacharya (1992), functions of singular diffusions generally do not have any of the usual mixing properties, such as uniform or strong mixing, for which functional central limit theorems and laws of the iterated logarithm are available. In fact, the primary tools used by Basak (1991) to establish these results are Liapunov function methods, together with a functional central limit theorem and functional law of the iterated logarithm for ergodic Markov processes due to Bhattacharya (1982, Theorems 2.1 and 2.7).

In probability theory there is a class of limit results originating with Strassen (1964), called strong invariance principles, which are "master theorems" on the basis of which one can establish functional central limit theorems, functional laws of the iterated logarithm, and several other asymptotic properties, as straightforward corollaries. If $\{z_k, k=1,2,\dots\}$ is a sequence of $\mathfrak{R}^N$-valued random variables on the probability space $(\Omega,\mathscr{F},P)$ then a *multivariate strong invariance principle with good rate* is said to hold when there is some constant $\eta\in(0,\infty)$, and some $\mathfrak{R}^N$-valued Wiener process $\{B(t), t\in[0,\infty)\}$ on $(\Omega,\mathscr{F},P)$ (or an extension thereof), such that P-a.s.

$$\left|\sum_{1\leqslant k\leqslant t} z_k - B(t)\right| = \mathrm{O}(t^{1/2-\eta}) \quad (\text{as } t\to\infty). \tag{2}$$

This result says that one can approximate the sample paths of the partial-sum process $\{S_n, n=1,2,\dots\}$, $S_n \triangleq \sum_{1\leqslant k\leqslant n} z_k$, by the paths of some Wiener process $\{B(t)\}$ to within an error given by the right-hand side of (2). The advantage of this formulation is that the error is precise enough for many asymptotic properties, known for the Wiener process $\{B(t)\}$, to be readily extended to the partial sums $\{S_n\}$, including, for example, precise upper/lower-function estimates of the rate of increase of $|S_n|$ and its maximum $M_n \triangleq \max_{1\leqslant k\leqslant n}|S_k|$ as $n\to\infty$, vector-valued functional laws of the iterated logarithm for the partial sums $\{S_n\}$, and functional laws of the iterated logarithm for the maximum process $\{M_n\}$.

Motivated by (2) we are going to establish a multivariate strong invariance principle with good rate for an $\mathfrak{R}^N$-valued function $f(\xi(t))$ of the singular diffusion $\{\xi(t)\}$ given by (1), when the entries $f^i:\mathfrak{R}^d\to\mathfrak{R}$ of the N-fold vector $f \triangleq (f^1, f^2, \dots, f^N)$ are members of a class of real-valued Hölder continuous mappings on $\mathfrak{R}^d$, quite similar to the function class C_r introduced by Basak (see (3.15) of Basak, 1991), and subject to hypotheses on the coefficients $b(\cdot)$ and $\sigma(\cdot)$ that are similar to, although somewhat stronger than, the hypotheses assumed in Basak (1991). We shall also use this result to establish several asymptotic properties of the $\mathfrak{R}^N$-valued process $\{f(\xi(t))\}$, including an upper/lower-function estimate on the rate of increase of the magnitude of the process $\{\int_0^t f(\xi(s))\,\mathrm{d}s\}$, and a vector form of the functional law of the iterated logarithm, as more or less direct consequences of the strong invariance principle. Our basic approach will be to use the Markov property of (1) to verify the hypotheses of a result of Eberlein (1986), which establishes multivariate strong invariance principles of the form (2) subject to very general conditions on the process $\{z_k\}$ (see Theorem 3.1 which follows).

In Section 2 we state the basic hypotheses, the main result (a strong invariance principle given by Theorem 2.12), and several corollaries. All proofs are relegated to Section 3.

2. Conditions and main result

We use the following notation: $\mathfrak{R}^d$ denotes the space of real d-dimensional column vectors with Euclidean norm $|x| \triangleq [\sum_{i=1}^d (x^i)^2]^{1/2}$ and inner product $(x, y) \triangleq \sum_{i=1}^d x^i y^i$ for all $x, y \in \mathfrak{R}^d$, while $\mathfrak{R}^{d\times D}$ denotes the space of real d by D matrices with operator norm $\|A\| \triangleq \max_{x\in\mathfrak{R}^D, |x|=1} |Ax|$ for all $A \in \mathfrak{R}^{d\times D}$. Write $(A)'$ for the transpose of a matrix A; $\mathrm{Tr}\{A\}$ for the trace of a square matrix A; and $\Lambda_{\min}\{A\}$, $\Lambda_{\max}\{A\}$, for the minimum and the maximum eigenvalues, respectively, of a real symmetric matrix A.

The following basic Conditions 2.1 and 2.2 will always be assumed for (1):

Condition 2.1. $\{W(t), t \in [0,\infty)\}$ is an $\mathfrak{R}^D$-valued standard Wiener process on the complete probability space $(\Omega, \mathscr{F}, P)$, and the mappings $b^i : \mathfrak{R}^d \to \mathfrak{R}$, $\sigma^{i,j} : \mathfrak{R}^d \to \mathfrak{R}$, for $i = 1, 2, \ldots, d$, and $j = 1, 2, \ldots, D$, are globally Lipschitz continuous; in particular,

$$\lambda_0 \triangleq \sup_{x,y\in\mathfrak{R}^d, x\neq y} \frac{\|\sigma(x)-\sigma(y)\|}{|x-y|} < \infty. \tag{3}$$

To formulate the next condition put

$$a(x) \triangleq \sigma(x)(\sigma(x))', \quad a(x, y) \triangleq (\sigma(x)-\sigma(y))(\sigma(x)-\sigma(y))', \tag{4}$$

for all $x, y \in \mathfrak{R}^d$.

Condition 2.2. For some constant $r \in [1,\infty)$, some symmetric and positive definite matrix $D \in \mathfrak{R}^{d\times d}$, and some constant $\beta \in (2(r-1)\Lambda_{\max}\{D\}\lambda_0^2, \infty)$, we have

$$2[b(x)-b(y)]'D(x-y) + \mathrm{Tr}\{a(x,y)D\} - \frac{2(x-y)'[Da(x,y)D](x-y)}{(x-y)'D(x-y)} \leqslant -\beta|x-y|^2, \tag{5}$$

for all distinct $x, y \in \mathfrak{R}^d$.

Remark 2.3. For later use note that $\beta \in (2(r-1)\Lambda_{\max}\{D\}\lambda_0^2, \infty)$ if and only if

$$\kappa(r,\beta,D) \triangleq r - \frac{\beta}{2\Lambda_{\max}\{D\}\lambda_0^2} < 1. \tag{6}$$

Remark 2.4. For each $x \in \mathfrak{R}^d$, let $\{\xi(t,x), t \in [0,\infty)\}$ denote the (pathwise-unique) $\mathfrak{R}^d$-valued continuous process on $(\Omega, \mathscr{F}, P)$ which is adapted to the filtration $\{\mathscr{F}_t^W\}$ defined by $\mathscr{F}_t^W \triangleq \sigma\{W(\tau), \tau \in [0,t]\} \vee \{P\text{-null sets in } \mathscr{F}\}$ and which solves (1) with the nonrandom initial condition $\xi(0) = x$.

Remark 2.5. Conditions 2.1 and 2.2 do not postulate nonsingularity of $a(x)$, and therefore include the case of singular diffusions. These conditions should be compared with the following Assumptions (A_1) (A_2), and (A_{3r}), which are the essential hypotheses adopted by Basak (1991) to secure a functional central limit theorem and law of the iterated logarithm for functions of a singular diffusion given by (1):

(A_1) same as our Condition 2.1;

(A_2) For some positive definite symmetric matrix $C \in \mathfrak{R}^{d\times d}$, and some constant $\gamma \in (0,\infty)$, we have

$$2[b(x)-b(y)]'C(x-y) + \mathrm{Tr}\{a(x,y)C\} - \frac{2(x-y)'[Ca(x,y)C](x-y)}{(x-y)'C(x-y)} \leqslant -\gamma|x-y|^2,$$

for all distinct $x, y \in \mathfrak{R}^d$;

(A_{3r}) For some constant $r \in [1,\infty)$, some symmetric and positive definite matrix $D \in \mathfrak{R}^{d\times d}$, and some constant $\beta_1 \in (2(r-1)\Lambda_{\max}\{D\}\lambda_0^2, \infty)$, we have

$$2[b(x)]'Dx + \mathrm{Tr}\{a(x)D\} - \frac{2x'[Da(x)D]x)}{x'Dx} \leqslant -\beta_1|x|^2,$$

for all large $|x|$.

Now Condition 2.2 of course implies Assumption (A_2) (with $C \triangleq D$ and $\gamma \triangleq \beta$). Moreover, from the elementary inequality

$$z^{c_1} \leqslant \delta^{(c_2-c_1)} z^{c_2} + \frac{1}{\delta^{c_1}} \quad \forall z \in [0,\infty) \tag{7}$$

(which holds for constants $0 \leqslant c_1 \leqslant c_2 < \infty$ and $\delta \in (0,\infty)$), and Condition 2.2, it is easily seen that, for each constant $\beta_1 \in (2(r-1)\Lambda_{\max}\{D\}\lambda_0^2, \beta)$, the inequality in Assumption (A_{3r}) holds for all sufficiently large $|x|$. Consequently, our Conditions 2.1 and 2.2 imply assumptions (A_1), (A_2) and (A_{3r}), and therefore we can use the results of Basak (1991) to conclude the following facts, which will be essential in the sequel:

(I) Theorem 2.1 of Basak (1991) shows that (i) there exists a unique invariant probability measure $\bar{m}$ on $\mathfrak{R}^d$ for the Markov process defined by (1), and (ii) for each uniformly bounded and globally Lipschitz continuous mapping $f: \mathfrak{R}^d \to \mathfrak{R}$, we have

$$\lim_{t\to\infty} E[f(\xi(t,x))] = \int_{\mathfrak{R}^d} f \,\mathrm{d}\bar{m} \quad \forall x \in \mathfrak{R}^d. \tag{8}$$

Thus (8) holds, in particular, for each f in $C_c^\infty(\mathfrak{R}^d)$ (the set of all infinitely smooth mappings from $\mathfrak{R}^d$ into $\mathfrak{R}$ with compact support), and therefore, since $C_c^\infty(\mathfrak{R}^d)$ is convergence-determining (see e.g. Theorem 5.1 on p. 645 of Bhattacharya and Waymire, 1990 or Problem 3.11.11 on p. 151 of Ethier and Kurtz, 1986), we see that (8) holds for each uniformly bounded and continuous f.

(II) By Proposition 3.1 of Basak (1991) (see line (3.8) in particular), for each $\varepsilon \in (\kappa(r,\beta,D),1)$ (recall Remark 2.3) there is a constant $C_1(\varepsilon) \in [0,\infty)$ such that

$$\sup_{t\in[0,\infty)} E[(\xi(t,x), D\xi(t,x))^{(r-\varepsilon)}] \leqslant (x,Dx)^{(r-\varepsilon)} + C_1(\varepsilon) \quad \forall x \in \mathfrak{R}^d.$$

(III) From Lemma 3.2 of Basak (1991), for each $\varepsilon \in (\kappa(r,\beta,D),1)$, we have

$$\int_{\mathfrak{R}^d} |x|^{2(r-\varepsilon)}\,\mathrm{d}\bar{m}(x) < \infty. \tag{9}$$

Remark 2.6. Our version of Assumption (A_2) in Remark 2.5 is slightly different from the statement of (A_2) on p. 46 of Basak (1991), and takes into account Remark 2.2 on p. 47 of Basak (1991)—but note that the term $2b(x,y)$ occurring in the inequality of Basak (1991, Remark 2.2) is misprinted and should be corrected to read $2[b(x)-b(y)]'C(x-y)$. Also, our statement of Assumption (A_1) looks a bit weaker than the statement of (A_1) on p. 46 of Basak (1991), which requires that $b(\cdot)$ be smooth with uniformly bounded first-order partial derivatives. In fact, global Lipschitz continuity of $b(\cdot)$ is enough for nearly all of the results established in Basak (1991) (including the results that we used in Remark 2.5), the smoothness of $b(\cdot)$ with bounded first derivatives being used only for Corollary 2.3 on p. 49 of Basak (1991), which is not needed here.

Remark 2.7. In light of Remark 2.5(I) one sees that Conditions 2.1 and 2.2 are sufficient for existence of a unique invariant probability measure for the diffusion given by (1), even when the matrix $\sigma\sigma^{\mathrm{T}}(x)$ is singular for some values of x. This is in contrast to the integral criteria of Khas'Minskii (1960) and Bhattacharya (1978) which give existence of a unique invariant probability in the case of *nondegenerate* diffusions (that is, $\sigma\sigma^{\mathrm{T}}(x)$ is nonsingular for each $x \in \mathfrak{R}^d$). Notice that Example 2.1 of Basak and Bhattacharya (1992) gives a diffusion which is both nondegenerate and satisfies Conditions 2.1 and 2.2 (thus, in view of Remark 2.5(I), has a unique invariant probability) but which, despite being nondegenerate, falls outside the scope of the integral test given by Theorem 3.5(a) of Bhattacharya (1978), which generalizes Khas'Minskii's integral criterion for existence of a unique invariant probability measure.

Remark 2.8. Singular diffusions arise quite naturally in physics. Consider a particle of unit mass with position $\{\eta(t)\}$ and velocity $\{v(t)\}$ in $\mathfrak{R}^3$, on which three forces are acting, namely (i) an externally imposed force field $K(\eta)$ depending on the position η of the particle, (ii) linear friction with coefficient β, and (iii) a random force modeled by a standard "white noise". Then $\xi(t) \triangleq (\eta(t), v(t))$ is given by the singular diffusion

(1) with the coefficients

$$b(\xi) \equiv b(\eta, v) \triangleq \begin{bmatrix} v \\ K(\eta) - \beta v \end{bmatrix}, \quad \sigma(\xi) \equiv \sigma(\eta, v) \triangleq \begin{bmatrix} 0_3 \\ I_3 \end{bmatrix},$$

and where $\{W(t)\}$ is an $\mathfrak{R}^3$-valued standard Wiener process (see Section 10 of Nelson, 1967). Thus, if Conditions 2.1 and 2.2 are verified for these coefficients (notice that $\lambda_0 = 0$, since $\sigma(\cdot)$ is a constant function), then the singular diffusion $\xi(t) \triangleq (\eta(t), v(t))$ has a unique invariant probability $\bar{m}$ (see Remark 2.5(I)), and the conclusions given by Theorem 2.12 as well as Corollaries 2.13 and 2.15 hold for $\{\xi(t)\}$.

Remark 2.9. Without loss of generality we will suppose that there is an $\mathfrak{R}^d$-valued random vector Z on the probability space $(\Omega, \mathscr{F}, P)$ in Condition 2.1, which is P-independent of $\{W(t), t \in [0, \infty)\}$ and has the invariant probability measure $\bar{m}$, given by Remark 2.5(I), for its distribution. We shall denote by $\{\bar{\xi}(t), t \in [0, \infty)\}$ the $\mathfrak{R}^d$-valued continuous process on $(\Omega, \mathscr{F}, P)$ which is adapted to the filtration $\{\mathscr{F}_t^{Z,W}\}$ defined by $\mathscr{F}_t^{Z,W} \triangleq \sigma\{Z, W(\tau), \tau \in [0, t]\} \vee \{P\text{-null sets in } \mathscr{F}\}$, and which solves (1) with the initial condition $\xi(0) = Z$ a.s. Then $\{\bar{\xi}(t), t \in [0, \infty)\}$ is a *strictly stationary solution* of (1) with one-dimensional marginal distribution given by the probability $\bar{m}$.

Remark 2.10. Our goal is to establish a strong invariance principle with good rate for processes of the form $\{f(\xi(t, x)), t \in [0, \infty)\}$ or $\{f(\bar{\xi}(t)), t \in [0, \infty)\}$, for a class of functions f which we introduce next: Fix some $r \in [1, \infty)$ and a pair $(\varepsilon, \theta) \in \mathfrak{R}^2$ such that $\varepsilon \in (-\infty, 1)$ and $\theta \in (0, 1 - \varepsilon]$; then $(r - \varepsilon - \theta) \in [0, \infty)$. For a Borel-measurable mapping $f : \mathfrak{R}^d \to \mathfrak{R}$, put

$$\|f\|_{r-\varepsilon} \triangleq \sup_{x \in \mathfrak{R}^d} \left[\frac{|f(x)|}{1 + |x|^{(r-\varepsilon)}} \right], \tag{10}$$

$$[f]_{r-\varepsilon,\theta} \triangleq \sup_{x,y \in \mathfrak{R}^d, x \neq y} \left[\frac{|f(x) - f(y)|}{|x - y|^{\theta}[1 + |x|^{(r-\varepsilon-\theta)} + |y|^{(r-\varepsilon-\theta)}]} \right], \tag{11}$$

$$M_{r,\varepsilon,\theta}(f) \triangleq \max\{\|f\|_{r-\varepsilon}, [f]_{r-\varepsilon,\theta}\}, \tag{12}$$

and

$$\mathscr{C}_0(r, \varepsilon, \theta) \triangleq \{f : \mathfrak{R}^d \to \mathfrak{R}\colon M_{r,\varepsilon,\theta}(f) < \infty\}. \tag{13}$$

Now suppose that Condition 2.2 holds (for some r, β, and D), and let

$$\mathscr{C}(r, \beta, D) \triangleq \bigcup_{(\varepsilon,\theta)} \mathscr{C}_0(r, \varepsilon, \theta), \tag{14}$$

where the set-union in (14) is over all pairs (ε, θ) such that

$$\varepsilon \in (\kappa(r, \beta, D), 1), \quad \theta \in (0, 1 - \varepsilon], \tag{15}$$

(see Remark 2.3). In view of Remark 2.5(III) we have $\int_{\mathfrak{R}^d} |f|^2 \, d\bar{m} < \infty$ for each $f \in \mathscr{C}(r, \beta, D)$. Observe that $\mathscr{C}(r, \beta, D)$ is essentially the same as the function class C_r defined by (3.15) of Basak (1991).

Remark 2.11. Without loss of generality we shall suppose that there is a uniformly distributed random variable $U : (\Omega, \mathscr{F}, P) \to [0, 1]$ which is P-independent of $\sigma\{Z, W(t), t \in [0, \infty)\}$, where Z is specified in Remark 2.9. This will ensure that $(\Omega, \mathscr{F}, P)$ carries an "approximating" Wiener process $\{B(t)\}$ in the following multivariate strong invariance principle with good rate, and that no "extension" of this probability space is necessary:

Theorem 2.12. *Suppose Conditions* 2.1 *and* 2.2. *Fix finitely many functions* $f^i \in \mathscr{C}(r, \beta, D)$, $i = 1, 2, \ldots, N$, *such that* $\int_{\mathfrak{R}^d} f^i \, d\bar{m} = 0$, *and define the mapping* $f : \mathfrak{R}^d \to \mathfrak{R}^N$ *by* $f(x) \triangleq (f^1(x), f^2(x), \ldots, f^N(x))'$, $\forall x \in \mathfrak{R}^d$. *Let* $\{\xi(t), t \in [0, \infty)\}$ *be the* $\mathfrak{R}^d$-*valued process which is either* (i) *a solution* $\{\xi(t, \tilde{x}), t \in [0, \infty)\}$ *of* (1) *with nonrandom initial condition* $\xi(0) = \tilde{x}$, *for some arbitrary but fixed* $\tilde{x} \in \mathfrak{R}^d$, *or* (ii) *a strictly stationary solution* $\{\bar{\xi}(t), t \in [0, \infty)\}$ *of* (1) *with one-dimensional marginal* $\bar{m}$ (*see Remark* 2.9). *Then the integrals in*

$$G^{i,j} \triangleq \int_0^\infty E[f^i(\xi(0)) f^j(\xi(t))] \, dt + \int_0^\infty E[f^j(\xi(0)) f^i(\xi(t))] \, dt, \tag{16}$$

$i, j = 1, 2, \ldots, N$, *converge absolutely*, G *is a real symmetric positive semidefinite* $N \times N$-*matrix, and there exists a constant* $\eta \in (0, \infty)$, *and an* $\mathfrak{R}^N$-*valued Wiener process* $\{B(t), t \in [0, \infty)\}$ *on* $(\Omega, \mathscr{F}, P)$, *such that* $E[B(1)B'(1)] = G$, *and* P-*a.s.*

$$\left| \int_0^t f(\xi(s)) \, ds - B(t) \right| = O(t^{1/2 - \eta}) \quad (\textit{as } t \to \infty). \tag{17}$$

The strong invariance principle of Theorem 2.12 enables one to apply known asymptotic properties of the multivariate Wiener process $\{B(t), t \in [0, \infty)\}$ to the multivariate process $\{\int_0^t f(\xi(s)) \, ds, t \in [0, \infty)\}$. The following corollaries illustrate two examples of this:

Corollary 2.13. *Suppose Conditions* 2.1 *and* 2.2. *Let the mapping* $f : \mathfrak{R}^d \to \mathfrak{R}^N$, *the* $\mathfrak{R}^d$-*valued process* $\{\xi(t), t \in [0, \infty)\}$, *and the* $N \times N$-*matrix* G, *be as in Theorem* 2.12, *let* n_1 *denote the multiplicity of the largest eigenvalue of* G, *and put* $\alpha \triangleq \sup\{|G^{1/2} z|: \ z \in \mathfrak{R}^N, |z| \leqslant 1\}$. *Then, for each continuous and nondecreasing* $\psi : [1, \infty) \to (0, \infty)$, *we have*

$$P\left[\left| \int_0^t f(\xi(s)) \, ds \right| > \alpha t^{1/2} \psi(t) \textit{ i.o. as } t \to \infty \right] = 0 \textit{ or } 1,$$

according as

$$I(\psi) \triangleq \int_1^\infty \frac{[\psi(t)]^{n_1}}{t} \exp\{-\psi^2(t)/2\} \, dt < \infty \textit{ or } = \infty.$$

Remark 2.14. Corollary 2.13 is similar to the multivariate version of the Kolmogorov upper/lower-function test for the rate of increase of a Brownian motion (see (14), (15) on p. 163 of Itô and McKean, 1974). The notation $G^{1/2}$ indicates the unique real symmetric positive semidefinite square root of G (see Theorem 7.2.6 of Horn and Johnson, 1985, p. 405). For the next result let $C[0,1]$ be the normed vector space of all continuous mappings $h:[0,1]\to\mathfrak{R}^N$ with the usual supremum norm, and let $AC_0[0,1]$ be the set of absolutely continuous mappings $h:[0,1]\to\mathfrak{R}^N$ with $h(0)=0$. Also, put

$$K \triangleq \left\{ h\in AC_0[0,1]\colon\ \int_0^1 (\dot{h}(\tau))'\dot{h}(\tau)\,\mathrm{d}\tau \leqslant 1 \right\}. \tag{18}$$

Corollary 2.15. *Suppose Conditions* 2.1 *and* 2.2. *Let the mapping* $f:\mathfrak{R}^d\to\mathfrak{R}^N$, *the* $\mathfrak{R}^d$*-valued process* $\{\xi(t), t\in[0,\infty)\}$, *and the* $N\times N$*-matrix* G, *be as in Theorem* 2.12, *and put*

$$\Xi_n(t) \triangleq n^{-1/2}\int_0^{nt} f(\xi(s))\,\mathrm{d}s \quad \forall t\in[0,\infty),\ \ n=1,2,\ldots\ . \tag{19}$$

Then

(i) *for P-almost all* ω, *the sequence of* $\mathfrak{R}^N$*-valued functions defined by*

$$\left\{ \frac{\Xi_n(t,\omega)}{\sqrt{2\log\log n}}, t\in[0,1] \right\}$$

is relatively compact in $C[0,1]$ *and the set of its limit points is given by* $G^{1/2}[K]\subset AC_0[0,1]$.

(ii) $\{\Xi_n(t), t\in[0,\infty)\}$ *converges weakly to* $\mathfrak{R}^N$*-valued Wiener measure with zero drift and covariance matrix* G *as* $n\to\infty$.

Remark 2.16. The preceding corollaries are included only to illustrate the type of limiting properties which follow from a strong invariance principle, and several other asymptotic results besides these are possible. For example, in the case where $N\triangleq 1$ (i.e. $f(\xi(t))$ is real-valued) one can establish upper/lower-function estimates, similar in form to Corollary 2.13, for the rate of increase (as $t\to\infty$) of the *maximum process* $\{\max_{0\leqslant\tau\leqslant t}|\int_0^\tau f(\xi(s))\,\mathrm{d}s|\}$ and a functional law of the iterated logarithm for the sequence $\{\max_{0\leqslant\tau\leqslant t}|\int_0^{n\tau} f(\xi(s))\,\mathrm{d}s|, t\in[0,1]\}$, $n=1,2,\ldots$. These follow by trivial modification of the arguments used to establish Theorem B on p. 3 and Theorem D on p. 4 of Philipp and Stout (1975).

Remark 2.17. The main results of Basak (1991) are essentially the functional law of the iterated logarithm and central limit theorem given by Corollary 2.15 in the particular case where $N=1$ (see Theorem 3.3 and Theorem 3.4 in Basak, 1991), but established subject to Assumptions (A_1), (A_2) and (A_{3r}) of Remark 2.5 which, as we have noted, are more general than our Conditions 2.1 and 2.2. On the other hand, in return for our rather stronger conditions, we get a strong invariance principle which provides a comprehensive characterization of the asymptotic properties of vector-valued functions of the singular diffusion $\{\xi(t)\}$, as illustrated by Corollary 2.13, Corollary 2.15, and Remark 2.16. Moreover, Conditions 2.1 and 2.2 are only a rather mild strengthening of Assumptions (A_1), (A_2) and (A_{3r}). In fact, the main difference is that (A_2) is formulated in terms of a real symmetric positive definite matrix C, while (A_{3r}) is formulated in terms of another real symmetric positive definite matrix D. In contrast, Condition 2.2 allows the choice of just one real symmetric positive definite matrix D, so that, in principle, one has the benefit of an additional degree of freedom when verifying (A_2) and (A_{3r}). In practice however, this extra degree of freedom is typically not used, and the easiest way to check Assumptions (A_2) and (A_{3r}) is actually to first verify Condition 2.2 (e.g. by using the sufficient conditions of Proposition 2.20 to follow). Then, as noted in Remark 2.5, (A_2) and (A_{3r}) follow at once, with the matrix C in (A_2) and the matrix D in (A_{3r}) both being given by the matrix D in Condition 2.2.

Remark 2.18. Theorem 2.12 and its corollaries involve a trade-off between the parameters (r,β,D) in Condition 2.2 and the class of functions $\mathscr{C}(r,\beta,D)$ of which the f^i are members. To see this, fix a real symmetric positive definite matrix D, and let (β_1,β_2) and (r_1,r_2) be such that $\kappa(r_1,\beta_1,D) \leqslant \kappa(r_2,\beta_2,D) < 1$ and $1 \leqslant r_2 \leqslant r_1$. Then it follows that $\beta_2 \leqslant \beta_1$. That is, to postulate Condition 2.2 for the parameters $r \triangleq r_1$, $\beta \triangleq \beta_1$ and D, is to make a more restrictive "stability" hypothesis on the coefficients of (1) than if one postulated Condition 2.2 for the parameters $r \triangleq r_2$, $\beta \triangleq \beta_2$ and D. On the other hand, it is easily seen from Remark 2.10 that $\mathscr{C}(r_2,\beta_2,D) \subset \mathscr{C}(r_1,\beta_1,D)$. Thus, in return for the more restrictive hypothesis on the coefficients of (1), we get Theorem 2.12 for functions f of the Markov process $\{\xi(t)\}$ which belong to the larger class $\mathscr{C}(r_1,\beta_1,D)$.

Remark 2.19. Suppose that Condition 2.1 holds, and the inequality (5) holds for some real, symmetric, positive definite matrix $D\in\mathfrak{R}^{d\times d}$, and some constant $\beta > 2\Lambda_{\max}\{D\}\lambda_0^2$. Then $\kappa(1,\beta,D) < 0$ (see (6)). For $f^i : \mathfrak{R}^d \to \mathfrak{R}$ defined by $f^i(x) \triangleq x^i$, $\forall x\in\mathfrak{R}^d$, $\forall i = 1,2,\ldots,d$, one sees from (13) that $f^i\in\mathscr{C}_0(1,0,1)$, and, since $\kappa(1,\beta,D) < 0$, the pair $(\varepsilon,\theta)=(0,1)$ satisfies (15) with $r \triangleq 1$, and therefore $f^i\in\mathscr{C}(1,\beta,D)$, $i=1,2,\ldots,d$. Moreover, since $\beta > 0$, we see that Condition 2.2 certainly holds for $r=1$. Thus, Theorem 2.12 and its Corollaries apply in the case where $r \triangleq 1$, $N \triangleq d$, $f(x) \triangleq x$, $\forall x\in\mathfrak{R}^d$, and thus we have a strong invariance principle with good rate, an upper/lower-function test, and a vector-valued functional law of the iterated logarithm for the $\mathfrak{R}^d$-valued process $\{\xi(t), t\in[0,\infty)\}$.

The following variant of Corollary 2.3 of Basak (1991) gives sufficient conditions on the drift term $b(x)$ of (1) which imply Condition 2.2:

Proposition 2.20. *Suppose that (3) holds for some $\lambda_0\in[0,\infty)$ and that the $b^i : \mathfrak{R}^d \to \mathfrak{R}$ are C^1-mappings with uniformly bounded first-order derivatives $J^{i,j}(x) \triangleq \partial_i b^j(x)$, $i,j = 1,2,\ldots,d$. If, for some constant $r\in[1,\infty)$ and some symmetric positive definite matrix $D\in\mathfrak{R}^{d\times d}$, we have*

$$\vartheta \triangleq \sup_{x\in\mathfrak{R}^d} \Lambda_{\max}\{DJ(x) + (J(x))'D\} < (3-d-2r)\Lambda_{\max}\{D\}\lambda_0^2, \tag{20}$$

then there is a constant $\beta \in (2(r-1)\Lambda_{\max}\{D\}\lambda_0^2, \infty)$ *such that Condition* 2.2 *holds* (*for the same symmetric positive definite D and constant r*).

Of particular interest is the case where the drift in (1) is *linear* (the stability-in-distribution properties of (1) with linear drift were established by Basak and Bhattacharya, 1992). We have

Corollary 2.21. *Suppose that* (3) *holds for some* $\lambda_0 \in [0, \infty)$, *and suppose that* $b(x) \triangleq Bx$, $\forall x \in \mathfrak{R}^d$, *in* (1), *for a constant matrix* $B \in \mathfrak{R}^{d\times d}$, *all eigenvalues of which have strictly negative real parts. If*

$$(d + 2r - 3)\lambda_0^2 \Lambda_{\max}\{D\} < 1, \tag{21}$$

for some constant $r \in [1, \infty)$, *and symmetric positive definite matrix* $D \in \mathfrak{R}^{d\times d}$ *given by*

$$D \triangleq \int_0^\infty \exp\{sB'\}\exp\{sB\}\,\mathrm{d}s, \tag{22}$$

then there is a constant $\beta \in (2(r-1)\Lambda_{\max}\{D\}\lambda_0^2, \infty)$ *such that Condition* 2.2 *holds* (*for the same matrix D and constant r*).

Remark 2.22. The significance of Proposition 2.20 is that one must compensate for rapid variations in $\sigma(\cdot)$ (indicated by a large value for λ_0 in (3)) by having all eigenvalues of the symmetric matrix $DJ(x) + (J(x))'D$ sufficiently far on the left side of the complex plane uniformly in $x \in \mathfrak{R}^d$.

3. Proofs

In this section we establish the results stated in Section 2. The main result used to establish Theorem 2.12 is the following multivariate strong invariance principle with good rate due to Eberlein (1986, Theorem 1):

Theorem 3.1. *Suppose that* $\{z_k, k = 1, 2, \ldots\}$ *is an* $\mathfrak{R}^N$*-valued zero-mean sequence adapted to the filtration* $\{\mathscr{G}_k, k = 1, 2, \ldots\}$ *on the probability space* $(\Omega, \mathscr{F}, P)$, *and there is a uniformly distributed random variable* $U : (\Omega, \mathscr{F}, P) \to [0, 1]$ *which is P-independent of* $\sigma\{\mathscr{G}_k, k = 1, 2, \ldots\}$. *Put* $S_n^i(m) \triangleq \sum_{m+1}^{n+m} z_k^i$, $\forall n, m = 1, 2, \ldots$, $\forall i = 1, 2, \ldots, N$, *and suppose there are constants* $c_1, c_2, c_3, c_4 \in (0, \infty)$, *and a real symmetric positive semidefinite* $N \times N$*-matrix G*, *such that*

$$\sup_{m\geqslant 1} E|E[S_n^i(m) \mid \mathscr{G}_m]| = \mathrm{O}(n^{1/2-c_1}), \tag{23}$$

$$\sup_{m\geqslant 1} E|E[S_n^i(m)S_n^j(m) \mid \mathscr{G}_m] - E[S_n^i(m)S_n^j(m)]| = \mathrm{O}(n^{1-c_2}), \tag{24}$$

$$\sup_{m\geqslant 1} |n^{-1}E[S_n^i(m)S_n^j(m)] - G^{i,j}| = \mathrm{O}(n^{-c_3}), \tag{25}$$

$$\sup_{k\geqslant 1} E[|z_k^i|^{2+c_4}] < \infty, \tag{26}$$

for all $i,j = 1,2,\dots,N$. *Then there exists a constant* $\eta \in (0,\infty)$, *and an* $\mathfrak{R}^N$-*Wiener process* $\{B(t), t\in[0,\infty)\}$ *on* $(\Omega,\mathscr{F},P)$, *such that* $E[B(1)B'(1)] = G$, *and P-a.s.*

$$\left|\sum_{1\leqslant k\leqslant t} z_k - B(t)\right| = \mathrm{O}(t^{1/2-\eta}) \quad (as\ t\to\infty). \tag{27}$$

Remark 3.2. Notice that Theorem 3.1 does not postulate any specific dependency property, such as a mixing or martingale-difference structure, for the process $\{z_k\}$, but only assumes rate-conditions on the first and second conditional moments of the partial sum $S_n(m)$. Consequently, Theorem 3.1 applies to a very general class of dependency structures. As we shall see, the Markov property of (1), together with stability properties of $\{\xi(t)\}$ that are implicit in Conditions 2.1 and 2.2, are the essential things that we will need in order to verify the conditions of Theorem 3.1. This approach should be contrasted with that used by Basak (1991) to establish functional central limit theorems and laws of the iterated logarithm, which is to identify and characterize a large subset of the range of the infinitesimal generator of the Markov process given by (1), and then use results of Bhattacharya (1982) which give a central limit theorem and law of the iterated logarithm for functions of a general Markov process that belong to the range of its infinitesimal generator.

Remark 3.3. Suppose Conditions 2.1 and 2.2. It follows at once from Remark 2.5(II), the Liapunov L^p-inequality (see Shiryaev, 1996, p. 193), Rayleigh's principle (see Theorem 4.2.2 of Horn and Johnson, 1985, p. 176), and $\Lambda_{\min}\{D\} > 0$, that for each $\varepsilon\in(\kappa(r,\beta,D),1)$ and $\rho\in(0,2(r-\varepsilon)]$, there is a constant $C\in[0,\infty)$, depending only on ε and ρ, such that

$$E|\xi(t,x)|^\rho \leqslant C[1+|x|^\rho] \quad \forall t\in[0,\infty),\ x\in\mathfrak{R}^d.$$

This inequality will often be used in the following proofs.

Proof of Theorem 2.12. (i) Fix some $\tilde{x}\in\mathfrak{R}^d$, and put $\tilde{\xi}(t) \triangleq \xi(t,\tilde{x})$. We first verify (23) for appropriately defined $S_n^i(m)$. In view of (14) we have $f^i\in\mathscr{C}_0(r,\varepsilon_i,\theta_i)$ for a pair (ε_i,θ_i) which satisfies (15), and so, from Remark 3.3 and (9), we have

$$\sup_{t\in[0,\infty)} E|f^i(\tilde{\xi}(t))|^2 \leqslant C_1, \tag{28}$$

for some constant $C_1\in[0,\infty)$. Now put

$$Q_T^i(t_0) \triangleq \int_{t_0}^{T+t_0} [f^i(\tilde{\xi}(t)) - Ef^i(\tilde{\xi}(t))]\,\mathrm{d}t \quad \forall T,t_0\in[0,\infty). \tag{29}$$

From (28) and the Fubini Theorem for conditional expectations and ordinary integrals (Ethier and Kurtz, 1986, no. 2.4.6 and 2.4.7, p. 74), we have

$$E[Q_T^i(t_0)\,|\,\mathscr{F}_{t_0}^W] = \int_{t_0}^{T+t_0} E[f^i(\tilde{\xi}(t))\,|\,\mathscr{F}_{t_0}^W]\,\mathrm{d}t - \int_{t_0}^{T+t_0} Ef^i(\tilde{\xi}(t))\,\mathrm{d}t \tag{30}$$

(where $\mathscr{F}^W_{t_0}$ is defined in Remark 2.4). For each $i = 1, 2, \dots, N$, put

$$\Psi^i(t,x) \triangleq E[f^i(\xi(t,x))] \quad \forall t \in [0,\infty) \ \forall x \in \mathfrak{R}^d, \tag{31}$$

for $\xi(t,x)$ as in Remark 2.4, and observe from the Markov property of (1) (see e.g. Theorem 14.27 of Elliott, 1982, p. 196) that

$$E[f^i(\tilde{\xi}(t)) \,|\, \mathscr{F}^W_{t_0}] = \Psi^i(t - t_0, \tilde{\xi}(t_0)) \quad \forall t \in [t_0, \infty). \tag{32}$$

We need the following result, which is established later in the present section:

Lemma 3.4. *Suppose Conditions* 2.1 *and* 2.2, *and fix a pair* (ε, θ) *such that* (15) *holds. Then there are constants* $C \in [0,\infty)$ *and* $\gamma \in (0,\infty)$, *depending only on* (ε, θ), *such that*

$$\left| Ef(\xi(t,x)) - \int_{\mathfrak{R}^d} f \, d\bar{m} \right| \leqslant C[f]_{r-\varepsilon,\theta} \exp\{-\gamma t\}[1 + |x|^{(r-\varepsilon)}], \tag{33}$$

for all $f \in \mathscr{C}_0(r, \varepsilon, \theta)$ *and* $(t,x) \in [0,\infty) \times \mathfrak{R}^d$.

Since $\int_{\mathfrak{R}^d} f^i \, d\bar{m} = 0$, from (32) and Lemma 3.4 it follows that

$$|\Psi^i(t,x)| \leqslant C_2 \exp\{-\gamma_1 t\}[1 + |x|^{(r-\varepsilon_i)}] \quad \forall (t,x) \in [0,\infty) \times \mathfrak{R}^d,$$

for some constants $C_2 \in [0,\infty)$ and $\gamma_1 \in (0,\infty)$, and therefore, from (32),

$$|E[f^i(\tilde{\xi}(t)) \,|\, \mathscr{F}^W_{t_0}]| \leqslant C_2 \exp\{-\gamma_1(t - t_0)\}[1 + |\tilde{\xi}(t_0)|^{(r-\varepsilon_i)}], \tag{34}$$

for all $t \in [t_0, \infty)$. Now, from Remark 3.3 and the fact that $\tilde{x}$ is fixed, we have $\sup_{t_0 \in [0,\infty)} E|\tilde{\xi}(t_0)|^{(r-\varepsilon_i)} < \infty$, and so, in view of (34) and $\gamma_1 > 0$, there is a constant $C_3 \in [0,\infty)$ such that

$$\int_{t_0}^{T+t_0} E|E[f^i(\tilde{\xi}(t)) \,|\, \mathscr{F}^W_{t_0}]| \, dt \leqslant C_3 \quad \forall T, t_0 \in [0,\infty), \tag{35}$$

and thus, from Jensen's inequality,

$$\int_{t_0}^{T+t_0} |Ef^i(\tilde{\xi}(t))| \, dt \leqslant C_3 \quad \forall T, t_0 \in [0,\infty). \tag{36}$$

In view of (30), (35), and (36),

$$E|E[Q^i_T(t_0) \,|\, \mathscr{F}^W_{t_0}]| \leqslant 2C_3 \quad \forall T, t_0 \in [0,\infty). \tag{37}$$

Now define the $\mathfrak{R}^N$-valued, zero-mean, random vectors

$$z_k \triangleq \int_{k-1}^{k} [f(\tilde{\xi}(t)) - Ef(\tilde{\xi}(t))] \, dt \quad \forall k = 1, 2, 3, \dots \, . \tag{38}$$

Then

$$Q_n^i(m) = S_n^i(m) \triangleq \sum_{m+1}^{n+m} z_k^i \quad \forall m, n = 1, 2, \ldots, \tag{39}$$

so (23) follows from (37) (with $c_1 \triangleq 1/2$, $\mathscr{G}_m \triangleq \mathscr{F}_m^W$, and $S_n(m)$ given by (39)).

We next verify (24): In view of (28) and the Cauchy–Schwarz inequality, we have the finite bound $E|f^i(\tilde{\xi}(s))f^j(\tilde{\xi}(t))| \leqslant C_1$, $\forall s, t \in [0, \infty)$, so we can certainly define

$$g^{i,j}(s, t; t_0) \triangleq E[f^i(\tilde{\xi}(s))f^j(\tilde{\xi}(t)) \mid \mathscr{F}_{t_0}^W] - E[f^i(\tilde{\xi}(s))f^j(\tilde{\xi}(t))], \tag{40}$$

for all $t_0 \in [0, \infty)$ and $s, t \in [t_0, \infty)$. From (29) and the Fubini theorem for conditional expectations and ordinary integrals, we have

$$\begin{aligned} E[Q_T^i(t_0)Q_T^j(t_0) \mid \mathscr{F}_{t_0}^W] = {} & \int_{t_0}^{T+t_0} E[f^i(\tilde{\xi}(s))f^j(\tilde{\xi}(t)) \mid \mathscr{F}_{t_0}^W]\,\mathrm{d}s\,\mathrm{d}t \\ & - \left(\int_{t_0}^{T+t_0} Ef^j(\tilde{\xi}(t))\,\mathrm{d}t\right) \int_{t_0}^{T+t_0} E[f^i(\tilde{\xi}(s)) \mid \mathscr{F}_{t_0}^W]\,\mathrm{d}s \\ & - \left(\int_{t_0}^{T+t_0} Ef^i(\tilde{\xi}(s))\,\mathrm{d}s\right) \int_{t_0}^{T+t_0} E[f^j(\tilde{\xi}(t)) \mid \mathscr{F}_{t_0}^W]\,\mathrm{d}t \\ & + \left(\int_{t_0}^{T+t_0} Ef^i(\tilde{\xi}(s))\,\mathrm{d}s\right)\left(\int_{t_0}^{T+t_0} Ef^j(\tilde{\xi}(t))\,\mathrm{d}t\right). \end{aligned} \tag{41}$$

Now center (41) at its expectation, then use (40) and Fubini's theorem, to get

$$\begin{aligned} & E[Q_T^i(t_0)Q_T^j(t_0) \mid \mathscr{F}_{t_0}^W] - E[Q_T^i(t_0)Q_T^j(t_0)] \\ & \quad = \int_{t_0}^{T+t_0} \left[\int_{t_0}^{t} g^{i,j}(s, t; t_0)\,\mathrm{d}s\right] \mathrm{d}t + \int_{t_0}^{T+t_0} \left[\int_{t}^{T+t_0} g^{i,j}(s, t; t_0)\,\mathrm{d}s\right] \mathrm{d}t \\ & \quad\quad - \left(\int_{t_0}^{T+t_0} Ef^i(\tilde{\xi}(s))\,\mathrm{d}s\right) E[Q_T^j(t_0) \mid \mathscr{F}_{t_0}^W] \\ & \quad\quad - \left(\int_{t_0}^{T+t_0} Ef^j(\tilde{\xi}(t))\,\mathrm{d}t\right) E[Q_T^i(t_0) \mid \mathscr{F}_{t_0}^W], \end{aligned} \tag{42}$$

for all $T, t_0 \in [0, \infty)$. We next upper-bound the expectation of the magnitude of the first term on the right-hand side of (42). From the composition rule for conditional expectation and (32)

$$E[f^i(\tilde{\xi}(s))f^j(\tilde{\xi}(t)) \mid \mathscr{F}_{t_0}^W] = E[f^i(\tilde{\xi}(s))\Psi^j(t - s, \tilde{\xi}(s)) \mid \mathscr{F}_{t_0}^W], \tag{43}$$

for all $0 \leqslant t_0 \leqslant s \leqslant t < \infty$. For each $i, j = 1, 2, \ldots, N$, put

$$\Phi^{i,j}(t_1, t_2, x) \triangleq E[f^i(\xi(t_1, x))\Psi^j(t_2, \xi(t_1, x))], \tag{44}$$

for all $t_1, t_2 \in [0, \infty)$, $x \in \mathfrak{R}^d$, so that, recalling $\tilde{\xi}(s) \triangleq \xi(s, \tilde{x})$, we have

$$\Phi^{i,j}(s, t-s, \tilde{x}) = E[f^i(\tilde{\xi}(s))\Psi^j(t-s, \tilde{\xi}(s))], \tag{45}$$

for all $0 \leqslant s \leqslant t < \infty$. Also, from the Markov property for (1), we get

$$E[f^i(\tilde{\xi}(s))\Psi^j(t-s, \tilde{\xi}(s)) \mid \mathscr{F}^W_{t_0}] = \Phi^{i,j}(s-t_0, t-s, \tilde{\xi}(t_0)), \tag{46}$$

for all $0 \leqslant t_0 \leqslant s \leqslant t < \infty$. Now combine (40), (43), (45), and (46) to get

$$g^{i,j}(s,t;t_0) = \left[\Phi^{i,j}(s-t_0, t-s, \tilde{\xi}(t_0)) - \int_{\mathfrak{R}^d} f^i(\xi)\Psi^j(t-s,\xi)\,\mathrm{d}\bar{m}(\xi)\right]$$
$$+ \left[\int_{\mathfrak{R}^d} f^i(\xi)\Psi^j(t-s,\xi)\,\mathrm{d}\bar{m}(\xi) - \Phi^{i,j}(s, t-s, \tilde{x})\right], \tag{47}$$

for all $0 \leqslant t_0 \leqslant s \leqslant t < \infty$. Next, we need the following result, which is proved later in the present section:

Lemma 3.5. *Suppose that Conditions* 2.1 *and* 2.2 *hold, let* f^i, $i = 1, 2, \ldots, N$, *be as in Theorem* 2.12, *and define* $\Psi^j(\cdot)$ *and* $\Phi^{i,j}(\cdot)$ *by* (31) *and* (44), *respectively. Then there are constants* $C \in [0, \infty)$ *and* $\gamma \in (0, \infty)$, *depending only on* $(f^i, \varepsilon_i, \theta_i)$ *and* $(f^j, \varepsilon_j, \theta_j)$, *such that*

$$|f^i(x)\Psi^j(t,x)| \leqslant C\exp\{-\gamma t\}[1 + |x|^{(2r-\varepsilon_i-\varepsilon_j)}], \tag{48}$$

for all $t \in [0, \infty)$ *and* $x \in \mathfrak{R}^d$, *and*

$$\left|\Phi^{i,j}(t_1, t_2, x) - \int_{\mathfrak{R}^d} f^i(\xi)\Psi^j(t_2,\xi)\,\mathrm{d}\bar{m}(\xi)\right|$$
$$\leqslant C\exp\{-\gamma(t_1+t_2)\}[1 + |x|^{(2r-\varepsilon_i-\varepsilon_j)}] \quad \forall t_1, t_2 \in [0,\infty),\ x \in \mathfrak{R}^d. \tag{49}$$

From Lemma 3.5 and (47), there are constants $C_4 \in [0, \infty)$ and $\gamma_2 \in (0, \infty)$ such that

$$|g^{i,j}(s,t;t_0)| \leqslant C_4\exp\{-\gamma_2(t-t_0)\}[1 + |\tilde{\xi}(t_0)|^{(2r-\varepsilon_i-\varepsilon_j)}]$$
$$+ C_4\exp\{-\gamma_2 t\}[1 + |\tilde{x}|^{(2r-\varepsilon_i-\varepsilon_j)}], \tag{50}$$

for all $0 \leqslant t_0 \leqslant s \leqslant t < \infty$. Now, put $\varepsilon_{i,j} \triangleq (\varepsilon_i + \varepsilon_j)/2$. We have $\varepsilon_i, \varepsilon_j \in (\kappa(r, \beta, D), 1)$ (since (15) holds for the pairs $(\varepsilon_i, \theta_i)$ and $(\varepsilon_j, \theta_j)$), thus $\varepsilon_{i,j} \in (\kappa(r, \beta, D), 1)$. In view of Remark 3.3, we have

$$\sup_{t_0} E[|\tilde{\xi}(t_0)|^{(2r-\varepsilon_i-\varepsilon_j)}] = \sup_{t_0} E[|\xi(t_0, \tilde{x})|^{2(r-\varepsilon_{i,j})}] \leqslant C_5[1 + |\tilde{x}|^{(2r-\varepsilon_i-\varepsilon_j)}],$$

for some constant $C_5 \in [0, \infty)$. From this, together with (50) and the fact that $\tilde{x}$ is fixed, we have

$$E[|g^{i,j}(s,t;t_0)|] \leqslant C_6\exp\{-\gamma_2(t-t_0)\} \quad \forall\, 0 \leqslant t_0 \leqslant s \leqslant t < \infty, \tag{51}$$

for some constant $C_6 \in [0,\infty)$; and therefore

$$E\left|\int_{t_0}^{T+t_0}\left[\int_{t_0}^{t} g^{i,j}(s,t;t_0)\,\mathrm{d}s\right]\mathrm{d}t\right| \leqslant C_6\int_{t_0}^{T+t_0}(t-t_0)\exp\{-\gamma_2(t-t_0)\}\,\mathrm{d}t$$
$$\leqslant C_7, \tag{52}$$

for all $T, t_0 \in [0,\infty)$ and some constant $C_7 \in [0,\infty)$. Since an upper-bound identical in form to (51) holds with i and j interchanged, we can similarly bound the expectation of the magnitude of the second term on the right-hand side of (42). From this, together with (37), (36), (52), and (42), we find a constant $C_8 \in [0,\infty)$ such that

$$E|E[Q_T^i(t_0)Q_T^j(t_0)\,|\,\mathscr{F}_{t_0}^W] - E[Q_T^i(t_0)Q_T^j(t_0)]| \leqslant C_8,$$

for all $T, t_0 \in [0,\infty)$, which, in view of (38) and (39), verifies (24) (with $c_2 \triangleq 1$ and $\mathscr{G}_m \triangleq \mathscr{F}_m^W$).

We next verify (25). Let $\{\bar{\xi}(t), t\in[0,\infty)\}$ be a strictly stationary solution of (1) with one-dimensional marginal $\bar{m}$ (see Remark 2.9). Then $\{f^i(\bar{\xi}(t)), t\in[0,\infty)\}$ is zero-mean (since we assume $\int_{\mathfrak{R}^d} f^i\,\mathrm{d}\bar{m} = 0$) and strictly stationary. Put

$$\Gamma^{i,j}(t) \triangleq \int_{\mathfrak{R}^d} f^i(\xi)\Psi^j(t,\xi)\,\mathrm{d}\bar{m}(\xi) \quad \forall t\in[0,\infty) \tag{53}$$

(see (31)) and note, from the Markov property of (1), that

$$E[f^i(\bar{\xi}(s))f^j(\bar{\xi}(t))] = E[f^i(\bar{\xi}(s))\Psi^j(t-s,\bar{\xi}(s))] = \Gamma^{i,j}(t-s), \tag{54}$$

for all $0 \leqslant s \leqslant t < \infty$. From Remark 2.5(III) we have $E|\bar{\xi}(0)^{(2r-\varepsilon_i-\varepsilon_j)} < \infty$, and thus, in the light of Lemma 3.5 (see (48)), there are constants $\gamma_3 \in (0,\infty)$ and $C_9 \in [0,\infty)$ such that

$$|\Gamma^{i,j}(t)| = |E[f^i(\bar{\xi}(0))\Psi^j(t,\bar{\xi}(0))]| \leqslant C_9\exp\{-\gamma_3 t\} \quad \forall t\in[0,\infty), \tag{55}$$

so we can put

$$G^{i,j} \triangleq \int_0^\infty [\Gamma^{i,j}(t) + \Gamma^{j,i}(t)]\,\mathrm{d}t, \quad i,j = 1,2,\ldots,N. \tag{56}$$

Note that $\Gamma^{i,j}(t) = E[f^i(\bar{\xi}(0))f^j(\bar{\xi}(t))]$, $\forall t\in[0,\infty)$ (see (54)), so that G is a symmetric positive semidefinite $N\times N$-matrix. From (54), (55), and standard rotation of coordinates, one sees that there is a constant $C_{10} \in [0,\infty)$ such that

$$\left|\frac{1}{T}\int_{t_0}^{T+t_0}\int_{t_0}^{T+t_0} E[f^i(\bar{\xi}(s))f^j(\bar{\xi}(t))]\,\mathrm{d}s\,\mathrm{d}t - G^{i,j}\right| \leqslant \frac{C_{10}}{T}, \tag{57}$$

for all $T\in(0,\infty)$ and $t_0\in[0,\infty)$. Define

$$h^{i,j}(s,t) \triangleq E[f^i(\tilde{\xi}(s))f^j(\tilde{\xi}(t))] - E[f^i(\bar{\xi}(s))f^j(\bar{\xi}(t))], \tag{58}$$

for all $s,t\in[0,\infty)$. Then, from (29) and (58),

$$\frac{1}{T}E[Q_T^i(t_0)Q_T^j(t_0)]-G^{i,j}$$
$$=\frac{1}{T}\int_{t_0}^{T+t_0}\int_{t_0}^{T+t_0}h^{i,j}(s,t)\,\mathrm{d}s\,\mathrm{d}t$$
$$+\left[\frac{1}{T}\int_{t_0}^{T+t_0}\int_{t_0}^{T+t_0}E[f^i(\bar{\xi}(s))f^j(\bar{\xi}(t))]\,\mathrm{d}s\,\mathrm{d}t-G^{i,j}\right]$$
$$-\frac{1}{T}\left(\int_{t_0}^{T+t_0}Ef^i(\tilde{\xi}(t))\,\mathrm{d}t\right)\left(\int_{t_0}^{T+t_0}Ef^j(\tilde{\xi}(t))\,\mathrm{d}t\right). \tag{59}$$

From (58), (43), (45), (54), and (53)

$$h^{i,j}(s,t)=\Phi^{i,j}(s,t-s,\tilde{x})-\int_{\mathfrak{R}^d}f^i(\xi)\Psi^j(t-s,\xi)\,\mathrm{d}\bar{m}(\xi), \tag{60}$$

for all $0\leqslant s\leqslant t<\infty$, thus, from Lemma 3.5, $|h^{i,j}(s,t)|\leqslant C_{11}\exp\{-\gamma_4 t\}$, for all $0\leqslant s\leqslant t<\infty$, for some constants $C_{11}\in[0,\infty)$ and $\gamma_4\in(0,\infty)$; since an identical upper-bound holds with i and j interchanged, we then get

$$\int_{t_0}^{T+t_0}\int_{t_0}^{T+t_0}|h^{i,j}(s,t)|\,\mathrm{d}s\,\mathrm{d}t\leqslant C_{12}\quad\forall T,t_0\in[0,\infty), \tag{61}$$

for some constant $C_{12}\in[0,\infty)$. Now combine (36), (57), (59), and (61), to get

$$\left|\frac{1}{T}E[Q_T^i(t_0)Q_T^j(t_0)]-G^{i,j}\right|\leqslant\frac{C_{13}}{T},$$

for some constant $C_{13}\in[0,\infty)$, and all $T\in(0,\infty)$ and $t_0\in[0,\infty)$. In view of (38) and (39), this verifies (25) (with $c_3\triangleq 1$ and $\mathscr{G}_m\triangleq\mathscr{F}_m^W$).

We next verify (26). It is enough to show that

$$\sup_{t\in[0,\infty)}E[|f^i(\tilde{\xi}(t))|^{2+\delta_i}]<\infty, \tag{62}$$

for some $\delta_i\in(0,\infty)$, for then (26) follows (with $c_4\triangleq\delta_i$) from (38), (62), and Jensen's inequality. Since $f^i\in\mathscr{C}_0(r,\varepsilon_i,\theta_i)$, where (15) holds for the pair (ε_i,θ_i), there is some $\rho_i\in(0,\infty)$ such that $\bar{\varepsilon}_i\triangleq(\varepsilon_i-\rho_i)\in(\kappa(r,\beta,D),1)$. Put $\delta_i\triangleq 2\rho_i/(r-\varepsilon_i)$, and observe that $(2+\delta_i)(r-\varepsilon_i)=2(r-\bar{\varepsilon}_i)$. Then, from (10)

$$|f^i(x)|^{(2+\delta_i)}\leqslant[2\|f^i\|_{r-\varepsilon_i}]^{(2+\varepsilon_i)}[1+|x|^{2(r-\bar{\varepsilon}_i)}]\quad\forall x\in\mathfrak{R}^d, \tag{63}$$

so that (62) follows from (63) and Remark 3.3.

All conditions for Theorem 3.1 have now been verified, and consequently there is a constant $\eta\in(0,\infty)$ and an $\mathfrak{R}^N$-valued Wiener process $\{B(t)\}$ on $(\Omega,\mathscr{F},P)$, such that $E[B'(1)B(1)]=G$, and (27) holds P-a.s. for $\{z_k\}$ defined by (38). That is, in view of (36), we have P-a.s.

$$\left|\int_0^{\lfloor t\rfloor} f(\tilde{\bar{\xi}}(t))\,\mathrm{d}t - B(t)\right| = \mathrm{O}(t^{1/2-\eta}) \quad (\text{as } t\to\infty). \tag{64}$$

Now put

$$\chi_n^i \triangleq \max_{n\leqslant t\leqslant n+1}\left|\int_n^t f^i(\tilde{\bar{\xi}}(s))\,\mathrm{d}s\right| \qquad \forall n=1,2,\dots\,. \tag{65}$$

By Jensen's inequality, together with (62), there exists some constant $C_{14}\in[0,\infty)$ such that

$$E|\chi_n^i|^{(2+\delta_i)} \leqslant \int_n^{n+1} E|f^i(\tilde{\bar{\xi}}(s))|^{(2+\delta_i)}\,\mathrm{d}s \leqslant C_{14},$$

for all $n=1,2,\dots$, thus, with $q_i\triangleq 2+(\delta_i/2)$, we have $\sum_n P[|\chi_n^i|>n^{1/q_i}]<\infty$ a.s. Then, from the Borel–Cantelli Theorem, there is a constant $\alpha_i\in(0,\infty)$ such that $|\chi_n^i| = \mathrm{O}(n^{1/2-\alpha_i})$ $(n\to\infty)$, that is,

$$\max_{\lfloor t\rfloor\leqslant\tau\leqslant 1+\lfloor t\rfloor}\left|\int_{\lfloor t\rfloor}^{\tau} f^i(\tilde{\bar{\xi}}(s))\,\mathrm{d}s\right| = \mathrm{O}(t^{1/2-\alpha_i}) \quad (\text{as } t\to\infty). \tag{66}$$

Now (17) follows from (64) and (66).

(ii) The case where $\{\xi(t)\}$ is identified with a strictly stationary solution $\{\bar{\xi}(t)\}$ of (1) is an obvious simplification of case (i) just considered. □

Proof of Corollary 2.13. This is an immediate consequence of Theorem 2.12, together with Theorem 2.4 of Kuelbs (1975) (which extends the Kolmogorov upper/lower-function test for scalar Brownian motion to vector-valued Brownian motion), and the argument used to establish Theorem 5.1 of Jain et al. (1975, pp. 130–131), which generalizes trivially to the vector-valued case. □

Proof of Corollary 2.15. This is identical to the proof of Theorem C on p. 3 of Philipp and Stout (1975). □

Proof of Proposition 2.20. The proof is quite similar to the proofs of Corollary 2.3 of Basak (1991) and Corollary 2.2 of Basak and Bhattacharya (1992), so we just summarize the main calculations. Fix arbitrary distinct $x,y\in\mathfrak{R}^d$. From the Rayleigh principle (see Theorem 4.2.2 of Horn and Johnson, 1985, p. 176) we have

$$\mathrm{Tr}\{a(x,y)D\} - \frac{2(x-y)'[Da(x,y)D](x-y)}{(x-y)'D(x-y)} \leqslant (d-1)\Lambda_{\max}\{D^{1/2}a(x,y)D^{1/2}\},$$

and, from (3) and (4), we have the inequality $\Lambda_{\max}\{D^{1/2}a(x,y)D^{1/2}\} \leqslant \lambda_0^2\Lambda_{\max}\{D\}|x-y|^2$. Also, from the mean-value theorem, the Rayleigh principle, and (20), we have

$$\begin{aligned}
&2[b(x)-b(y)]'D(x-y)\\
&\quad=\int_0^1 (x-y)'[DJ(y+\alpha(x-y))+J'(y+\alpha(x-y))D](x-y)\,\mathrm{d}\alpha\\
&\quad\leqslant \vartheta|x-y|^2.
\end{aligned}$$

Since $\vartheta<(3-d-2r)\Lambda_{\max}\{D\}\lambda_0^2$, and the preceding inequalities hold for arbitrary distinct $x,y\in\Re^d$, we obtain (5) for $\beta \triangleq -\vartheta+(1-d)\Lambda_{\max}\{D\}\lambda_0^2>2(r-1)\lambda_0^2\Lambda_{\max}\{D\}$. □

Proof of Corollary 2.21. From Bellman (1960, Theorem 6, p. 175) we see that the matrix D given by (22) solves the identity $DB+B'D=-I$, and is symmetric positive definite. Thus $\Lambda_{\max}\{DB+B'D\}=-1$, and (20) of Proposition 2.20 is an immediate consequence of (21). □

We establish Lemmas 3.4 and 3.5 in the remainder of this section. To this end, we need the following preliminary results:

Lemma 3.6. *Suppose Conditions* 2.1 *and* 2.2. *Then, for each* $\varepsilon\in(\kappa(r,\beta,D),1)$ (*recall Remark* 2.3), *there is a constant* $\alpha\in(0,\infty)$, *depending only on* ε, *such that*

$$\begin{aligned}
&E[[(\xi(t,x)-\xi(t,y))'D(\xi(t,x)-\xi(t,y))]^{(r-\varepsilon)}]\\
&\quad\leqslant \exp\{-\alpha t\}[(x-y)'D(x-y)]^{(r-\varepsilon)}\quad \forall t\in[0,\infty),\ x,y\in\Re^d.
\end{aligned}$$

Remark 3.7. Basak and Bhattacharya (1992, see (2.17)) establish this result in the special case where $r\triangleq 1$ in Condition 2.2, and $b(\cdot)$ has the linear form $b(x)\triangleq Bx$, $x\in\Re^d$. An identical computation (involving Itô's formula) works for the more general conditions of Lemma 3.6, so the proof is omitted.

Lemma 3.8. *Suppose Conditions* 2.1 *and* 2.2, *and fix* (ε,θ) *such that* (15) *holds. Then there are constants* $C\in[0,\infty)$ *and* $\gamma\in(0,\infty)$, *depending only on* (ε,θ), *with the following property*: *if, for some* $f\in\mathscr{C}_0(r,\varepsilon,\theta)$, *we define*

$$\Psi(t,x)\triangleq E[f(\xi(t,x)]\quad \forall(t,x)\in[0,\infty)\times\Re^d, \tag{67}$$

then (*recalling* (11)) *we have*

$$[\Psi(t,\cdot)]_{r-\varepsilon,\theta}\leqslant C[f]_{r-\varepsilon,\theta}\exp\{-\gamma t\}\quad \forall t\in[0,\infty). \tag{68}$$

Proof of Lemma 3.8. Since $r\in[1,\infty)$ and $\theta\in(0,1-\varepsilon]$, we have $r-\varepsilon-\theta\geqslant 0$. Suppose, to begin with, that $r-\varepsilon-\theta>0$, and define $p\triangleq(r-\varepsilon)/\theta$, $q\triangleq(r-\varepsilon)/(r-\varepsilon-\theta)$. Then we have $p,q\in(1,\infty)$ with $p^{-1}+q^{-1}=1$. Since $\Lambda_{\min}\{D\}>0$, it follows from Lemma 3.6, Rayleigh's inequality, and Liapunov's L^p-inequality that there is constant

$\alpha \in (0,\infty)$, depending only on ε, and a constant $C_1 \in [0,\infty)$, such that

$$[E|\xi(t,x)-\xi(t,y)|^{p\theta}]^{1/(p\theta)}$$
$$\leqslant [E|\xi(t,x)-\xi(t,y)|^{2(r-\varepsilon)}]^{1/(2r-2\varepsilon)}$$
$$\leqslant C_1 \exp\left\{\frac{-\alpha t}{2(r-\varepsilon)}\right\}|x-y| \quad \forall t\in[0,\infty)\ \forall x,y\in\mathfrak{R}^d. \tag{69}$$

Now, in view of Remark 3.3, there is a constant $C_2 \in [0,\infty)$, depending only on ε, such that $\sup_t E|\xi(t,x)|^{(r-\varepsilon)} \leqslant C_2[1+|x|^{(r-\varepsilon)}]$, $x\in\mathfrak{R}^d$, thus there is a constant $C_3\in[0,\infty)$, again depending only on ε, such that

$$E[1+|\xi(t,x)|^{(r-\varepsilon-\theta)}+|\xi(t,y)|^{(r-\varepsilon-\theta)}]^q \leqslant 3^q C_3[1+|x|^{(r-\varepsilon)}+|y|^{(r-\varepsilon)}], \tag{70}$$

for all $t\in[0,\infty)$ and $x,y\in\mathfrak{R}^d$. Since $f\in\mathscr{C}_0(r,\varepsilon,\theta)$, it follows from (11), (67), (69), (70), and Hölder's inequality with conjugate exponents (p,q), that

$$|\Psi(t,x)-\Psi(t,y)| \leqslant E|f(\xi(t,x))-f(\xi(t,y))|$$
$$\leqslant [f]_{r-\varepsilon,\theta}\{E|\xi(t,x)-\xi(t,y)|^{p\theta}\}^{1/p}$$
$$\times\{E[1+|\xi(t,x)|^{(r-\varepsilon-\theta)}+|\xi(t,y)|^{(r-\varepsilon-\theta)}]^q\}^{1/q}$$
$$\leqslant [f]_{r-\varepsilon,\theta}C_4\exp\left\{\frac{-\alpha\theta t}{2(r-\varepsilon)}\right\}$$
$$\times|x-y|^\theta[1+|x|^{(r-\varepsilon)}+|y|^{(r-\varepsilon)}]^{1/q}, \tag{71}$$

for all $t\in[0,\infty)$ and $x,y\in\mathfrak{R}^d$, where $C_4\in[0,\infty)$ is a constant, depending only on ε. Now the result follows from (11) and (71). In the case where $r-\varepsilon-\theta=0$, it necessarily follows from $r\geqslant 1$ and (15) that $r=1$ and $\varepsilon+\theta=1$. Then $|f(x)-f(y)|\leqslant 3[f]_{r-\varepsilon,\theta}|x-y|^\theta$, $\forall x,y\in\mathfrak{R}^d$. Now an upper bound of the form on the right-hand side of (71) follows from the second inequality of (69) and Liapunov's L^p-inequality. □

Using Lemma 3.6 and Lemma 3.8 we can establish Lemma 3.4 and Lemma 3.5:

Proof of Lemma 3.4. Fix some $f\in\mathscr{C}_0(r,\varepsilon,\theta)$, and define $\Psi(\cdot,\cdot)$ by (67). By the Markov property of (1) we have $Ef(\xi(s+t,x))=E\Psi(t,\xi(s,x))$, $\forall s,t\in[0,\infty), x\in\mathfrak{R}^d$, so that

$$|Ef(\xi(s+t,x))-Ef(\xi(t,x))| = |E[\Psi(t,\xi(s,x))-Ef(\xi(t,x))]|$$
$$\leqslant E|\Psi(t,\xi(s,x))-Ef(\xi(t,x))|, \tag{72}$$

for all $s,t\in[0,\infty), x\in\mathfrak{R}^d$. From Lemma 3.8 there are constants $C_1\in[0,\infty)$ and $\gamma_1\in(0,\infty)$, depending only on (ε,θ), such that

$$|\Psi(t,\tilde{x})-Ef(\xi(t,x))| \leqslant [f]_{r-\varepsilon,\theta}C_1\exp\{-\gamma_1 t\}|\tilde{x}-x|^\theta$$
$$\times[1+|\tilde{x}|^{(r-\varepsilon-\theta)}+|x|^{(r-\varepsilon-\theta)}], \tag{73}$$

for all $t \in [0,\infty)$ and $x, \tilde{x} \in \mathfrak{R}^d$. From (7) we have $|x|^\theta \leqslant 1 + |x|^{(r-\varepsilon)}$, $\forall x \in \mathfrak{R}^d$, so that

$$|\tilde{x} - x|^\theta [1 + |\tilde{x}|^{(r-\varepsilon-\theta)} + |x|^{(r-\varepsilon-\theta)}]$$
$$\leqslant 2^{1+\theta}[1 + |\tilde{x}|^{(r-\varepsilon)} + |x|^{(r-\varepsilon)} + |x|^\theta |\tilde{x}|^{(r-\varepsilon-\theta)} + |\tilde{x}|^\theta |x|^{(r-\varepsilon-\theta)}], \tag{74}$$

for all $x, \tilde{x} \in \mathfrak{R}^d$. From Remark 3.3 there is a constant $C_2 \in [0,\infty)$, depending only on (ε,θ), such that we have $\sup_s E|\xi(s,x)|^{(r-\varepsilon-\theta)} \leqslant C_2[1+|x|^{(r-\varepsilon-\theta)}]$ and $\sup_s E|\xi(s,x)|^\theta \leqslant C_2[1+|x|^\theta]$ for all $x \in \mathfrak{R}^d$. In view of these uniform bounds, taking $\tilde{x} \triangleq \xi(s,x)$ in (73) and (74), and using (72), (7), and $r - \varepsilon - \theta \geqslant 0$, we get

$$|Ef(\xi(s+t,x)) - Ef(\xi(t,x))| \leqslant C_3[f]_{r-\varepsilon,\theta} \exp\{-\gamma_1 t\}[1 + |x|^{(r-\varepsilon)}], \tag{75}$$

for all $s,t \in [0,\infty)$ and $x \in \mathfrak{R}^d$, where $C_3 \in [0,\infty)$ is a constant depending only on (ε,θ). Now suppose that f is *uniformly bounded*. Then, since f is necessarily continuous (being a member of $\mathscr{C}_0(r,\varepsilon,\theta)$), from Remark 2.5(I) we have

$$\lim_{s\to\infty} Ef(\xi(s+t,x)) = \int_{\mathfrak{R}^d} f \, d\bar{m}, \tag{76}$$

and then, upon taking $s \to \infty$ in (75), we get (33) for all $f \in \mathscr{C}_0(r,\varepsilon,\theta)$ that are also uniformly bounded, for constants $C \triangleq C_3$ and $\gamma \triangleq \gamma_1$ which have been seen to depend only on (ε,θ). It therefore remains to consider the case where f is *unbounded*. To this end, for some arbitrary $f \in \mathscr{C}_0(r,\varepsilon,\theta)$, put $f^+(x) \triangleq \max\{0, f(x)\}$ and $f_n^+(x) \triangleq \min\{f^+(x), n\}$, $\forall x \in \mathfrak{R}^d$, $n = 1,2,\ldots$. Then, it is easily seen that $\|f_n^+\|_{r-\varepsilon} \leqslant \|f^+\|_{r-\varepsilon} \leqslant \|f\|_{r-\varepsilon}$ and $[f_n^+]_{r-\varepsilon,\theta} \leqslant [f^+]_{r-\varepsilon,\theta} \leqslant [f]_{r-\varepsilon,\theta}$, so that (see (12) and (13)) we have $f_n^+ \in \mathscr{C}_0(r,\varepsilon,\theta)$, and, of course, f_n^+ is uniformly bounded. Since (33) has been shown to hold for all members of $\mathscr{C}_0(r,\varepsilon,\theta)$ that are uniformly bounded, we get

$$\left| Ef_n^+(\xi(t,x)) - \int_{\mathfrak{R}^d} f_n^+ \, d\bar{m} \right| \leqslant C_3[f]_{r-\varepsilon,\theta} \exp\{-\gamma t\}[1 + |x|^{(r-\varepsilon)}], \tag{77}$$

for all $n = 1,2,\ldots,$ $s,t \in [0,\infty)$, and $x \in \mathfrak{R}^d$ (recall that $[f_n^+]_{r-\varepsilon,\theta} \leqslant [f]_{r-\varepsilon,\theta}$), where $C_3 \in [0,\infty)$ is the constant from (75), which depends only on (ε,θ). Upon taking $n \to \infty$ in (77) and using the monotone convergence theorem, we get

$$\left| Ef^+(\xi(t,x)) - \int_{\mathfrak{R}^d} f^+ \, d\bar{m} \right| \leqslant C_3[f]_{r-\varepsilon,\theta} \exp\{-\gamma t\}[1 + |x|^{(r-\varepsilon)}], \tag{78}$$

for all $s,t \in [0,\infty)$ and $x \in \mathfrak{R}^d$. By the same reasoning, an upper-bound identical to (78) also holds with f^+ replaced with f^- (the negative part of f), so that (33) follows for arbitrary $f \in \mathscr{C}_0(r,\varepsilon,\theta)$, with $C \triangleq 2C_3$. □

Proof of Lemma 3.5. To simplify the notation, put

$$F(t,x) \triangleq f^i(x)\Psi^j(t,x) \quad \forall (t,x) \in [0,\infty) \times \mathfrak{R}^d, \tag{79}$$

where $\Psi^j(t,x)$ is defined by (31). Since $f^j \in \mathscr{C}_0(r,\varepsilon_j,\theta_j)$, where (ε_j,θ_j) satisfies (15), and $\int_{\mathfrak{R}^d} f^j \, \mathrm{d}\bar{m} = 0$, from Lemma 3.4 we see that there are constants $C_1 \in [0,\infty)$ and $\gamma_1 \in (0,\infty)$, depending only on (ε_j,θ_j), such that

$$|\Psi^j(t,x)| \leqslant C_1[f^j]_{r-\varepsilon_j,\theta_j} \exp\{-\gamma_1 t\}[1+|x|^{(r-\varepsilon_j)}], \tag{80}$$

for all $t \in [0,\infty)$, $x \in \mathfrak{R}^d$, thus, from (10),

$$|F(t,x)| \leqslant \{\|f^i\|_{r-\varepsilon_i}[1+|x|^{(r-\varepsilon_i)}]\} \times \{C_1[f^j]_{r-\varepsilon_j,\theta_j} \exp\{-\gamma_1 t\}[1+|x|^{(r-\varepsilon_j)}]\}, \tag{81}$$

for all $t \in [0,\infty)$, $x \in \mathfrak{R}^d$. Since $r \geqslant 1$ and $\varepsilon_i, \varepsilon_j < 1$, we have $2r - \varepsilon_i - \varepsilon_j > \max\{r - \varepsilon_i, r - \varepsilon_j\} > 0$, and therefore (7) (with $\delta \triangleq 1$) gives

$$[1+|x|^{(r-\varepsilon_i)}][1+|x|^{(r-\varepsilon_j)}] \leqslant 3[1+|x|^{(2r-\varepsilon_i-\varepsilon_j)}] \quad \forall x \in \mathfrak{R}^d,$$

so that we get (48) from (81), where the constant C in (48) depends on f^i, f^j, and (ε_j,θ_j), and the constant $\gamma \triangleq \gamma_1$ depends on (ε_j,θ_j).

To establish (49), note from (79), (11) and (10) that

$$\begin{aligned}
&|F(t,x) - F(t,y)| \\
&\leqslant |f^i(x) - f^i(y)||\Psi^j(t,x)| + |f^i(y)||\Psi^j(t,x) - \Psi^j(t,y)| \\
&\leqslant \{[f^i]_{r-\varepsilon_i,\theta_i}|x-y|^{\theta_i}[1+|x|^{(r-\varepsilon_i-\theta_i)} + |y|^{(r-\varepsilon_i-\theta_i)}]\} \\
&\quad \times \{\|\Psi^j(t,\cdot)\|_{r-\varepsilon_j}[1+|x|^{(r-\varepsilon_j)}]\} + \{\|f^i\|_{r-\varepsilon_i}[1+|y|^{(r-\varepsilon_i)}]\} \\
&\quad \times \{[\Psi^j(t,\cdot)]_{r-\varepsilon_j,\theta_j}|x-y|^{\theta_j}[1+|x|^{(r-\varepsilon_j-\theta_j)} + |y|^{(r-\varepsilon_j-\theta_j)}]\},
\end{aligned} \tag{82}$$

for all $t \in [0,\infty)$, $x,y \in \mathfrak{R}^d$. Since (15) holds for (ε_i,θ_i) and (ε_j,θ_j) we have that $\varepsilon_i + \theta_i \leqslant 1$, thus $r - \varepsilon_i - \theta_i \geqslant 0$, thus $2r - \varepsilon_i - \varepsilon_j - \theta_i \geqslant r - \varepsilon_j > 0$ (since $r \geqslant 1$ and $\varepsilon_j < 1$), so that $2r - \varepsilon_i - \varepsilon_j - \theta_i > r - \varepsilon_i - \theta_i$. In view of these facts, we can use (7) (with $\delta \triangleq 1$) to find

$$\begin{aligned}
&[1+|x|^{(r-\varepsilon_i-\theta_i)} + |y|^{(r-\varepsilon_i-\theta_i)}][1+|x|^{(r-\varepsilon_j)}] \\
&\leqslant 4[1+|x|^{(2r-\varepsilon_i-\varepsilon_j-\theta_i)} + |y|^{(2r-\varepsilon_i-\varepsilon_j-\theta_i)}],
\end{aligned}$$

for all $x,y \in \mathfrak{R}^d$. Combining this inequality with (82) gives

$$\begin{aligned}
&|F(t,x) - F(t,y)| \\
&\leqslant 4[f^i]_{r-\varepsilon_i,\theta_i}\|\Psi^j(t,\cdot)\|_{r-\varepsilon_j}|x-y|^{\theta_i}[1+|x|^{(2r-\varepsilon_i-\varepsilon_j-\theta_i)} + |y|^{(2r-\varepsilon_i-\varepsilon_j-\theta_i)}] \\
&\quad + 4\|f^i\|_{r-\varepsilon_i}[\Psi^j(t,\cdot)]_{r-\varepsilon_j,\theta_j}|x-y|^{\theta_j}[1+|x|^{(2r-\varepsilon_i-\varepsilon_j-\theta_j)} + |y|^{(2r-\varepsilon_i-\varepsilon_j-\theta_j)}],
\end{aligned} \tag{83}$$

for all $t \in [0,\infty)$, $x,y \in \mathfrak{R}^d$. Now put

$$F^+(t,x) \triangleq \max\{0, F(t,x)\}, \quad F_n^+(t,x) \triangleq \min\{F^+(t,x), n\}, \tag{84}$$

for all $n=1,2,\ldots,$ $(t,x)\in[0,\infty)\times\Re^d$. For arbitrary $n=1,2,\ldots,$ define

$$\Theta_n(t_1,t_2,x) \triangleq E[F_n^+(t_2,\xi(t_1,x))] \quad \forall t_1,t_2\in[0,\infty),\ x\in\Re^d. \tag{85}$$

It is easily checked that $|F_n^+(t,x)-F_n^+(t,y)|\leqslant|F(t,x)-F(t,y)|$, for all $t\in[0,\infty)$ and $x,y\in\Re^d$. From this, with (85) and (83),

$$\begin{aligned}&|\Theta_n(t_1,t_2,\tilde{x})-EF_n^+(t_2,\xi(t_1,x))|\\&\quad\leqslant[f^i]_{r-\varepsilon_i,\theta_i}\|\Psi^j(t_2,\cdot)\|_{r-\varepsilon_j}H_1(t_1,x,\tilde{x})+\|f^i\|_{r-\varepsilon_i}[\Psi^j(t_2,\cdot)]_{r-\varepsilon_j,\theta_j}H_2(t_1,x,\tilde{x}),\end{aligned} \tag{86}$$

for all $t_1,t_2\in[0,\infty)$ and $x,\tilde{x}\in\Re^d$, where

$$\begin{aligned}H_1(t,x,\tilde{x}) &\triangleq 4E[|\xi(t,\tilde{x})-\xi(t,x)|^{\theta_i}\\&\qquad\times[1+|\xi(t,\tilde{x})|^{(2r-\varepsilon_i-\varepsilon_j-\theta_i)}+|\xi(t,x)|^{(2r-\varepsilon_i-\varepsilon_j-\theta_i)}]],\end{aligned} \tag{87}$$

and $H_2(t,x,\tilde{x})$ is defined in the same way as $H_1(t,x,\tilde{x})$, except that θ_i is replaced with θ_j. We next establish upper-bounds for H_1 and H_2. Define the numbers $p\triangleq(2r-\varepsilon_i-\varepsilon_j)/\theta_i$ and $q\triangleq(2r-\varepsilon_i-\varepsilon_j)/(2r-\varepsilon_i-\varepsilon_j-\theta_i)$; since (15) holds for (ε_i,θ_i) and (ε_j,θ_j), it follows that $p,q\in(1,\infty)$, and $p^{-1}+q^{-1}=1$. Now put $\varepsilon_{i,j}\triangleq(\varepsilon_i+\varepsilon_j)/2$ and observe that $\varepsilon_{i,j}\in(\kappa(r,\beta,D),1)$ and $p\theta_i=2(r-\varepsilon_{i,j})$. Then, since $\Lambda_{\min}\{D\}>0$, from Lemma 3.6 and Rayleigh's principle we get a constant $\alpha_2\in(0,\infty)$, depending only on $\varepsilon_{i,j}$, and a constant $C_2\in[0,\infty)$, such that

$$[E|\xi(t,\tilde{x})-\xi(t,x)|^{p\theta_i}]^{1/p}\leqslant C_2\exp\left\{\frac{-\alpha_2\theta_i t}{2(r-\varepsilon_{i,j})}\right\}|\tilde{x}-x|^{\theta_i}, \tag{88}$$

for all $t\in[0,\infty)$, $x,\tilde{x}\in\Re^d$. Also, in view of Remark 3.3, there is a constant $C_3\in[0,\infty)$, depending only on ε_i, ε_j and θ_i, such that

$$\begin{aligned}&\{E[1+|\xi(t,\tilde{x})|^{(2r-\varepsilon_i-\varepsilon_j-\theta_i)}+|\xi(t,x)|^{(2r-\varepsilon_i-\varepsilon_j-\theta_i)}]^q\}^{1/q}\\&\quad\leqslant C_3[1+|\tilde{x}|^{(2r-\varepsilon_i-\varepsilon_j-\theta_i)}+|x|^{(2r-\varepsilon_i-\varepsilon_j-\theta_i)}],\end{aligned} \tag{89}$$

for all $t\in[0,\infty)$, $x,\tilde{x}\in\Re^d$. Combining (87), (88), and (89), and using Hölder's inequality, we get constants $C_4\in[0,\infty)$ and $\gamma_4\in(0,\infty)$, depending only on ε_i, ε_j, and θ_i, such that

$$H_1(t,x,\tilde{x})\leqslant C_4\exp\{-\gamma_4 t\}B_1(x,\tilde{x}) \quad \forall t\in[0,\infty),\ x,\tilde{x}\in\Re^d, \tag{90}$$

where, from (7) with $\delta\triangleq 1$,

$$\begin{aligned}B_1(x,\tilde{x}) &\triangleq |\tilde{x}-x|^{\theta_i}[1+|\tilde{x}|^{(2r-\varepsilon_i-\varepsilon_j-\theta_i)}+|x|^{(2r-\varepsilon_i-\varepsilon_j-\theta_i)}]\\&\leqslant C_5[1+|\tilde{x}|^{(2r-\varepsilon_i-\varepsilon_j)}+|x|^{(2r-\varepsilon_i-\varepsilon_j)}] \quad \forall x,\tilde{x}\in\Re^d,\end{aligned} \tag{91}$$

for a constant $C_5\in[0,\infty)$ which depends only on θ_i. An identical upper-bound holds for $H_2(t,x,\tilde{x})$, except that θ_i is replaced with θ_j in (91), and the constants depend on ε_i, ε_j, and θ_j. Next, by (85) and the Markov property of (1), we have

$EF_n^+(t_2, \xi(s+t_1, x)) = E\Theta_n(t_1, t_2, \xi(s, x))$, for all $s, t_1, t_2 \in [0, \infty)$, $x \in \mathfrak{R}^d$, and therefore

$$|EF_n^+(t_2, \xi(s+t_1, x)) - EF_n^+(t_2, \xi(t_1, x))|$$
$$\leqslant E|\Theta_n(t_1, t_2, \xi(s, x)) - EF_n^+(t_2, \xi(t_1, x))|, \qquad (92)$$

for all $s, t_1, t_2 \in [0, \infty)$ and $x \in \mathfrak{R}^d$. Using Remark 3.3, (90), and (91), it is seen that there are constants $\gamma_6 \in (0, \infty)$ and $C_6 \in [0, \infty)$, depending only on ε_i, ε_j, θ_i, θ_j, such that

$$\sup_{s \geqslant 0} EH_k(t, x, \xi(s, x)) \leqslant C_6 \exp\{-\gamma_6 t\}[1 + |x|^{(2r-\varepsilon_i-\varepsilon_j)}], \qquad (93)$$

for all $t \in [0, \infty)$, $x \in \mathfrak{R}^d$, $k = 1, 2$. Combining (93), (92), (86), (80), and using Lemma 3.8, we get

$$|EF_n^+(t_2, \xi(s+t_1, x)) - EF_n^+(t_2, \xi(t_1, x))|$$
$$\leqslant C_7 \exp\{-\gamma_7(t_1 + t_2)\}[1 + |x|^{(2r-\varepsilon_i-\varepsilon_j)}], \qquad (94)$$

for all $s, t_1, t_2 \in [0, \infty)$ and $x \in \mathfrak{R}^d$, where $C_7 \in [0, \infty)$ and $\gamma_7 \in (0, \infty)$ are constants depending only on $(\varepsilon_i, \theta_i, f^i)$ and $(\varepsilon_j, \theta_j, f^j)$. Now, for fixed $t_2 \in [0, \infty)$, the function $F_n^+(t_2, \cdot)$ is continuous and uniformly bounded on $\mathfrak{R}^d$ (see (83) and (84)), thus, upon taking $s \to \infty$ in (94), and using Remark 2.5(I), we get

$$\left| EF_n^+(t_2, \xi(t_1, x)) - \int_{\mathfrak{R}^d} F_n^+(t_2, \xi)\, \mathrm{d}\bar{m}(\xi) \right|$$
$$\leqslant C_7 \exp\{-\gamma_7(t_1 + t_2)\}[1 + |x|^{(2r-\varepsilon_i-\varepsilon_j)}], \qquad (95)$$

for all $t_1, t_2 \in [0, \infty)$ and $x \in \mathfrak{R}^d$. An upper-bound identical to (95) holds with F_n^+ replaced by F_n^-. Now take $n \to \infty$ in (95) and its analogue for F_n^-, and use the monotone convergence theorem, together with (79) and (44), to get (49). □

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