Continuous Empirical Characteristic Function Estimation of GARCH Models

Dinghai Xu [*]

Abstract

This paper develops a simple alternative estimation method for the GARCH models based on the empirical characteristic function. A set of Monte Carlo experiments is carried out to assess the performance of the proposed estimator. The results reveal that the proposed estimator has good finite sample properties and is comparable to the conventional maximum likelihood estimator. The method is applied to the foreign exchange data for empirical illustration.

Keyword: Empirical Characteristic Function, GARCH, Maximum Likelihood Estimator

---

[*]Department of Economics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada; Tel: 001-519-888-4567 ext. 32047; Fax: 001-519-725-0530; Email: dhxu@uwaterloo.ca

1 Introduction

Empirical research has shown that financial asset returns exhibit time-varying volatility and volatility clustering. Consequently, time-dependent models have gained much attention in the empirical finance literature. A benchmark model family for capturing these features was developed by Engle (1982) and Bollerslev (1986), and is popularly known as (Generalized) Autoregressive Conditional Heteroscedasticity (GARCH) model. In the GARCH specification, the underlying conditional variance is modelled as a function of both the past squared errors and past conditional variances. Empirical evidence shows that the GARCH model captures the dynamic properties of financial data.

Methods for estimating parameters of GARCH models may be classified as being either: (i) a likelihood based method, such as Maximum Likelihood Estimation (MLE), Quasi-MLE (QMLE), see, for example, Bollerslev (1986) or (ii)a moment-based method, such as Generalized Method of Moments (GMM), see, for example, Rich, Raymond and Butler (1991). The purpose of this paper is to provide an alternative estimator based on the Empirical Characteristic Function (ECF). This estimator has several advantages. First, under certain weighting measures, a closed form objective distance function is available. This simplifies the estimation procedure and can be easily implemented in practice. Second, the proposed estimator has strong consistency and asymptotic normality properties, see Heathcote (1977), Feuerverger (1990), Knight and Yu (2002). Third, the Fourier inversion theorem implies a one-to-one mapping between the characteristic function (CF) and the likelihood function, which then means that the CF contains the same amount of information as the distribution function. In addition, the CF is always uniformly bounded due to the Fourier transformation. The likelihood function is not always bounded over its parameter space in mixture models, such as discrete mixtures of normal and switching regression models, which means the MLE procedure may break down in practice. The CF based estimator does not suffer from this problem. Specifically, Xu (2007) demonstrates that the CF based estimator performs well when the MLE fails to converge in the estimation of discrete mixture of normal and switching regression models. This paper illustrates the estimation procedure via a conventional GARCH model , which deserves more attention[1] in future research.

This paper is organized as follows. Section 2 discusses the CF based estimator in the general GARCH (p,q) specification. Section 3 conducts several Monte Carlo simulations. Section 4 applies the proposed estimator to foreign exchange rate (FX) data. Section 5 concludes the paper.

---

[1]For instance, if the data are drawn from different distributions (such as mixtures of normal) in the GARCH settings, see Alexander and Lazar (2006), the MLE is not a suitable estimator due to the singularities of the likelihood surface. Insisting on a likelihood based method may lead to numerical instability (i.e., failure in convergence globally).

2 GARCH Specification and CF-based Estimator

Define P_t as the closing price on the trading day t. The daily return X_t is calculated as logarithmic closing price differences:

$$X_t = 100(\log P_t - \log P_{t-1}) \quad t = 1, 2, ..., T. \tag{1}$$

The normal GARCH (p,q) model assumes that the time-varying conditional variance of X_t is function of past information.

$$X_t | I_{t-1} \quad \sim \quad N(\mu, \sigma_t^2) \tag{2}$$

$$\sigma_t^2 \quad = \quad \lambda + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \tag{3}$$

where I_{t-1} stands for the information set up to $t - 1$. The unknown parameter vector is $\theta = (\mu, \lambda, \alpha_1, ..., \alpha_p, \beta_1, ..., \beta_q)$. The stationarity and positivity conditions require that

$$\lambda > 0; \quad \alpha_i, \beta_j \quad \geq \quad 0 \quad i = 1, ..., p, \quad j = 1, ..., q$$

$$\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j \quad < \quad 1.$$

The MLE is based on the following conditional normal density,

$$f(X_t | I_{t-1}) = (2\pi\sigma_t^2)^{-\frac{1}{2}} \exp\left(-\frac{(X_t - \mu)^2}{2\sigma_t^2}\right). \tag{4}$$

The corresponding theoretical CF of (4) is defined as,

$$C(r, \theta | I_{t-1}) = E(e^{irX_t} | I_{t-1}) = \exp(i\mu r - \frac{1}{2}\sigma_t^2 r^2) \tag{5}$$

where $i = \sqrt{-1}$.

The empirical counterpart (ECF) associated with (5) is defined as,

$$C_t(r, X_t) = \exp(irX_t). \tag{6}$$

The main idea of our proposed estimation strategy is to match continuously the theoretical CF, (5), with its corresponding empirical component, (6), under certain weighting measures. We refer to this estimation method as the continuous ECF (CECF) procedure. The objective distance function is constructed as follows,

$$D_t(\theta; X) = \int |C_t(r; X_t) - C(r; \theta | I_{t-1})|^2 w(r) dr. \tag{7}$$

where $w(r)$ is the weighting function, which ensures that the integral in (7) is well defined.

In this paper, we use $w(r) = \exp(-br^2)$, where b is a non-negative real number. This weighting measure[2] retains certain properties of the Gaussian kernel. With this weighting function, we are able to derive a closed form expression of the objective distance function in (7), which simplifies estimation.

Proposition 1. If X_t is generated from (2) and (3) and the distance measure based on the CECF is defined as in (7), then the closed form expression for (7) is given by:

$$D_t(\theta; X) = \sqrt{\frac{\pi}{b}} + \sqrt{\frac{\pi}{b + \sigma_t^2}} - 2\sqrt{\frac{\pi}{b + \frac{1}{2}\sigma_t^2}} \exp\left(-\frac{(X_t - \mu)^2}{4b + 2\sigma_t^2}\right). \qquad (8)$$

Proof: *See the Appendix.*

The implementation of CECF requires minimization of $D(\theta) = \sum_{t=1}^{T} D_t(\theta; X)$ with respect to the unknown parameters in the model.

Heathcote (1977) establishes the asymptotic normality of the CECF estimator,

$$\sqrt{T}(\hat{\theta} - \theta) \sim N(0, \Lambda^{-1}\Omega\Lambda^{-1}) \qquad (9)$$

where $\Lambda = E\left(\frac{\partial D^2(\theta)}{\partial\theta\partial\theta'}\right)$ and $\Omega = E\left(\frac{\partial D(\theta)}{\partial\theta}\frac{\partial D(\theta)}{\partial\theta'}\right)$.

The parameter b may impact the efficiency of the estimation. Xu (2007) proposes an efficient iterative procedure to estimate both θ and b. The idea is to iteratively minimize certain measure (such as the trace or determinant) of the covariance matrix for $\hat{\theta}$ with respect to b. However, in the GARCH settings, the covariance matrix in (9) is not available in an analytical form and the iterative procedure is not straightforward to apply. In the simulation section, we experiment with different b values to assess the impact on efficiency. The results show that in our cases, the effects are not very significant. In the empirical application, we set the b value to be 1.

3   Experimental Design and Results

In this section, we conduct several Monte Carlo experiments (Exp.) to compare the performance of the CECF estimator and the conventional MLE. Each experiment is replicated 200 times. The characteristics of the simulations are presented in Table 1.

---

[2]This weighting function form has been used in the literature, see Heathcote (1977), Knight and Yu (2002) and Xu (2007).

Table 1: Monte Carlo Experiments

Exp.	μ	λ	α_1	α_2	β_1	β_2	b	T
1	0.001	0.001	0.02	-	0.9	-	1	3000
2	0.001	0.001	0.02	-	0.9	-	1	1000
3	0.001	0.001	0.02	-	0.9	-	2	3000
4	0.001	0.001	0.02	-	0.9	-	3.5	3000
5	0.001	0.001	0.15	-	0.7	-	1	3000
6	-0.1	0.001	0.05	-	0.9	-	1	3000
7	0.001	0.001	0.01	0.02	0.9	-	1	3000
8	0.001	0.001	0.01	0.02	0.5	0.4	1	10000

The experiment results are reported in Table 2 (Exp. 1-4) and Table 3 (Exp. 5-8)[3]. We construct the measures of mean, bias and root of mean squared error (RMSE) for comparison purposes[4]. First, we want to examine the performance of the CECF estimator across different GARCH environments. Exp.1 and 2 differ only in the sample size in each replication. We find that, in general, large samples produce smaller biases and RMSEs as expected. This raises some practical concerns about the impact of the limited samples on the CECF estimation. In the empirical section, we will discuss this issue in more details. As mentioned, the parameter b may impact the efficiency of the CECF estimator. We conduct Exp.1, 3 and 4 with different b values.[5] We find that the means are all close to the true parameter values and we observe no significant differences in terms of the bias and the RMSE. This implies that the marginal contribution of parameter b to the estimation efficiency is relatively small in our cases. For this reason, we set $b = 1$ in the empirical estimation. In Exp.5 and Exp.6, we change some parameter values. The results show that the CECF estimator generates stable estimates. It is worth noting that in Exp.6, when μ is significantly different from 0, we observe slightly bigger bias and RMSE. We find a similar pattern in the MLE estimates. However, this problem is not serious in practice because it can be solved by transforming the GARCH structure with a conditional mean. Exp.7 and Exp.8 are carried out to assess the performance of the CECF estimator in GARCH (2,1) and GARCH(2,2) models. As expected, the results are qualitatively unchanged in terms of the bias and RMSE measures.

---

[3]The capital letters C and M in Table 2 and 3 stand for CECF and MLE respectively.

[4]In the working paper version, other measures were also constructed for more detailed comparison, such as range, standard deviation, coverage rate etc. To save space, the results are not reported in this paper.

[5]In the working paper version, we also experiment with other b values across different GARCH settings. The results are very similar to those reported in Table 1.

Table 2: Simulation Results Comparison (Exp. 1 – 4)

	μ	λ	α_1	β_1
Exp. 1				
Mean-C	-0.6e-04	0.0017	0.0219	0.8438
Mean-M	-0.2e-04	0.0017	0.0223	0.8433
Bias-C	-0.0011	0.0007	0.0019	-0.0569
Bias-M	-0.0010	0.001	0.0023	-0.0567
Rmse-C	0.0022	0.0019	0.0112	0.1581
Rmse-M	0.0022	0.0020	0.0115	0.1639
Exp. 2				
Mean-C	0.0001	0.0022	0.0294	0.7927
Mean-M	0.0004	0.0022	0.0293	0.7910
Bias-C	-0.0009	0.0012	0.0094	-0.1073
Bias-M	-0.0006	0.0012	0.0093	-0.1090
Rmse-C	0.0032	0.0026	0.0220	0.2257
Rmse-M	0.0034	0.0032	0.0233	0.2688
Exp. 3				
Mean-C	-0.3e-04	0.0016	0.0217	0.8463
Bias-C	-0.0010	0.0006	0.0017	-0.0537
Rmse-C	0.0022	0.0018	0.0112	0.1524
Exp. 4				
Mean-C	-0.3e-04	0.0017	0.0218	0.8445
Bias-C	-0.0010	0.0007	0.0018	-0.0555
Rmse-C	0.0022	0.0018	0.0110	0.1526

Second, the CECF estimator is compared to the conventional MLE in each experiment. We find that in many cases, the MLE produces slightly smaller RMSE than the CECF estimator. The reason is intuitive in that the MLE is the most asymptotically efficient under certain regularity conditions. However, overall, the CECF estimator performs as efficiently as the MLE with small differences between their bias and RMSE. We also expect that the differences would decrease as the sample size and the number of replications increase.

4   Empirical Application

The data contains 21 years (January 02, 1985 to December 30, 2005) of daily trading prices for 4 FX series including Canadian dollar (CAD), Euro (EUR), British pound (GBP) and Japanese yen (JPY). All of the currencies are in terms of US dollars. The daily returns are constructed as the logarithmic price differences in the usual way based on (1). Some summary statistics of the sample data are presented in Table 4.

We apply the proposed CECF estimator and the MLE with the GARCH(1,1) model to all four currencies. The empirical results are reported in Table 5.

Overall, the empirical results from Table 5 are consistent with the empirical literature. We find that the GARCH coefficients are generally high (> 0.9) and the sum of the ARCH and GARCH coefficients is close to one. This indicates

Table 3: Simulation Results Comparison (Exp. 5 – 8)

	μ	λ	α_1	α_2	β_1	β_2
Exp. 5						
Mean-C	-0.0002	0.0011	0.1439	-	0.6933	-
Mean-M	-0.0001	0.0010	0.1479	-	0.6962	-
Bias-C	-0.0012	0.0001	-0.0061	-	-0.0067	-
Bias-M	-0.0011	0.0000	-0.0021	-	-0.0038	-
Rmse-C	0.0018	0.0003	0.0259	-	0.0599	-
Rmse-M	0.0017	0.0002	0.0191	-	0.0409	-
Exp. 6						
Mean-C	-0.0003	0.0013	0.0517	-	0.8853	-
Mean-M	-0.0002	0.0011	0.0506	-	0.8933	-
Bias-C	0.0997	0.0003	0.0017	-	-0.0147	-
Bias-M	0.0998	0.0001	0.0006	-	-0.0067	-
Rmse-C	0.0997	0.0006	0.0138	-	0.0360	-
Rmse-M	0.0998	0.0005	0.0114	-	0.0302	-
Exp. 7						
Mean-C	0.0002	0.0016	0.0071	0.0247	0.8580	-
Mean-M	0.0002	0.0014	0.0081	0.0240	0.8703	-
Bias-C	-0.0008	0.0006	-0.0029	0.0047	-0.0420	-
Bias-M	-0.0008	0.0004	-0.0019	0.0040	-0.0297	-
Rmse-C	0.0023	0.0025	0.0196	0.0222	0.1847	-
Rmse-M	0.0023	0.0015	0.0193	0.0223	0.1092	-
Exp. 8						
Mean-C	-0.0002	0.0012	0.0119	0.0210	0.3858	0.4953
Mean-M	-0.0001	0.0012	0.0116	0.0197	0.4495	0.4370
Bias-C	-0.0012	0.0002	0.0019	0.0010	-0.0142	-0.0047
Bias-M	-0.0011	0.0002	0.0016	-0.0003	0.0495	-0.0630
Rmse-C	0.0017	0.0006	0.0073	0.0102	0.1535	0.1539
Rmse-M	0.0017	0.0005	0.0080	0.0112	0.3207	0.3090

Table 4: Summary Statistics of Sample Data

	CAD	EUR	GBP	JPY
Length	5371	5403	5372	5383
Mean	-0.0025	0.0097	0.0077	-0.0140
Variance	0.1267	0.4577	0.4351	0.5068
Skewness	0.0795	-0.0328	0.0724	-0.5125
Kurtosis	5.4452	5.2681	6.8027	8.3715
Minimum	-1.9887	-4.1874	-4.4760	-6.9075
Maximum	1.7964	4.8272	4.5529	4.2060

the high persistent pattern of the underlying conditional volatilities. A direct comparison between our CECF estimator and the MLE shows that our proposed estimation method generates comparable estimates to the MLE.

As mentioned earlier, in this empirical application section, we discuss a practi-

Table 5: Empirical Estimates

	μ	λ	α_1	β_1
CAD				
CECF	-0.0028	0.0012	0.0435	0.9453
	(0.0001)	(0.0002)	(0.0014)	(0.0033)
MLE	-0.0014	0.0012	0.0526	0.9380
	(0.0041)	(0.0012)	(0.0033)	(0.0039)
EUR				
CECF	0.0105	0.0124	0.0498	0.9168
	(0.0006)	(0.0035)	(0.0009)	(0.0060)
MLE	0.0087	0.0067	0.0437	0.9418
	(0.0083)	(0.0011)	(0.0038)	(0.0056)
GBP				
CECF	0.0109	0.0045	0.0332	0.9512
	(0.0004)	(0.0024)	(0.0017)	(0.0043)
MLE	0.0098	0.0026	0.0358	0.9580
	(0.0074)	(0.0004)	(0.0023)	(0.0028)
JPY				
CECF	0.2e05	0.0127	0.0339	0.9301
	(0.0001)	(0.0037)	(0.0007)	(0.0108)
MLE	-0.0081	0.0147	0.0493	0.9221
	(0.0090)	(0.0012)	(0.0032)	(0.0048)

Note: The standard errors are in the parentheses.

cal issue regarding to the impact of limited sample sizes on the CECF estimation. Following Ng and Lam (2006), we construct the correlation measures of the conditional volatilities implied from different sample sizes with those from the whole sample data, including {500, 700, 1000, 1500, 2000, 2500, 4000, whole sample}. The results are in Table 6.

Table 6: Correlation Measures

	500	700	1000	1500	2000	2500	4000	whole sample
CAD	0.6641	0.6598	0.7883	0.7804	0.8839	0.9024	0.9233	1.0000
EUR	0.9784	0.9775	0.9779	0.9896	0.9710	0.9818	0.9967	1.0000
GBP	0.9360	0.9768	0.9859	0.9809	0.9777	0.9948	0.9997	1.0000
JPY	0.8406	0.7981	0.7974	0.9143	0.9371	0.9715	0.9957	1.0000

Table 6 indicates that for CECF estimation of the conventional GARCH models, a sample size of 1500 is, in general, enough to generate comparatively high correlations. However, it may not always be the case in practice.

5 Conclusion

This paper provides a simple estimation method based on the CF for GARCH models. Under the exponential weight function, the closed-form distance measure between the CF and the corresponding ECF is available, which simplifies estimation. Evidence from both Monte Carlo experiments and empirical applications shows that the CECF estimator is a comparable estimator to the MLE in the conventional GARCH settings. This estimation method deserves more attention in the literature, especially when the regularity conditions do not hold for the MLE, which is the case for mixture-GARCH models.

Appendix

Proof of Proposition 1

Noting that $\exp(irx) = \cos(rx) + i\sin(rx)$, (5) can be rewritten as:

$$\exp(i\mu r - \frac{1}{2}\sigma_t^2 r^2) = \cos(\mu r)\exp(-\frac{1}{2}\sigma^2 r^2) + i\sin(\mu r)\exp(-\frac{1}{2}\sigma^2 r^2).$$

Similarly, the ECF, (6) can also be decomposed into two parts as,

$$\exp(irX_t) = \cos(rX_t) + i\sin(rX_t).$$

Then, it is straightforward to get,

$$
\begin{aligned}
|C_t(r; X_t) - C(r; \theta|I_{t-1})|^2 &= \cos^2(rX_t) + \sin^2(rX_t) + \exp(-\sigma_t^2 r^2) \\
&- 2\exp(-\frac{1}{2}\sigma_t^2 r^2)\left[\cos(\mu r)\cos(rX_t) + \sin(\mu r)\sin(rX_t)\right].
\end{aligned}
$$

We evaluate the above expression in the one-dimensional integral with the exponential weighting function $\exp(-br^2)$. Using the result, $\cos(x) = \frac{\exp(ix) + \exp(-ix)}{2}$, it is easy to show:

$$
\int\left(1 + \exp(-\sigma_t^2 r^2) - 2\exp(-\frac{1}{2}\sigma_t^2 r^2)\cos(X_t r - \mu r)\right)\exp(-br^2)dr
$$
$$
= \sqrt{\frac{\pi}{b}} + \sqrt{\frac{\pi}{b + \sigma_t^2}} - 2\sqrt{\frac{\pi}{b + \frac{1}{2}\sigma_t^2}}\exp\left(-\frac{(X_t - \mu)^2}{4b + 2\sigma_t^2}\right)
$$

References

[1] Bollerslev, T., 1986. Generalized Autoregressive Conditional Heteroskedasticity, Journal of Econometrics, 31, 307-327.

[2] Engle, R. F., 1982. Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom inflation, Econometrica, 50, 987-1007.

[3] Feuerverger, A., 1990. An Efficiency Result for the Empirical Characteristic Function in Stationary Time-Series Models, Canadian Journal of Statistics 18, 155-161.

[4] Heathcote, C.R., 1977. Integrated Mean Square Error Estimation of Parameters, Biometrica, 64, 255-264.

[5] Knight, J. L. and Yu, J., 2002. Empirical Characteristic Function in Time Series Estimation, Econometric Theory, 18, 691 - 721.

[6] Ng, H. and Lam, K.P., 2006. How Does the Sample Size Affect GARCH Models?, Proceedings of the Joint Conference on Information Sciences.

[7] Rich, R.W., Raymond, J. and Butler, J.S., 1991. Generalized Instrumental variables Estimation of AutoRegressive Conditional Heteroskedastic Models, Economics Letters 35, 179-185.

[8] Xu, D., 2007. Asset Returns, Volatility and Value-at-Risk, Ph.D Thesis, The University of Western Ontario.