(De)Regulation and Market Thickness

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Abstract

Regulation is a set of constraints imposed on transactions between buyers and sellers. We introduce a dynamic frictional matching model with horizontal differentiation and nontransferable utility in which a regulator determines permissible transactions. We show the existence and uniqueness of a market equilibrium for any level of regulation and characterize the regulator's optimal choice of regulatory environment. We argue that in 'thin', markets, regulation can correct market failure arising from mismatch between buyers and sellers. However, in 'thick' markets, deregulation is optimal, as a regulator can rely on market participants' equilibrium behavior instead of explicit constraints on economic activities.

1 Introduction

Generally speaking, regulation imposes constraints on the set of economic activities in which market participants are permitted to engage. These constraints may directly affect prices or quantities in the form of, say, taxes or quotas, or they may indirectly affect market outcomes by serving as artificial barriers to entry through, say, licensing requirements. Regulation is intended to correct market imperfections and to align market outcomes with those preferred by a regulatory authority. This raises a natural question: why would a regulator intervene in a transaction between consenting parties?

In this paper, we show that regulation can be beneficial when market imperfections lead participants to consent to transactions that they would refuse in better-functioning markets. However, while regulation may direct transactions to benefit more market participants, the constraints it imposes may either diminish the surplus generated in some trades or prevent other welfare-enhancing trades that would otherwise occur. The regulatory burden is larger

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in markets with fewer imperfections, as trading parties can better seek out the transactions that they prefer on their own. This reveals a complementarity between the stringency of optimal regulation and market frictions.

A market's efficiency is naturally tied to the extent to which there is competition between market participants. The well-known literature on optimal regulation views competition as driven by the number and size of market participants (Armstrong and Sappington (2007), Baron (1989)).[1] But to the extent that markets are horizontally differentiated, competition between market participants should instead be captured by market *thickness*, which is a measure of the availability and variety of transaction opportunities. Although the concept of market thickness is an old one, dating back at least to Chamberlin (1937), it has been largely avoided in the theoretical literature on markets and has been irregularly measured by empirical researchers.

To formalize our ideas, we develop a tractable dynamic, frictional matching model with horizontally differentiated buyers and sellers. Following a match, the seller chooses an economic activity and proposes a transaction to the buyer at that activity. Economic activities are represented as match-specific production functions and determine the size and sharing of the (nontransferable) match surplus. If the buyer turns down this proposed transaction, it must wait for another match in future periods. The thickness of the market, or the scale of its imperfections, is given by the common discount factor of buyers and sellers, which measures the ex ante probability that buyers will be able to identify an agreeable seller in a fixed period of time (see, for example, McLaren (2003)). Our main modelling innovation is our introduction of regulation. Before the dynamic market opens, a regulator who has preferences over the types of transactions in which market participants engage restricts sellers to a set of permitted economic activities. In each match, the regulator has a targeted Pareto-efficient transaction that lies 'in between' the ideal match-specific transactions of the buyers and the sellers. Regulatory constraints directly affect the matching patterns and the activities at which transactions occur in the market equilibrium.

Our results build on well-known insights from dynamic frictional matching markets. On the one hand, when markets are thin, buyers are less likely to find compatible sellers and they settle for transactions with sellers that are a poor match. In equilibrium, this lack of alternatives available to buyers generates low continuation payoffs from market participation.

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[1] Another classic literature, following Stigler (1971) and Peltzman (1976), argues that regulation is primarily intended to divert surplus to special interests. While we recognize that rent-seeking is important in the implementation of regulation, we abstract from potential institutional failures and instead focus on the optimality of regulation.

Hence, buyers can transact at unfavourable activities even with sellers that are a good match. In thin markets, the regulator has an incentive to compress the set of permitted economic activities, which improves the surplus from trade in lower quality matches at the expense of the surplus available from trade in higher quality matches: excess variety is harmful and the regulator benefits from making transactions more uniform. On the other hand, when markets are thick, buyers can afford to be 'picky', and they reject unfavourable transactions in order to wait for better opportunities. In such markets, deregulation is optimal, since the welfare costs of over-regulating good matches are high relative to those of under-regulating poor matches. In short, although thin markets may require intervention, thick markets effectively regulate themselves.

Our results have both longitudinal and cross-sectional implications for the empirical study of regulation and can help explain seemingly contrasing experiences with regulation across industries and countries. In particular, our results provide insights into the lifecycle of regulatory intervention: young markets should be more regulated and deregulation should occur once they mature. For example, regulatory intensity in the US has varied substantially over the past thirty five years as several industries – e.g., airlines (Kahn (1987); Morrison and Winston (1989)), banks (DeYoung (1994)), and natural gas producers (MacAvoy (2000)) – have been deregulated, reducing average prices and improving cost efficiency in the process. Notably, these successful deregulations all occurred in mature industries with sufficient heterogeneity in sellers that had enjoyed improvements in the abilities of buyers to find compatible sellers. In the case of airlines, regulations took the form (in part) of minimum service requirements that effectively transferred surplus from busy routes ('good' matches) to sparse routes ('bad' matches), while deregulation was followed by large-scale entry of small firms willing to serve these local markets. At the same time, many observers have argued that lax regulations at the start of the financial crisis in 2007 contributed to the severe turmoil in the real estate and financial industries (Brunnermeier et al. (2009)).[2] Indeed, financial innovation has been blamed for contributing to the financial crisis as relatively young markets for new financial instruments (e.g. credit default swaps) featuring a small number of highly specialized sellers were singled out as particularly troublesome. Perhaps unsurprisingly, financial markets have since become more intensely regulated with the Dodd-Frank Act of 2010.

Our results can also help reconcile the substantial heterogeneity in regulatory intensity

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[2]In addition to mere observers, the circle of critics of the financial regulatory structure has grown to include regulators themselves. See, for example, Alan Greenspan's testimony before Congress on October 23nd, 2008.

observed both across countries and across industries within countries by generating predictions about which countries or industries should face heavier regulatory burdens at any point in time. To illustrate this point further, we offer broad evidence on the regulatory experiences of a variety of countries in Figure 1. For each of the 36 countries in our sample, we plot a proxy of the level of regulatory intensity against a proxy of the thickness of markets in their economy.[3] While we stress that we do not draw any causal inferences from this plot, there is a clear negative relationship between regulatory intensity and market thickness that is statistically significant at the 95% confidence level. This is suggestive of the fact that thicker product markets are less heavily regulated. Moreover, this relationship is not merely an artifact of thicker markets tending to be larger than thinner markets. Indeed, when we reproduce the same plot conditional upon market size, as measured by either GDP or GDP per capita, we find an even more steeply negative and statistically significant relationship between regulatory intensity and market thickness.

The related literature on dynamic matching has primarily focused on vertical differentiation, deriving conditions for positive assortative matching (Burdett and Coles (1997), Sattinger (1995), Shimer and Smith (2000)). In our model, horizontal differentiation plays a key role as the goal of regulation is to limit welfare losses from mismatch.[4] Another departure from assortative matching frameworks is that in our model, the value of a transaction within a match is not determined solely by the types of buyers and sellers, but rather it depends on post-match interactions (in our case on the economic activity that the seller offers the buyer.) This, along with our focus on how market outcomes vary with the level of market imperfections, relates to work on the convergence of decentralized dynamic bargaining equilibria to Walrasian outcomes (De Fraja and Sakovics (2001), Diamond (1971), Gale (1986), Lauermann (2011), Rubinstein and Wolinsky (1985, 1990), Satterthwaite and Shneyerov (2007)).[5] A main innovation in our framework is that sellers' choice sets are endogenously determined

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[3]These results are robust to alternative measures of regulatory intensity (inidices of administrative regulation and of domestic economic regulation). As an external validity check of our proxy for market thickness, we computed the OECD-wide and worldwide industry-level number of establishments per capita using data from Nicita and Olarreaga (2007) and found them to be highly correlated to OECD-wide and worldwide industry-level measures of market thickness as estimated in Knetter and Slaughter (2001) using data on internation trade flows ($\rho = 0.82$ and 0.79 respectively.)

[4]We are aware of very few models of frictional matching that focus explicitly on horizontal differentiation. See Clark (2007) and Hofmann and Qari (2011). Lauermann and Nöldeke (2012) study a general matching setup with nontransferable utility.

[5]In directed search models, Delacroix and Shi (2007) introduce a private quality investment by sellers, Doyle and Wong (2008) allow sellers to renege on their price postings following buyer visits, while Menzio (2005) has post-match outcomes be determined by an alternating offer bargaining game. In these static models, post-match actions by sellers affect buyers pre-match sorting decisions, whereas in our dynamic model, sellers' activity choices affect buyers' payoffs from continued search.

Figure 1: Regulation and Market Thickness Across Countries
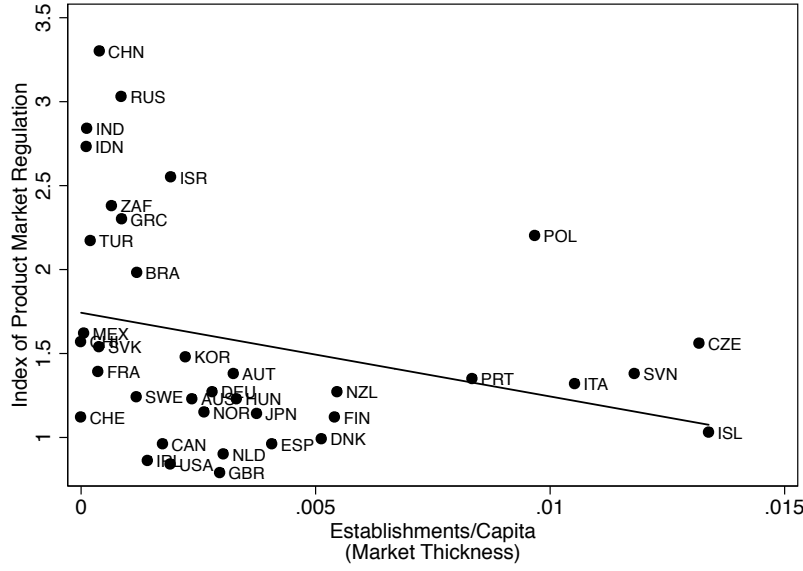
Sources: Index of product market regulation is taken from the OECD Product Market Regulation Database (2011) and is based on national data from 2008. The number of establishments in each country is computed from Nicita and Olarreaga (2007) and uses values for the most recent year in which a national economic census was undertaken in each country. Population figures are drawn from the *World Development Indicators* (WorldBank). The sample includes all countries for which the OECD measures product market regulation.

by an active regulator.

The remainder of the paper is organized as follows. In section 2, we present our model of exchange and regulation. In section 3, we describe the strategies of buyers, sellers and the regulator and define our equilibrium concept. We prove the existence of an equilibrium, which proceeds in two steps. First, given a regulatory environment chosen by the regulator, we establish the existence of a unique equilibrium of the dynamic matching game between buyers and sellers, which consists of both buyers' search strategies and the economic activities offered by sellers in different matches. Second, we establish the existence of an optimal choice of regulatory environment. In section 4, we first characterize market equilibria given a fixed regulatory environment and level of market thickness and describe the effect of regulation on market outcomes. Second, we derive our main results on the comparative statics of optimal regulation in market thickness. In section 5, we illustrate our results in a simple application of our model that is tailored to study over-specialization in the provision of medical services.

2 Model

2.1 The Environment

A continuum of long-lived, heterogeneous buyers and sellers participate in an infinite-horizon, dynamic, frictional matching market with nontransferable utility. Without loss of generality, we assume that there is a unit mass of both buyers and sellers. Buyer types, θ_B, are distributed on Θ_B with density f_B and seller types, θ_S, are distributed on Θ_S with density f_S, where Θ_B and Θ_S are compact subsets of $\mathbb{R}^n$. A *match* consists of a buyer-seller pair $(\theta_B, \theta_S) \in \Theta_B \times \Theta_S$. The outcome of a match depends on buyers' and sellers' types and also on the particular transaction they engage in. In any match, the seller chooses an *economic activity*, parametrized by $a \in A = [\underline{a}, \overline{a}]$, with $-\infty < \underline{a} < \overline{a} < \infty$. A transaction at activity a in match (θ_B, θ_S) generates payoffs according to $\pi_{\theta_B} : \Theta_S \times A \to [0, \overline{\pi}]$ for the buyer, according to $\pi_{\theta_S} : \Theta_B \times A \to [0, \overline{\pi}]$ for the seller and according to $\pi_R : \Theta_B \times \Theta_S \times A \to \overline{\pi}$ for the regulator, where $\overline{\pi} < \infty$ and π_{θ_B}, π_{θ_S} and π_R are continuous in (θ_B, θ_S, a). We introduce more structure on the preferences of the buyers, the sellers and the regulator in Section 2.4.

The model allows for many interpretations of economics activities, which index match-specific production functions. For example, activities can be thought of as match-specific investments made by sellers. This can include decisions about product quality in consumer markets or the provision of effort/competence in service markets. Alternatively, activities can represent the establishment of terms of trade within a match, insofar as this is consistent with nontransferable utility. This can include the outcomes of bargaining following joint projects or the setting of purchasing prices or quantities.

2.2 The Regulatory Stage

Prior to the market's opening $(t = 0)$, the regulator may restrict sellers' choices to a set of permitted activities. This is how we view regulation: some body outside the market can, ex ante, rule out certain transactions from occurring by prohibiting their corresponding production functions. A *regulatory environment* is described by a correspondence $P : \Theta_B \times \Theta_S \rightrightarrows A$, where $P(\theta_B, \theta_S)$ consists of the activities permitted by the regulator in match (θ_B, θ_S). Let $\mathcal{P}$ denote the set of regulatory environments. To ensure existence of optimal activity choices for sellers and the regulator, we assume that all regulatory environments $P \in \mathcal{P}$ are closed and convex valued and that $\mathcal{P}$ is compact.[6]

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[6]Hence, any regulatory environment P can be represented by a pair of functions $\ell, h : \Theta_B \times \Theta_S \to A$ where $a \in P(\theta_B, \theta_S)$ if and only if $a \in [\ell(\theta_B, \theta_S), h(\theta_B, \theta_S)]$. Also, since $\mathcal{P}$ consists of the family of such

We can allow for various degrees of regulatory power, which can be due to enforcement or political constraints imposed on the regulator. With no further restrictions, $\mathcal{P}$ describes the set of *match-specific* regulatory environments, in which the regulator can tailor rules to each market meeting. If we also require that, for all $P \in \mathcal{P}$ and any $\theta_B \in \Theta_B$ and $\theta_S, \theta'_S \in \Theta_S$, we have $P(\theta_B, \theta_S) = P(\theta_B, \theta'_S)$, then $\mathcal{P}$ describes the set of *buyer-specific* regulatory environments, in which the regulator specifies the transactions any seller can engage in with a given buyer. Finally, if we require that, for all $P \in \mathcal{P}$, any $\theta_B, \theta'_B \in \Theta_B$ and any $\theta_S, \theta'_S \in \Theta_S$, $P(\theta_B, \theta_S) = P(\theta'_B, \theta'_S)$, then $\mathcal{P}$ describes the set of *blanket* regulatory environments, in which the whole economy is subject to the same regulatory constraints. Unless explicitly noted, our results apply to all three forms of regulation.

2.3   The Matching Market

The market opens following a choice of regulatory environment, and in each period $t = 1, 2, \ldots$, buyers and sellers are randomly allocated into matches. In each match, once the seller has chosen an activity, a transaction occurs only if agreed to by both parties. Forgoing a transaction yields a period payoff of 0 to both buyer and seller. Buyers and sellers share a common discount factor $\delta < 1$, which parametrizes market thickness. When δ is small, it is costly for buyers to locate sellers, as in a thin market. When δ is large, buyers bear little cost to finding suitable transactions, as in a thick markets.[7]

As is standard in models of dynamic frictional matching, we assume that buyers have inelastic unitary demands for transactions and exit the market once their demand is met. Two further assumptions about the matching market considerably improve the tractability of our model and of our equilibrium characterization results. First, we assume that sellers have fully elastic supplies and can remain in the market following a transaction. Since $\pi_{\theta_S} \geq 0$, this assumption ensures that sellers accept transactions with any buyer at any activity and can be treated as static optimizers. Second, we assume that an exiting buyer is replaced in the market by a buyer of the same type, which ensures that the distribution of unmatched buyers is exogenous and stationary.[8] Both assumptions are technically convenient and do not affect the central trade-off introduced by our model, which focuses on how equilibrium activities adjust to the thickness of the market. Even if sellers are willing to trade with any buyer,

pairs of functions, it is compact if we assume that this family is equicontinuous, given that it is uniformly bounded since $A = [\underline{a}, \overline{a}]$.

[7]Alternatively, δ can be interpreted as a common matching rate in a model without discounting.

[8]This 'cloning' assumption is a common simplification in the matching literature (see Adachi (2003) and Eeckhout (1999)).

Figure 2: Timing in the Matching Market.
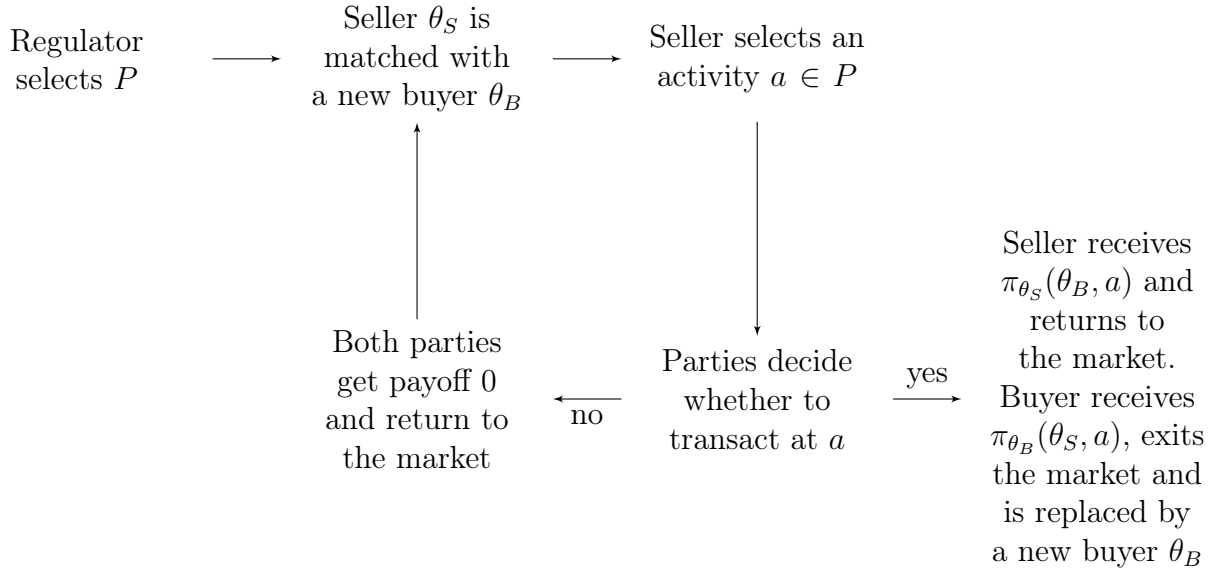
the activities at which transactions occur in equilibrium are still determined by the buyers'
trade-off between current and future trading opportunities, and it is the endogenous outflow
of buyers that this trade-off generates that the regulator tries to control. Both assumptions
ensure the uniqueness of market equilibria, which is of critical importance in our model to
avoid difficulties with equilibrium selection in the regulator's problem.[9] We summarize the
timing of the model in Figure 2.

2.4   Payoffs

Optimal regulatory intervention involves a trade-off between 'one-size-fits-all' constraints on
economic activities and horizontal differentiation in the tastes and skills of market partici-
pants. In this section, we impose more structure on the preferences of buyers, sellers and
the regulator to obtain predictions on equilibrium matching patterns and optimal regulatory
environments.

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[9]In the absence of the first assumption, equilibrium would involve two-sided dynamic acceptance decisions
(see Adachi (2003), Lauermann and Nöldeke (2012) or Shimer and Smith (2000)). In the absence of the second
assumption, our fixed point argument establishing existence would need to integrate the distribution of buyers
either directly, by adding it to the domain of the fixed point mapping (see Shimer and Smith (2000)), or
indirectly, by enfolding it into the current mapping (as we currently handle activity choices). Both options
would raise nontrivial technical issues.

Assumption 1. π_{θ_B} *is strictly quasiconcave in* a *and quasiconcave in* θ_S. π_{θ_S} *is strictly quasiconcave in* a *and quasiconcave in* θ_B. *Furthermore, there exists a mapping* $\sigma : \Theta_B \to \Theta_B$ *such that, for any* $(\theta_S, \theta_B, a) \in \Theta_S \times \Theta_B \times A$,

$$\sigma(\theta_B) \in \arg\max_{\theta'_S \in \Theta_S} \pi_{\theta_B}(\theta'_S, a), \text{ and}$$

$$\theta_B \in \arg\max_{\theta'_B \in \Theta_B} \pi_{\sigma(\theta_B)}(\theta'_B, a),$$

where $\sigma(\theta_B)$ *can be interpreted as the preferred seller for a buyer of type* θ_B.

Assumption 1 states that buyers and sellers have single-peaked preferences over both their trading partners and activities and that all agents' ideal matches are mutually agreeable; that is, a buyer's preferred seller is also that seller's preferred buyer. The critical feature the assumption captures is that in market environments, to each want of a particular buyer corresponds a (not necessarily unique) seller whose skills are best put to use in satisfying precisely that want. We view regulation as playing a role when market imperfections do not allow these mutually beneficial transactions to arise often enough.

Define the function $\hat{\alpha} : \Theta_B \times \Theta_S \times \mathcal{P} \to A$ such that

$$\hat{\alpha}_{\theta_S}(\theta_B, P) = \arg\max_{a \in P(\theta_B, \theta_S)} \pi_{\theta_S}(\theta_B, a),$$

where $\hat{\alpha}_{\theta_S}(\theta_B, P)$ is the ideal activity for seller θ_S when matched with buyer θ_B under regulatory environment P. Our assumptions ensure that $\hat{\alpha}$ is both well-defined and continuous.[10] With slight abuse of notation, let $\hat{\alpha}_{\theta_B}(\theta_S, P)$ denote the corresponding ideal for a buyer in match (θ_B, θ_S) given regulation P, which can be derived as for $\hat{\alpha}_{\theta_S}$. Also, define $\hat{\alpha}_{\theta_S}(\theta_B) \equiv \hat{\alpha}_{\theta_S}(\theta_B, A)$ and $\hat{\alpha}_{\theta_B}(\theta_S) \equiv \hat{\alpha}_{\theta_B}(\theta_S, A)$, as the ideal activities of, respectively, sellers and buyers in the absence of regulation.

As utility is nontransferable, the choice of an activity determines both the level of surplus in a match and the shares of this surplus accruing to buyers and sellers. Our next assumption relates the quality of a match to buyers' and sellers' preferences over activities.

Assumption 2. *For all matches* (θ_B, θ_S),

1. *If* $\theta'_B \in \Theta_B$ *is such that* $|\theta_B - \sigma^{-1}(\theta_S)| < |\theta'_B - \sigma^{-1}(\theta_S)|$, *then* $\hat{\alpha}_{\theta_S}(\theta_B) \geq \hat{\alpha}_{\theta_S}(\theta'_B)$. *If* $\theta'_S \in \Theta_S$ *is such that* $|\theta_S - \sigma(\theta_B)| < |\theta'_S - \sigma(\theta_B)|$, *then* $\hat{\alpha}_{\theta_B}(\theta_S) \geq \hat{\alpha}_{\theta_B}(\theta'_S)$.

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[10]Note that the correspondence $\phi : \Theta_B \times \Theta_S \times \mathcal{P} \rightrightarrows A$ defined such that $\phi(\theta_B, \theta_S, P) = P(\theta_B, \theta_S)$ is convex, compact-valued and continuous. By Assumption 1 and the continuity of π_{θ_S} on $\Theta_B \times A$, the maximum theorem under convexity (see Sundaram (1996)) ensures that $\hat{\alpha}$ is a continuous function.

2. $\hat{\alpha}_{\theta_S}(\theta_B) \geq \hat{\alpha}_{\theta_B}(\theta_S)$.

Assumption 2 ensures that higher activities generate greater buyer and seller surplus than lower activities in good matches (part 1). This implies that buyers and sellers have partially concordant interests over higher activities in good matches. Note that in principle, the preferred activities of buyers and sellers need not be Pareto-efficient. However, by the strict quasiconcavity of π_{θ_S} in a, a payoff-maximizing seller will choose a Pareto-efficient activity in all matches in equilibrium. Furthermore a key feature of our model is that utility is nontransferable and buyers and sellers do not have concordant interests over Pareto-efficient activities. To this end, we impose the convenient normalization that sellers always weakly prefer higher activities than buyers (part 2).

We consider markets in which sellers, when unconstrained by regulation, choose activities on the Pareto-efficient frontier for a given match that conflict with the regulator's targeted Pareto-efficient activity for that match, which we refer to as the *welfare-maximizing activity*. These are the relevant markets for our purposes as there is scope for regulation. More precisely, fix a continuous function $\bar{\pi} : \Theta_S \times \Theta_B \to \mathbb{R}$, and assume that for all matches (θ_B, θ_S), $\bar{\pi}_{\theta_S}(\theta_B) \in [\pi_{\theta_S}(\theta_B, \hat{\alpha}_{\theta_B}(\theta_S)), \pi_{\theta_S}(\theta_B, \hat{\alpha}_{\theta_S}(\theta_B))]$ and that $\bar{\pi}_{\theta_S}(\theta_B)$ is decreasing in $|\sigma(\theta_B) - \theta_S|$. Define the function $\hat{\alpha}_R : \Theta_B \times \Theta_S \to A$ such that

$$\hat{\alpha}_R(\theta_B, \theta_S) = \arg\max_{a \in A} \pi_{\theta_B}(\theta_S, a)$$

$$\text{subject to } \pi_{\theta_S}(\theta_B, a) \geq \bar{\pi}_{\theta_S}(\theta_B),$$

which describes the welfare-maximizing activity in a match (θ_B, θ_S), where $\bar{\pi}_{\theta_S}(\theta_B)$ is the share of the surplus in this match that the regulator prefers to allocate to the seller. Our assumptions ensure that $\hat{\alpha}_R$ is well-defined, continuous, and that for all matches (θ_B, θ_S), $\hat{\alpha}_R(\theta_S, \theta_B) \in [\hat{\alpha}_{\theta_B}(\theta_S), \hat{\alpha}_{\theta_S}(\theta_B)]$.[11] Hence, welfare is maximized when transactions occur at some compromise activity lying between the ideal activities of buyers and sellers.

Assumption 3. *For all matches (θ_S, θ_B),*

1. *π_R is strictly quasiconcave in a, with $\hat{\alpha}_R(\theta_B, \theta_S) = \arg\max_{a \in A} \pi_R(\theta_B, \theta_S, a)$.*

2. *$\hat{\alpha}_R(\theta_B, \theta_S)$ is decreasing in $|\sigma(\theta_B) - \theta_S|$.*

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[11] The correspondence $\phi : \Theta_B \times \Theta_S \rightrightarrows A$ defined such that $\phi(\theta_B, \theta_S) = \{a \in A : \pi_{\theta_S}(\theta_B, a) \geq \bar{\pi}_{\theta_S}(\theta_B)\}$ is nonempty, convex compact-valued and continuous. The maximum theorem under convexity (see Sundaram (1996)) ensures that $\hat{\alpha}_R$ is a continuous function. That $\hat{\alpha}_R(\theta_S, \theta_B) \in (\hat{\alpha}_{\theta_B}(\theta_S), \hat{\alpha}_{\theta_S}(\theta_B))$ follows since we assume that the regulator's targeted level of utility for selles, $\bar{\pi}_{\theta_S}$, satisfies $\bar{\pi}_{\theta_S}(\theta_B) \in (\pi_{\theta_S}(\theta_B, \hat{\alpha}_{\theta_B}(\theta_S)), \pi_{\theta_S}(\theta_B, \hat{\alpha}_{\theta_S}(\theta_B)))$.

3. $\pi_{\theta_B}(\theta_S, \hat{\alpha}_R(\theta_B, \theta_S))$ and $\pi_{\theta_S}(\theta_B, \hat{\alpha}_R(\theta_B, \theta_S))$ are decreasing in $|\sigma(\theta_B) - \theta_S|$.

The regulator's preferences over activities in any match are single-peaked around the welfare-maximizing activity for that match (part 1). For worse matches, welfare-maximizing activities are lower (part 2), while welfare losses from more severe mismatch are (at least weakly) shared between buyers and sellers (part 3). This is, indirectly, an assumption on the function $\bar{\pi}$, which states that $\bar{\pi}_{\theta_S}(\theta_B)$ does not decline so fast in $|\sigma(\theta_B) - \theta_S|$ that the buyer is made better off under welfare-optimal activities in worse matches.

Finally, we introduce technical requirements that are important for establishing the existence of an equilibrium and the uniqueness of equilibrium payoffs.

Assumption 4. *For all matches (θ_B, θ_S) and any regulatory environment P,*

1. $\pi_{\theta_S}(\theta_B, \hat{\alpha}_{\theta_B}(\theta_S)) = 0$.

2. $\hat{\alpha}_{\theta_B}(\theta_S) \in P(\theta_B, \theta_S)$.

We assume that sellers weakly prefer not trading to offering buyers their preferred activity within a match (part 1). Note that this is consistent with horizontally differentiated markets. To illustrate this, suppose that an activity $a \in A$ describes a particular good to be supplied by the seller *and* a transaction price. Given any activity, both buyers and sellers are better off transacting with their mutually agreeable types, e.g., a buyer has a higher willingness to pay for the good of its preferred seller type. However, given a particular seller, a buyer may prefer the offered good with a large price cut, while the seller may prefer a higher price.[12] In addition, in any match, the buyers' ideal activity can be offered in any regulatory environment (part 2).

3   Strategies and Equilibrium

At time $t = 0$, a strategy for the regulator is a regulatory environment $P \in \mathcal{P}$. At time $t = 1$, the market opens under regulatory environment P, and we focus on stationary strategies for buyers and sellers. At any time $t \geq 1$, a strategy for seller type θ_S is the choice of a permitted activity in each of its possible matches, represented by $\alpha_{\theta_S} : \Theta_B \times \mathcal{P} \to A$, with the restriction

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[12] If we had allowed $\pi_{\theta_S} < 0$, then a slight generalization of part 1 of Assumption 4 would allow for $\pi_{\theta_S}(\theta_B, \hat{\alpha}_{\theta_B}(\theta_S)) \leq 0$. This would complicate notation but present no difficulties. Since sellers always refuse trades yielding negative payoffs, an interpretation of our present setup is that $\hat{\alpha}_{\theta_B}(\theta_S)$ represents the ideal feasible trade for the buyer in match (θ_B, θ_S).

that for all buyer types θ_B and regulatory activity P, $\alpha_{\theta_S}(\theta_B, P) \in P(\theta_B, \theta_S)$. Similarly, at any time $t \geq 1$, a strategy for buyer type θ_B is the choice of accepting or refusing to transact with possible seller-activity pairs, represented by acceptance indicator $\mathbb{I}_{\theta_B} : \Theta_S \times A \to \{0, 1\}$.

Given a strategy profile $(P, \alpha, \mathbb{I})$, the expected payoff at any time t to a seller of type θ_S is

$$U_{\theta_S}(P, \alpha, \mathbb{I}) = \frac{1}{1-\delta} \int_{\Theta_B} \mathbb{I}_{\theta_B}(\theta_S, \alpha_{\theta_S}(\theta_B, P)) \pi_{\theta_S}(\theta_B, \alpha_{\theta_S}(\theta_B, P)) f_B(\theta_B) d\theta_B, \quad (1)$$

that to a buyer of type θ_S is

$$V_{\theta_B}(P, \alpha, \mathbb{I}) = \frac{\int_{\Theta_S} \mathbb{I}_{\theta_B}(\theta_S, \alpha_{\theta_S}(\theta_B, P)) \pi_{\theta_B}(\theta_S, \alpha_{\theta_S}(\theta_B, P)) f_S(\theta_S) d\theta_S}{1 - \delta \int_{\Theta_S} \mathbb{I}_{\theta_B}(\theta_S, \alpha_{\theta_S}(\theta_B, P)) f_S(\theta_S) d\theta_S}, \quad (2)$$

while the regulator's payoff is

$$W(P, \alpha, \mathbb{I}) = \frac{1}{1-\delta} \int_{\Theta_S} \int_{\Theta_B} \mathbb{I}_{\theta_B}(\theta_S, \alpha_{\theta_S}(\theta_B, P)) \pi_R(\theta_B, \theta_S, \alpha_{\theta_S}(\theta_B, P)) f_B(\theta_B) f_S(\theta_S) d\theta_B d\theta_S. \quad (3)$$

Definition 1. A strategy profile $(P, \alpha, \mathbb{I})$ is an equilibrium if and only if

1. For all $(\theta_B, \theta_S) \in \Theta_B \times \Theta_S$ and $a \in A$, buyers' acceptance strategies satisfy

$$\mathbb{I}_{\theta_B}(\theta_S, a) = 1 \text{ if and only if } \pi_{\theta_B}(\theta_S, a) \geq \delta V_{\theta_B}(P, \alpha, \mathbb{I}). \quad (4)$$

2. For all $(\theta_B, \theta_S) \in \Theta_B \times \Theta_S$ and $P \in \mathcal{P}$,

$$\alpha_{\theta_S}(\theta_B, P) \begin{cases} \in \arg\max_{\{a \in P(\theta_B, \theta_S) : \mathbb{I}_{\theta_B}(\theta_S, a) = 1\}} \pi_{\theta_S}(\theta_B, a), & \text{if } \{a \in P(\theta_B, \theta_S) : \mathbb{I}_{\theta_B}(\theta_S, a) = 1\} \neq \emptyset, \\ = \hat{\alpha}_{\theta_B}(\theta_S, P) & \text{otherwise.} \end{cases}$$
$$(5)$$

3. The regulator's choice of regulatory environment satisfies

$$P \in \arg\max_{P' \in \mathcal{P}} W(P', \alpha, \mathbb{I}).$$

In an equilibrium, buyers accept all transactions that yield them at least their expected discounted payoff from continued search, sellers choose economic activities that maximize their stage payoffs and transactions occur under a welfare-maximizing regulatory regime. Note that our definition of equilibrium implicitly imposes a selection criterion; when buyers

are indifferent, they accept trades, and when sellers are indifferent, they propose acceptable trades. Furthermore, the requirement that the seller proposes the buyer's ideal activity in matches in which the buyer refuses all transactions eliminates a trivial source of equilibrium multiplicity.

We establish the existence of an equilibrium in two steps. First, we show that under any regulatory environment P and any market thickness δ, there exists a unique *market equilibrium* of the dynamic matching game between buyers and sellers. Second, given anticipated market equilibria, we establish the existence of an optimal choice of regulatory environment.

Proposition 1. *Given any regulatory environment P and market thickness δ, a unique market equilibrium $(\alpha^\star(P, \cdot), \mathbb{I}^\star)$ exists. Furthermore, equilibrium payoffs are unique and continuous in (P, δ).*[13]

Given a regulatory environment P, the existence of a market equilibrium follows from a fixed point argument on a space of bounded and continuous functions representing buyers' payoffs. First, given any payoff function $V : \Theta_B \to \mathbb{R}$ for the buyers, we derive the associated buyers' acceptance strategies for each match (θ_B, θ_S) and any activity $a \in P(\theta_B, \theta_S)$, in a way that is consistent with (4). This step is standard. Second, we derive a candidate for the associated seller's activity strategy $\tilde{\alpha}_{\theta_S}(\theta_B, P; \delta, V)$ in a way that is consistent with (5). This step is potentially more challenging, as it could introduce (a) a source of equilibrium multiplicity, since sellers may have many optimal activity choices, as well as (b) a source of discontinuity in buyers' payoffs, mirroring discontinuities in the sellers' correspondence of optimal activity proposals. However, the assumption that sellers are long-lived, quasiconcavity of π_{θ_B} and π_{θ_S}, our selection assumptions embedded in (4) and (5) and Assumption 4 allow us to sidestep these issues by ensuring that $\tilde{\alpha}_{\theta_S}(\theta_B, P; \delta, V)$ is single-valued and continuous. Third, given $\tilde{\alpha}_\cdot(\cdot, P; \delta, V)$, we solve buyers' optimal search problem. This step uses standard dynamic programming techniques, and it yields an updated value function $\tilde{V} : \Theta_B \to \mathbb{R}$. The mapping $V \mapsto \tilde{V}$ generated above satisfies the conditions of the contraction mapping theorem, and its unique fixed point defines the unique market equilibrium payoffs associated with regulatory environment P and thickness δ.[14] That market equilibrium payoffs can be delivered as fixed points of a contraction mapping is somewhat surprising, as an essential difference from standard search results is that our fixed point mapping encompasses sellers'

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[13]The proofs of all our results are in the Appendix.

[14]Our model has a continuous set of buyer types, which rules out appealing to a general existence result (with finite type spaces) for dynamic frictional matching markets due to Manea (2011). It also makes appealing to the contraction mapping theorem particularly handy, as we avoid difficult technical issues relating to appropriate topologies in infinite-dimensional spaces.

optimal activity choices. The result is critical for the tractability of our model as well as for interpretation of our results on the comparative statics of optimal regulation. If a given regulatory environment P was consistent with multiple market equilibria, ensuring that the government's problem is well-defined would present a problem of equilibrium selection which would be both theoretically and conceptually challenging.

By Proposition 1, we can define continuous maps $V_{\theta_B}^\star : \mathcal{P} \times [0,1) \to \mathbb{R}$ and $U_{\theta_S}^\star : \mathcal{P} \times [0,1) \to \mathbb{R}$ such that $V_{\theta_B}^\star(P, \delta)$ and $U_{\theta_S}^\star(P, \delta)$ denote, respectively, the unique market equilibrium payoffs to the buyers and sellers under regulatory regime P and market thickness δ. We highlight the dependence of equilibrium payoffs on market thickness since it will play a predominant role in our main results. Similarly, we denote equilibrium policies and highlight their dependence on δ by maps $\alpha^\star : \Theta_S \times \Theta_B \times \mathcal{P} \times [0,1) \to A$ and $\mathbb{I}^\star : \Theta_B \times \Theta_S \times A \times [0,1) \to \{0,1\}$. Finally, Proposition 1 yields a map $\bar{W} : \mathcal{P} \times [0,1) \to \mathbb{R}$, where $\bar{W}(P, \delta)$ is the welfare associated to the unique equilibrium in a market with regulation P and thickness δ. That is, if $(\alpha^\star, \mathbb{I}^\star)$ is the market equilibrium corresponding to regulatory environment P, then $\bar{W}(P, \delta) = W(P, \alpha^\star, \mathbb{I}^\star)$. The solution to the regulator's problem

$$W^\star(\delta) = \max_{P \in \mathcal{P}} \bar{W}(P, \delta)$$

yields the welfare associated to the optimal regulatory environment under market thickness δ.

Proposition 2. *Given market thickness δ, an equilibrium $(P^\star, \alpha^\star, \mathbb{I}^\star)$ exists.*

Our assumptions on $\mathcal{P}$ ensure that it is a compact set. Hence, the proof of Proposition 2 reduces to showing that $\bar{W}$ is a continuous function of P, which uses continuity arguments developed in the proof of Proposition 1.[15]

4   Equilibrium Characterization and Comparative Statics

In this Section, we present our main results on the comparative statics of optimal regulation. As a first step, we characterize the properties of market equilibria for fixed regulatory environments. Given a regulatory environment P, market thickness δ, corresponding market equilibrium $(\alpha^\star, \mathbb{I}^\star)$ and buyer type θ_B, we denote the set of seller types with whom this buyer transacts by $M_{\theta_B}(P, \delta) \subseteq \Theta_S$. More precisely, $\theta_S \in M_{\theta_B}(P, \delta)$ if and only if $\mathbb{I}_{\theta_B}^\star(\theta_S, \alpha_{\theta_S}^\star(\theta_B, P, \delta), \delta) = 1$.

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[15]Note that equilibria need not be unique, since the regulator may have multiple optimal choices.

Proposition 3. *Given a regulatory environment P, market thickness δ, corresponding market equilibrium $(\alpha^\star, \mathbb{I}^\star)$ and buyer type θ_B,*

1. *$M_{\theta_B}(P, \delta)$ contains seller type $\sigma(\theta_B)$ and is given by*

$$M_{\theta_B}(P, \delta) = \{\theta_S \in \Theta_S; \pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S)) \geq \delta V_{\theta_B}^\star(P, \delta)\}.$$

    In particular, if $\theta_S \in M_{\theta_B}(P, \delta)$ and $\theta'_S \in \Theta_S$ is such that $|\theta'_S - \sigma(\theta_B)| \leq |\theta_S - \sigma(\theta_B)|$, then $\theta'_S \in M_{\theta_B}(P, \delta)$.

2. *Fix any $\theta_S \in M_{\theta_B}(P, \delta)$.*

    (a) *If $\pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_S}(\theta_B, P)) \geq \delta V_{\theta_B}^\star(P, \delta)$, then $\alpha_{\theta_S}^\star(\theta_B, P, \delta) = \hat{\alpha}_{\theta_S}(\theta_B, P)$.*

    (b) *If, on the other hand, $\pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_S}(\theta_B, P)) < \delta V_{\theta_B}^\star(P, \delta)$, then $\alpha_{\theta_S}^\star(\theta_B, P, \delta)$ is given by the unique value of $a \in [\hat{\alpha}_{\theta_B}, \hat{\alpha}_{\theta_S})$ for which $\pi_{\theta_B}(\theta_S, a) = \delta V_{\theta_B}^\star(P, \delta)$.*

3. *If $\delta' < \delta$, then $V_{\theta_B}^\star(P, \delta') \leq V_{\theta_B}^\star(P, \delta)$.*

4. *If $P' \subset P$, then $V_{\theta_B}^\star(P', \delta) \geq V_{\theta_B}^\star(P, \delta)$.*

We illustrate market equilibria in Figure 3, in which we fix a buyer of type θ_B and depict seller types θ_S on the horizontal axis and activities a on the vertical axis. The interval of activities P represents the regulatory environment. We also depict the ideal activities $\hat{\alpha}_{\theta_B}(\cdot)$ of buyer θ_B, the ideal activities $\hat{\alpha}.(\theta_B)$ of sellers when matched with this buyer, and the ideal activities $\hat{\alpha}_R(\theta_B, \cdot)$ of the regulator in matches involving buyer θ_B. These are all single-peaked around the buyer's ideal seller of type $\sigma(\theta_B)$. The dotted ring in the centre of the figure depicts the indifference curve of the buyer at level $\delta V_{\theta_B}^\star$. Indifference curves at higher levels of utility lie inside this ring, with the highest indifference curve being the single point $(\sigma(\theta_B), \hat{\alpha}_{\theta_B}(\sigma(\theta_B)))$. The equilibrium activity-seller pairs involving buyer θ_B are highlighted in bold.

In equilibrium, a buyer accepts transactions from sellers whose types are sufficiently close to its ideal seller's type (part 1). In particular, the buyer's acceptance set $M_{\theta_B}(P, \delta)$ is convex whenever Θ_S is convex. Furthermore, a buyer transacts only with those sellers who, were they to provide it with its ideal match-specific activity, would offer a payoff at least as high as its discounted continuation payoff. In Figure 3, the set $M_{\theta_B}(P, \delta)$ corresponds to those seller types θ_S such that the points $(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S))$ lie inside the area defined by the indifference curve of buyer θ_B at $\delta V_{\theta_B}^\star$ (those sellers in $[\underline{\theta}_S, \overline{\theta}_S]$). However, only sellers whose types are located

Figure 3: Market Equilibria under Regulation
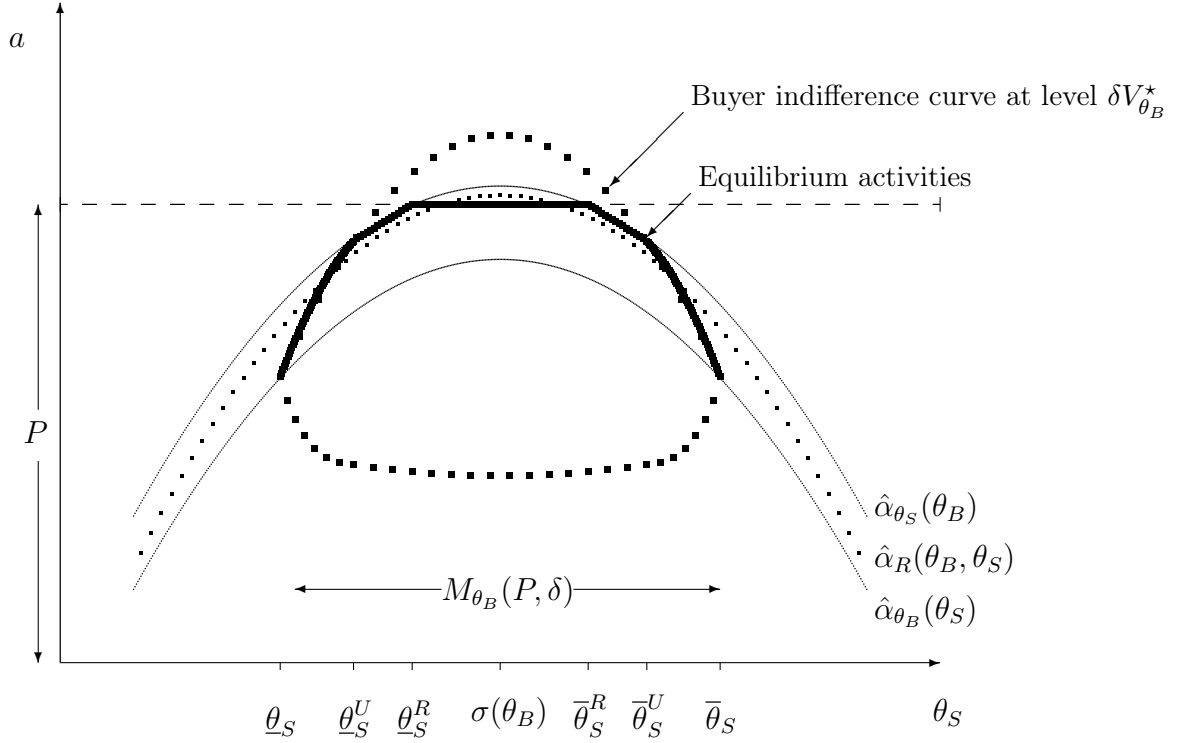
on the boundaries of the matching set of buyer θ_B will actually offer its ideal match-specific activity (in Figure 3, the two sellers $\underline{\theta}_S$ and $\overline{\theta}_S$). If close enough to the buyer's ideal type, a seller can offer to transact at its ideal activity among those permitted under regulatory environment P and still deliver to the buyer a payoff preferred to continued search (part 2. In Figure 3, these are the sellers in $[\underline{\theta}_S^U, \overline{\theta}_S^U]$). Such a seller need not be particularly well-off in such a match, since regulation may substantially hinder its choice of activity (in Figure 3, sellers in $[\underline{\theta}_S^R, \overline{\theta}_S^R]$ are constrained by the regulatory environment). A type of seller sufficiently far from the buyer's ideal seller type (in Figure 3, those sellers in $[\underline{\theta}_S, \underline{\theta}_S^U] \cup [\overline{\theta}_S^U, \overline{\theta}_S]$) provides an activity between the ideal match-specific activities of the buyer and the seller, keeping the buyer indifferent between that transaction and continued search.

Buyers are (weakly) better off in thick markets (part 3). In particular, by part 1, this implies that for $\delta' < \delta$, we have that $M_{\theta_B}(P, \delta) \subseteq M_{\theta_B}(P, \delta')$. Since the marginal transactions that a buyer is involved with in equilibrium take place at its preferred activity with the marginal seller (in Figure 3, the transactions with sellers $\underline{\theta}_S$ and $\overline{\theta}_S$), the marginal transactions of buyers in thick markets arise with sellers closer to their ideal type. This result highlights the central intuition for why deregulation may be optimal in thick markets; buyers

are less prey to mismatch in thick markets since the availability of future trading opportunities increases their willingness to turn down unfavourable matches and search for better transactions. If the distribution of activities offered by sellers was exogenous, this result would be standard and follow from the fact that any buyer's acceptance strategy in a thin market could be mimicked by the same buyer in a thick market, yielding higher payoffs through a lower cost of delay. In our setting, the equilibrium distribution of activities offered by sellers varies with market thickness as sellers adjust their offers of activities when buyers adjust their search strategies. To counter this issue and prove part 3 of Proposition 3, we exploit the fact that market equilibrium payoffs correspond to fixed points of a contraction mapping.

Finally, buyers are worse off in less regulated markets (part 4). The proof of this result is related to that of part 3, but there are some key differences in intuition. Sellers close enough to a buyer's ideal seller are typically constrained by the regulatory environment; they would choose activities they prefer if these were permitted, and the buyer would accept them because its current transactions with these sellers yield payoffs strictly above that to continued search. Hence, with deregulation, the buyer gains less from transactions with sellers close to its ideal type. While sellers far from the buyer's ideal type are typically unconstrained by regulation, they also gain from deregulation because the drop in the buyer's continuation payoffs allows them to transact at activities they prefer. Hence, deregulation tilts activities towards those favored by sellers at the expense of buyers' payoffs.

We now present our main results, which address the relationship between optimal regulation and market thickness. Given a regulatory environment P, market thickness δ, corresponding market equilibrium $(\alpha^\star, \mathbb{I}^\star)$ and buyer type θ_B, let $R_{\theta_B}(P, \delta)$ be the set of equilibrium transactions under (P, δ) involving buyer θ_B that are under-regulated, which is given by

$$R_{\theta_B}(P, \delta) = \{\theta_S \in \Theta_S; \alpha^\star_{\theta_S}(\theta_B, P, \delta) \geq \hat{\alpha}_R(\theta_B, \theta_S)\}.$$

That is, a transaction at activity a in match (θ_B, θ_S) is under-regulated if the regulator has an incentive to intervene in that match to lower the equilibrium activity.

Proposition 4. *Fix regulatory environment P' and market thicknesses δ' and δ such that $\delta' < \delta$.*

1. *Given any buyer type θ_B, $R_{\theta_B}(P', \delta) \subseteq R_{\theta_B}(P', \delta')$.*

2. *Let P' be an optimal regulatory environment under δ' and suppose that, for all θ_B, $\alpha^\star_{\sigma(\theta_B)}(\theta_B, P', \delta') < \hat{\alpha}_R(\theta_B, \sigma(\theta_B))$.*

(a) *Under match or buyer-specific regulation, if δ is sufficiently close to δ', then $W^\star(\delta) \geq W^\star(\delta')$ and there exists $P \supset P'$ such that $\bar{W}(P, \delta) \geq \bar{W}(P', \delta')$.*

(b) *There exists $\bar{\delta} > \delta'$ such that if $\delta \geq \bar{\delta}$ and P is an optimal regulatory environment under δ, then $P \supset P'$.*

For a fixed regulatory environment, the set of transactions that could gain from additional regulation shrinks as markets thicken (part 1). Regulation is primarily beneficial by constraining the activities at which poor matches transact. In thicker markets, buyers find delaying transactions less costly and sellers offer lower activities to meet buyers' improved outside options. Market thickness, with its corresponding increasing ability of buyers to wait for advantageous transactions, is a substitute for regulatory activity.

When regulatory environments are binding, the regulator is better off in thicker markets, and furthermore, while not necessarily optimal, additional regulation is always beneficial (part 2a). The inequality $\alpha^\star_{\sigma(\theta_B)}(\theta_B, P', \delta') < \hat{\alpha}_R(\theta_B, \sigma(\theta_B))$ describes a *binding regulatory environment* as one for which even transactions at ideal matches are over-regulated (the market equilibrium illustrated in Figure 3 satisfies this condition).[16] Consider a type θ_B buyer, market thickness δ' and binding regulatory environment P'. If market thickness is higher, then for a fixed regulatory environment P', the buyer is better off and only accepts transactions from sellers closer to its ideal seller. On the other hand, when the regulatory environment is binding, deregulation always strictly decreases the buyer's payoffs, as sellers close to the buyer's ideal type take advantage of newly permitted activities and the buyer accepts transactions from sellers further from its ideal seller. Hence, for market thickness δ greater than but sufficiently close to δ', the regulator can find a deregulated environment $P \supset P'$ such that $M_{\theta_B}(P, \delta) = M_{\theta_B}(P', \delta')$, that is, under which the buyer accepts to transact with the same sellers under (P, δ) and (P', δ'). Under (P, δ), it is those sellers sufficiently close to the buyers ideal seller type that can take advantage of the newly deregulated activities. For all these sellers, P' is a binding regulatory environment by assumption, and hence welfare is higher under (P, δ).[17] Marginal deregulation following small changes in thickness, through its effects on matching market equilibrium outcomes, acts as a targeted loosening of regulatory constraints only for those matches for which welfare would be improved if more activities were allowed.

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[16]This condition could be disposed of if we assumed that the regulator's and the sellers' preferences over activities coincide at ideal matches. For an example, see Section 5.

[17]This illustrates why the statement is restricted to match or buyer-specific regulation. As the argument proceeds by adjusting regulatory environments for each type of buyer separately, the set of regulatory environments must allow the same flexibility. Otherwise, the regulatory change from P' to P could have different effects on different buyer types.

We obtain the limiting result that any binding regulatory environment is suboptimal if markets are sufficiently thick (part 2b).[18] In thicker markets buyers transact mostly with sellers close to their ideal seller. If a market is sufficiently thick, then buyers will transact only with constrained sellers whose activities are constrained by regulation. Assumption 3 ensures that if the regulatory environment is binding, these transactions will occur at activities systematically lower than those that are welfare-optimal. In such markets, deregulation is a necessary feature of any optimal regulatory environment.[19]

5 Application: A Model of Over-Prescription

In this section, we present a simple and numerically tractable application that illustrates our general results. Consider a market with heterogeneous buyers and sellers whose types (θ_B and θ_S respectively) are distributed uniformly on the unit ring. We interpret buyers as patients with distinct medical needs and sellers as doctors with distinct medical training. Economic activities are chosen from the set $A = [0, 1]$. We interpret these as specialized medical procedures unique to each seller type.

Given a match (θ_B, θ_S) and an activity a, payoffs to patients and doctors are given by

$$\pi_{\theta_B}(\theta_S, a) = u_B - ad(\theta_B, \theta_S),$$

and

$$\pi_{\theta_S}(\theta_B, a) = au_S,$$

where $d(\cdot, \cdot)$ is distance on the unit ring and $1 > u_B > u_S > 0$. Patients are best off when they are appropriately matched with a doctor, e.g., a patient suffering from chest pains is best off when matched with a cardiologist. Doctors are made better off when they order the most specialized procedures available (large values of a), but patients bear this cost, which is increasing in its medical irrelevance.

For simplicity, we assume that the Pareto-efficient activity preferred by the regulator in a match (θ_B, θ_S) is the activity maximizing the sum of patients' and doctors' payoffs, and

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[18]While our previous results support the intuition that deregulation is good in thick markets, a global comparative static result need not hold. When choosing a regulatory environment, the regulator balances the payoffs of buyers and sellers, which can depend on complex ways on the model's parameters. In Section 5 we present a simple numerical example in which optimal regulation is decreasing in market thickness.

[19]Note that this result is valid for any optimal regulatory environments at thicknesses δ and δ'.

that the regulator's payoff in a match is also given by the sum of agents' payoffs. That is,

$$\pi_R(\theta_B, \theta_S, a) = \pi_{\theta_B}(\theta_S, a) + \pi_{\theta_S}(\theta_B, a).$$

In this market, the regulator can be thought of as one of the many external oversight bodies that influence the degree of medical specialization in the health care industry. For example, the National Residential Match Program is a non-profit organization that ostensibly represents both patients and doctors and influences the extent to which specialized doctors are available in geographic health care markets. They do so by first setting target levels of residents and fellows in thousands of hospitals that vary by specialty and then by allocating prospective doctors accordingly. As another example, the Center for Medicare and Medicaid Services determines the types of procedures for which they offer insurance coverage, which serves as an implicit seller specific regulatory policy.

For numerical tractability, we consider blanket regulatory environments P that take the form of an interval $[0, a_R]$, where $a_R \leq 1$. Due to the symmetry of the example, the buyer-specific regulation is identical to blanket regulation. It can be verified that this example satisfies the assumptions of our model. In any equilibrium under any regulatory environment P and thickness δ, we have that $V^\star_{\theta_B}(P, \delta) = V^\star_{\theta'_B}(P, \delta) \equiv V^\star(P, \delta)$ for any patient types θ_B and θ'_B. Doctors' optimal activity choices are given by:

$$\alpha^\star_{\theta_S}(\theta_B, P) = \begin{cases} a_R & \text{if } d(\theta_B, \theta_S) \leq \frac{u_B - \delta V^\star}{a_R}, \\ \max\left\{\frac{u_B - \delta V^\star(P, \delta)}{d(\theta_B, \theta_S)}, 0\right\} & \text{otherwise.} \end{cases}$$

A doctor of type θ_S orders the most specialized procedure that is allowed under P whenever possible (i.e., when matched sufficiently well), while it chooses the procedure that leaves patients indifferent between agreeing to the procedure (and receiving a payoff of $u_B - ad$) and waiting for a preferable doctor (and receiving a payoff of $\delta V^\star(P, \delta)$) otherwise (if such an activity exists, i.e., if the match is not too poor). Given doctors' strategies, we can derive an implicit function for $V^\star(P, \delta)$ as

$$V^\star(P, \delta) = 2 \cdot \int_0^{\frac{u_B - \delta V^\star(P, \delta)}{a_R}} u_B - a_R x \, dx + 2 \cdot \int_{\frac{u_B - \delta V^\star(P, \delta)}{a_R}}^{1/2} \delta V(P, \delta) \, dx,$$

Figure 4: Buyers' Value Functions $V^\star(P, \delta)$
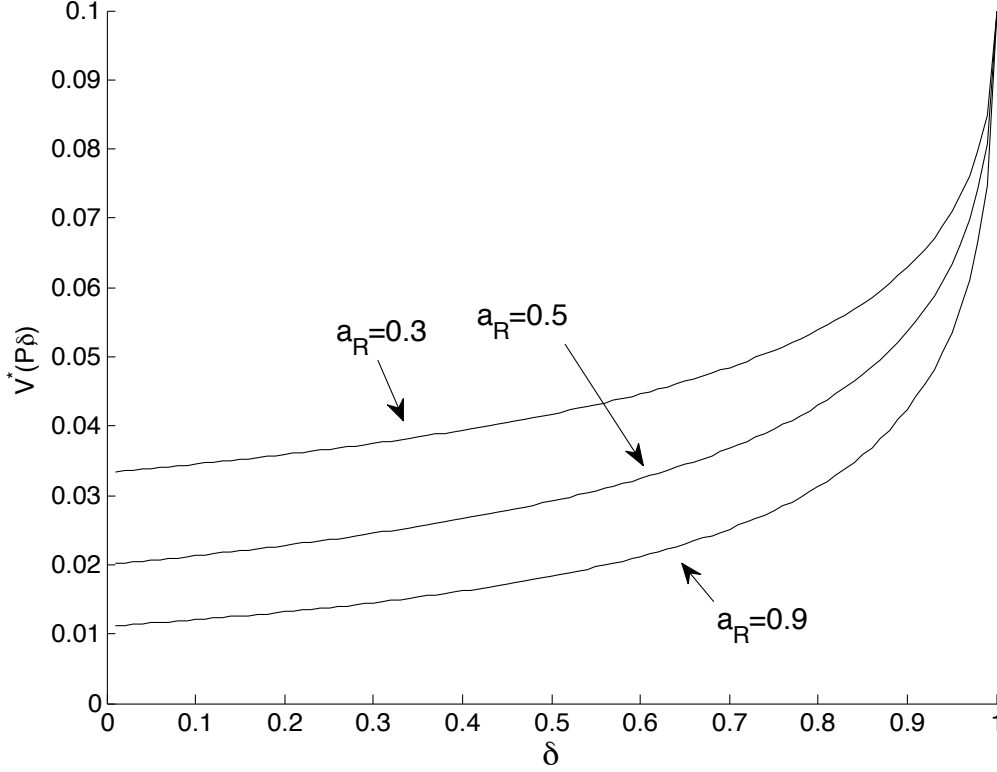
which gives us the following solution for $V^*(P, \delta)$

$$V^\star(P, \delta) = \frac{u_B}{\delta} + \frac{1}{2\delta^2} \left(a_R (1 - \delta) + (a_R^2(1 - \delta)^2 + 4u_B a_R(1 - \delta)\delta)^{1/2} \right)$$

The behavior of $V^\star(P, \delta)$ under regulation, illustrated in Figure 4, conforms to the results of Section 4. More restrictive regulations, i.e., lower a_R, improve patients' payoffs, and as markets thicken, their payoffs continue to improve as they choose better doctor matches (Proposition 3).

Under buyer-specific regulation, the regulator's optimal choice of regulation, $a_R^\star(\delta)$, satisfies

$$a_R^\star(\delta) = \underset{a_R \in [0,1]}{\arg\max} \, V^\star(P, \delta) + 2u_S(u_B - \delta V^\star(P, \delta)) \left(1 + \log \frac{a_R}{2(u_B - \delta V^\star(P, \delta))} \right)$$

Although an analytical solution for $a_R^\star(\delta)$ exists, it is algebraically complex. Instead, we numerically derive the optimal regulation as a function of market thickness, which we present in Figure 5. In thin markets, the regulator will restrict doctors from overspecializing by setting the maximum allowable economic activity $a_R \approx 0.07$. As markets thicken, the regulator

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Figure 5: Optimal Regulation $a_R^\star(\delta)$
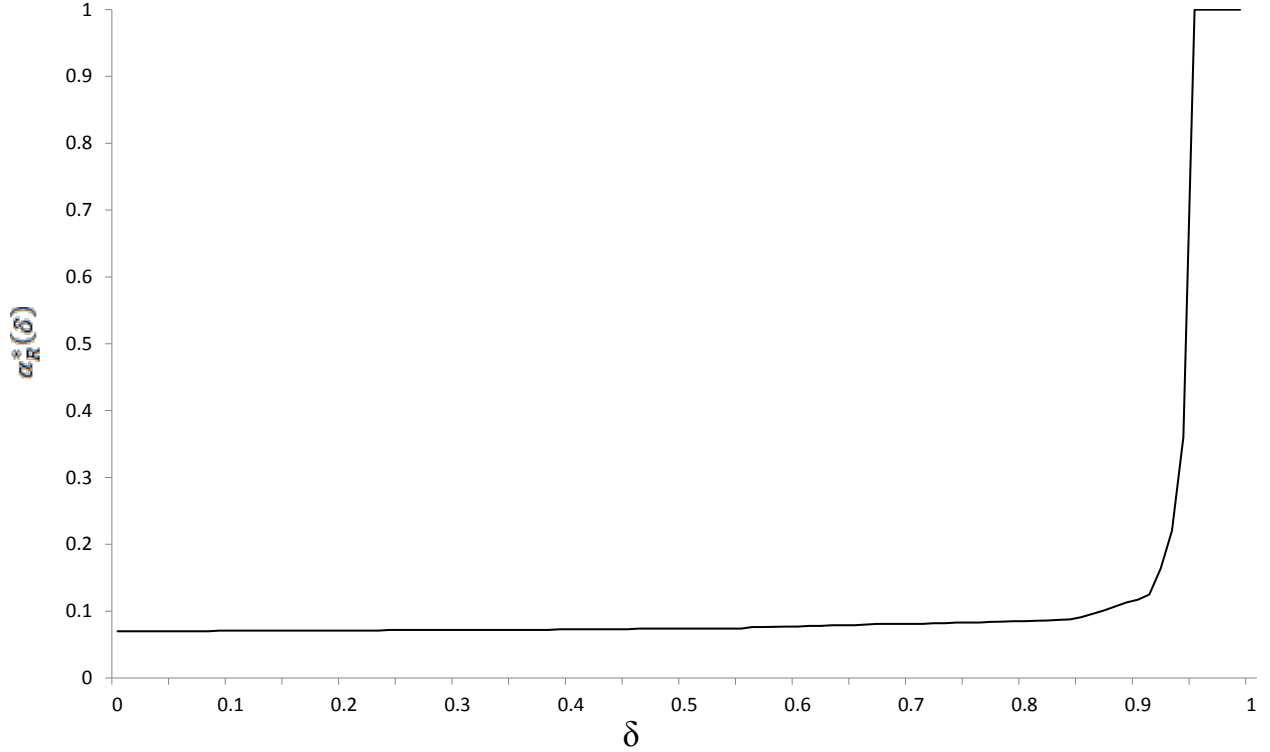
Solved numerically using $u_B = 0.1$ and $u_S = 0.25$.

will slightly increase a_R until $\delta \approx 0.95$ at which point the regulator will fully deregulate the market (Proposition 4).

6   Conclusion

This paper formalizes the idea that when buyers have specific tastes but access to few sellers (or costly access to many sellers), restrictions on the the types of economic activities that are permitted may facilitate trade and improve welfare. If, however, buyers gain inexpensive access to a variety of sellers, then regulatory constraints may reduce the surplus from trade. Our results highlight the fact that market thickness is a measure of the level of competitiveness across differentiated sellers, as thicker markets induce buyers to wait for better trades that may not have otherwise been offered to them. In competitive markets, regulatory intervention is both unwarranted, since market participants reject poor transactions, and even harmful, since it impedes beneficial transactions.

Our framework represents a novel approach to the study of regulatory (or deregulatory) policy. Accordingly, our primary goal is to develop a tractable yet sufficiently general model

that focuses on the relationship between market thickness and optimal regulation. Our results suggest many avenues for future work. Of these, we consider incorporating dynamics into regulators' policy choices as the most fruitful. In the model presented here, although the market for transactions is dynamic, the regulatory environment is fully determined before the market opens. If instead the regulatory environment could respond to evolution of the market, then several additional questions regarding market equilibria and optimal regulatory policy may arise. First, it is natural to ask how the regulatory environment itself affects market thickness. For example, if a market's thickness evolves according to the frequency and quality of transactions, then it would also depend on the dynamics of the regulatory environment. If regulation constrains market growth, then this would introduce an additional trade-off to the regulator between sustaining current markets and developing future ones. Such a trade-off could serve as the basis for a richer, dynamic theory of market development and deregulation. Second, our approach is silent on the politics surrounding regulatory intervention. Although our results suggest that deregulatory waves are called for when a market is sufficiently thick, it would be interesting to explore how the balance of political power between the winners and losers of regulation either hastens or delays such waves.

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A   Appendix

Proof of Proposition 1. We start by deriving a seller's optimal choice of activities given the buyer value function $V \in BC(\Theta_B)$. The seller's optimal value function, $\tilde{U} : \Theta_S \times \Theta_B \times \mathcal{P} \times [0,1) \times BC(\Theta_B) \to \Re$, is defined such that for all $P \in \mathcal{P}$, $\delta \in [0,1)$, $V \in BC(\Theta_B)$ and matches (θ_B, θ_S),

$$\hat{U}_{\theta_S}(\theta_B, P; \delta, V) = \max_{a \in P(\theta_B, \theta_S)} \pi_{\theta_S}(\theta_B, a) \mathbb{I}_{\pi_{\theta_B}(\theta_S, a) \geq \delta V_{\theta_B}}. \tag{6}$$

To simplify the analysis of the seller's problem in (6), we study a related problem. We artificially expand the set of activities to $A \cup \{r\}$, where r denotes an action by the seller that consists of refusing to transact with a buyer. We likewise extend the payoff functions to $A \cup \{r\}$ by specifying that for all (θ_B, θ_S), $\pi_{\theta_B}(\theta_S, r) = \pi_{\theta_S}(\theta_B, r) = 0$. Fix $P \in \mathcal{P}$, $\delta \in [0,1)$ and $V \in BC(\Theta_B)$. The optimal activity correspondence for sellers, $\check{\alpha} : \Theta_S \times \Theta_B \times \mathcal{P} \times [0,1) \times$

$BC(\Theta_B) \rightrightarrows A \cup \{r\}$, is such that, for all matches (θ_B, θ_S),

$$\check{\alpha}_{\theta_S}(\theta_B, P; \delta, V) \in \underset{a \in \{a' \in P(\theta_B, \theta_S): \pi_{\theta_B}(\theta_S, a') \geq \delta V_{\theta_B}\} \cup \{r\}}{\arg \max} \pi_{\theta_S}(\theta_B, a).$$

Note that we have assumed that either seller θ_S proposes an activity that buyer θ_B accepts or proposes refusal option r.

Define the correspondence $\varphi : \Theta_B \times \Theta_S \times \mathcal{P} \times [0, 1) \times BC(\Theta_B) \rightrightarrows A$ as $\varphi(\theta_B, \theta_S, P, \delta, V) = \{a' \in P(\theta_B, \theta_S) : \pi_{\theta_B}(\theta_S, a') \geq \delta V_{\theta_B}\} \cup \{r\}$. Although φ is upper hemicontinuous,[20] φ may fail to be lower hemicontinuous. In particular, such failures of lower hemicontinuity occur whenever the set of activities buyers accept, $\{a' \in P(\theta_B, \theta_S) : \pi_{\theta_B}(\theta_S, a') \geq \delta V_{\theta_B}\}$, is empty at $(\theta_B, \theta_S, P, \delta, V)$ but nonempty in some neighborhood of $(\theta_B, \theta_S, P, \delta, V)$. While we cannot apply the usual maximum theorem of Berge (see Aliprantis and Border (2006)) to the sellers' problem above to guarantee the continuity of their optimal value function, we can apply a generalization of the theorem due to Ausubel and Deneckere (1993). Since π_{θ_S} is continuous, applying their result reduces to proving the following lemma.

Lemma 1. *φ is upper hemicontinuous. Moreover, for any $(\theta_B, \theta_S, P, \delta, V) \in \Theta_B \times \Theta_S \times \mathcal{P} \times [0, 1) \times BC(\Theta_B)$, $a \in \varphi(\theta_B, \theta_S, P, \delta, V)$ and $\epsilon > 0$, there exists a neighborhood N of $(\theta_B, \theta_S, P, \delta, V)$ such that if $(\theta'_B, \theta'_S, P', \delta', V') \in N$, then there exists $a' \in \varphi(\theta'_B, \theta'_S, P', \delta', V')$ such that $\pi_{\theta'_S}(\theta'_B, a') > \pi_{\theta_S}(\theta_B, a) - \epsilon$.*

Proof. We first show that φ is upper hemicontinuous. Fix $\{(\theta_B^n, \theta_S^n, P^n, \delta^n, V^n)\} \to (\theta_B, \theta_S, P, \delta, V)$, along with $\{a^n\} \to a$ such that, for all n, $a^n \in \varphi(\theta_S^n, \theta_B^n, P^n, \delta^n, V^n)$. The nontrivial case is that in which there exists $n^\star$ such that $a^n \in A$ for all $n \geq n^\star$. Then, for all $n \geq n^\star$, $a^n \in P^n(\theta_B^n, \theta_S^n)$ and $\pi_{\theta_B^n}(\theta_S^n, a^n) \geq \delta^n V_{\theta_B^n}^n$. First, we need to establish that $a \in P(\theta_B, \theta_S)$. Since we can represent any correspondence P by functions ℓ and h, we have that $a^n \in [\ell^n(\theta_B^n, \theta_S^n), h^n(\theta_B^n, \theta_S^n)]$, and since $\{\ell^n(\theta_B^n, \theta_S^n)\} \to \ell(\theta_B, \theta_S)$ and $\{h^n(\theta_B^n, \theta_S^n)\} \to h(\theta_B, \theta_S)$, we have that $a \in [\ell(\theta_B, \theta_S), h(\theta_B, \theta_S)]$. Second, we need to establish that $\pi_{\theta_B}(\theta_S, a) \geq \delta V_{\theta_B}$, which follows from the continuity of π_{θ_B} and V_{θ_B}. Hence, $a \in \varphi(\theta_B, \theta_S, P, \delta, V)$ as desired.

We now show the second condition holds. Fix any $(\theta_B, \theta_S, P, \delta, V) \in \Theta_B \times \Theta_S \times \mathcal{P} \times [0, 1) \times BC(\Theta_B)$, $a \in \varphi(\theta_B, \theta_S, P, \delta, V)$ and $\epsilon > 0$. If $a \neq r$ and there exists a neighborhood N of $(\theta_B, \theta_S, P, \delta, V)$ such that, for any $(\theta'_B, \theta'_S, P', \delta', V') \in N$, there exists $a' \in \varphi(\theta'_B, \theta'_S, P', V')$ such that $a' \neq r$, then the condition holds by the continuity of ℓ, h and π_{θ_S}. Similarly if $a = r$, then since, for all $(\theta_B, \theta_S, P, \delta, V)$, $r \in \varphi(\theta_B, \theta_S, P, \delta, V)$ and $\pi_{\theta_S}(\theta_B, r) = 0$, the condition holds. The remaining case is that $a \neq r$ and that for all neighborhoods N of $(\theta_B, \theta_S, P, \delta, V)$,

---

[20]We show this below in Lemma 1.

there exists $(\theta'_B, \theta'_S, P', \delta', V') \in N$ such that $\{r\} = \varphi(\theta'_B, \theta'_S, P', \delta', V')$. We claim that it must be that $a = \alpha^\star_{\theta_B, P}(\theta_S)$ and that $\pi_{\theta_B}(\theta_S, a) = \delta V_{\theta_B}$. To see this, suppose, towards a contradiction, that $a \neq \hat{\alpha}_{\theta_B}(\theta_S, P)$ and note that since $a \neq r$ we must have $\pi_{\theta_B}(\theta_S, a) \geq \delta V_{\theta_B}$. By the strict quasiconcavity of π_{θ_B}, since $a \neq \hat{\alpha}_{\theta_B}(\theta_S, P)$, there exists $a' \in P(\theta_B, \theta_S)$ such that $\pi_{\theta_B}(\theta_S, a') > \delta V_{\theta_B}$. This implies that there exist $a'' \in A$ and a neighborhood N' of $(\theta_B, \theta_S, P, V)$ such that, for any $(\theta'_B, \theta'_S, P', \delta', V') \in N'$, $a'' \in \varphi(\theta'_B, \theta'_S, P', \delta', V')$, which yields the desired contradiction. This argument also shows that, since $a \neq r$, we must have $\pi_{\theta_B}(\theta_S, a) = \delta V_{\theta_B}$. Hence, since $\hat{\alpha}_{\theta_B}(\theta_S, P) = \hat{\alpha}_{\theta_B}(\theta_S)$ (item 2 of Assumption 4), it must be the case that $\pi_{\theta_S}(\theta_B, a) = 0$ (item 1 of Assumption 4). Since for any neighborhood N of $(\theta_B, \theta_S, P, \delta, V)$, given any $(\theta'_B, \theta'_S, P', \delta', V') \in N$ we have $r \in \varphi(\theta'_B, \theta'_S, P', \delta', V')$ and $\pi_{\theta'_S}(\theta'_B, r) = 0$, the condition from above holds and the proof is complete. $\qquad\square$

By the maximum theorem of Ausubel and Deneckere (1993), sellers' value functions

$$\check{U}_{\theta_S}(\theta_B, P; \delta, V) = \max_{a \in \{a' \in P(\theta_B, \theta_S): \pi_{\theta_B}(\theta_S, a') \geq \delta V_{\theta_B}\} \cup \{r\}} \pi_{\theta_S}(\theta_B, a) \tag{7}$$

are continuous and the correspondence $\check{\alpha}_{\theta_S}(\theta_B, P; \delta, V)$ is upper hemicontinuous. Note that by quasiconcavity, $\check{\alpha}$ is single-valued whenever $\{a' \in P(\theta_B, \theta_S) : \pi_{\theta_B}(\theta_S, a') \geq \delta V_{\theta_B}\}$ is empty or contains more than one element, and furthermore it never contains more than two elements $\{a, r\}$ with $a \in P(\theta_B, \theta_S)$.

To derive a solution to the sellers' problem in (6) through the solution $\check{\alpha}$ that conforms to equilibrium condition (5), we define $\tilde{\alpha}$ such that

$$\tilde{\alpha}_{\theta_S}(\theta_B, P; \delta, V) = \begin{cases} \hat{\alpha}_{\theta_B}(\theta_S) & \text{whenever } r \in \check{\alpha}_{\theta_S}(\theta_B, P, \delta; V) \\ \check{\alpha}_{\theta_S}(\theta_B, P; \delta, V) & \text{otherwise.} \end{cases}$$

First, note that, by definition, $\tilde{\alpha}_{\theta_S}(\theta_B, P; \delta, V)$ is single-valued everywhere. Second, note that $\tilde{\alpha}$ is continuous. This follows since $\check{\alpha}$ is upper hemicontinuous, with failures of lower hemicontinuity only when $\check{\alpha}_{\theta_S}(\theta_B, P; \delta, V) = \{\hat{\alpha}_{\theta_B}(\theta_S), r\}$. Finally, note that $\tilde{\alpha}$ achieves value $\tilde{U}$ from (6) since $\check{\alpha}$ achieves value $\check{U}$ from (7).

Given any $\tilde{\alpha}$, we compute, for each buyer type θ_B, a new value function $\tilde{V}_{\theta_B}$ through standard dynamic programming techniques. Consider the mapping $T : \mathcal{P} \times [0, 1) \times BC(\Theta_B) \to BC(\Theta_B)$ such that

$$T_{\theta_B}(P, \delta, V) = \int_{\Theta_S} \max\{\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P; \delta, V)), \delta V_{\theta_B}\} f_S(\theta_S) d\theta_S.$$

Lemma 2. *T is a continuous function.*

Proof. First, we show that T_{θ_B} is well-defined, that is, for any $(P, \delta, V) \in \mathcal{P} \times [0, 1) \times BC(\Theta_B)$, we have $T(P, \delta, V) \in BC(\Theta_B)$. $T(P, \delta, V)$ is bounded because $\pi_{\theta_B}(\theta_S, a) \leq \bar{\pi}$ and $V_{\theta_B} \leq \frac{\bar{\pi}}{1-\delta}$. Since $\tilde{\alpha}_{\theta_S}(\theta_B, P; \delta, V)$ is continuous in θ_B, then the continuity of π_{θ_B} and V ensures that, by the bounded convergence theorem, $T(P, \delta, V)$ is a continuous function of θ_B.

Second, we show that the mapping T is continuous. No issue arises with continuity in δ, so we focus on continuity in (P, V) and for convenience ignore the dependence of T and $\tilde{\alpha}$ on δ. Fix any $P, P' \in \mathcal{P}$ and $V, V' \in BC(\Theta_B)$. Then

$$
\begin{aligned}
\|T(P, V) - T(P', V')\|_\infty &= \sup_{\theta_B \in \Theta_B} \left| \int_{\Theta_S} \max\{\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P; V)), \delta V_{\theta_B}\} f_S(\theta_S) d\theta_S \right. \\
&\quad \left. - \int_{\Theta_S} \max\{\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P'; V')), \delta V'_{\theta_B}\} f_S(\theta_S) d\theta_S \right| \\
&\leq \sup_{\theta_B \in \Theta_B} \int_{\Theta_S} \left| \max\{\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P; V)), \delta V_{\theta_B}\} \right. \\
&\quad \left. - \max\{\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P'; V')), \delta V'_{\theta_B}\} \right| f_S(\theta_S) d\theta_S \\
&\leq \sup_{\theta_B \in \Theta_B} \int_{\Theta_S} \max \Big\{ |\pi_B(\theta_S, \hat{\alpha}_{\theta_S}(\theta_B, P)) - \pi_B(\theta_S, \hat{\alpha}_{\theta_S}(\theta_B, P'))|, \\
&\quad \delta |V_{\theta_B} - V'_{\theta_B}| \Big\} f_S(\theta_S) d\theta_S.
\end{aligned} \tag{8}
$$

To show the second inequality holds, we exploit the particular selection from $\check{\alpha}$ defining $\hat{\alpha}$. Fix θ_B. The inequality is certainly valid for those θ_S for which both $\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P; V)) \leq \delta V_{\theta_B}$ and $\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P'; V')) \leq \delta V'_{\theta_B}$. Note that if both $\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P; V)) > \delta V_{\theta_B}$ and $\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P'; V')) > \delta V'_{\theta_B}$, then the seller θ_S's choice is unconstrained under both (P, V) and (P', V'), and by the strict quasiconcavity of π_{θ_S}, $\tilde{\alpha}_{\theta_S}(\theta_B, P; V) = \hat{\alpha}_{\theta_S}(\theta_B, P)$ and $\tilde{\alpha}_{\theta_S}(\theta_B, P'; V') = \hat{\alpha}_{\theta_S}(\theta_B, P')$, the second inequality is valid. Now suppose that θ_S is such that $\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P; V)) > \delta V_{\theta_B}$ and $\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P'; V')) \leq \delta V'_{\theta_B}$. If $\pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_S}(\theta_B, P)) - \delta V'_{\theta_B} \leq 0$, then

$$
\left| \pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P; V)) - \delta V'_{\theta_B} \right| \leq \delta \left| V_{\theta_B} - V'_{\theta_B} \right|.
$$

as $\tilde{\alpha}_{\theta_S}(\theta_B, P; V) = \hat{\alpha}_{\theta_S}(\theta_B, P))$. Suppose instead that $\pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_S}(\theta_B, P)) - \delta V'_{\theta_B} > 0$. Note that since $\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P'; V')) \leq \delta V'_{\theta_B}$, it must be that $\pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_S}(\theta_B, P')) \leq \delta V'_{\theta_B}$, otherwise $\tilde{\alpha}_{\theta_S}(\theta_B, P'; V')$ would not be optimal for seller θ_S. Hence,

$$
\left| \pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P; V)) - \delta V'_{\theta_B} \right| \leq \left| \pi_B(\theta_S, \hat{\alpha}_{\theta_S}(\theta_B, P)) - \pi_B(\theta_S, \hat{\alpha}_{\theta_S}(\theta_B, P')) \right|,
$$

which implies that the second inequality is valid. A symmetric argument holds for the case of $\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P; V)) \leq \delta V_{\theta_B}$ and $\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P'; V')) > \delta V'_{\theta_B}$.

Since $\hat{\alpha}_{\theta_S}$ is continuous,

$$\lim_{P' \to P} \left| \pi_B(\theta_S, \hat{\alpha}_{\theta_S}(\theta_B, P)) - \pi_B(\theta_S, \hat{\alpha}_{\theta_S}(\theta_B, P')) \right| = 0.$$

Also, since $\lim_{V' \to V} ||V - V'||_\infty = 0$,

$$\lim_{V' \to V} \left| V_{\theta_B} - V'_{\theta_B} \right| = 0$$

for all θ_B. These two facts along with (8) imply

$$\lim_{(P',V') \to (P,V)} ||T(P,V) - T(P',V')||_\infty = 0,$$

which completes the proof. $\qquad\qquad\square$

Note that if, given any $(P, \delta) \in \mathcal{P} \times [0, 1)$, we consider the mapping $T_{P,\delta} : BC(\Theta_B) \to BC(\Theta_B)$ such that $T_{P,\delta}(V) = T(P, \delta, V)$, then $T_{P,\delta}$ is well-defined and, by (8)

$$\begin{aligned}
||T_{P,\delta}(V) - T_{P,\delta}(V')||_\infty &= ||T(P, \delta, V) - T(P, \delta, V')||_\infty \\
&\leq \delta \sup_{\theta_B \in \Theta_B} \int_{\Theta_S} \left| V_{\theta_B} - V'_{\theta_B} \right| f_S(\theta_S) d\theta_S \\
&= \delta ||V - V'||_\infty.
\end{aligned}$$

That is, $T_{P,\delta}$ is a contraction. Hence, given any $(P, \delta) \in \mathcal{P} \times [0, 1)$, $T_{P,\delta}$ has a unique fixed point, $\tilde{V}(P, \delta)$, which implies unique equilibrium payoffs for buyers and sellers. This in turn implies the uniqueness of equilibrium strategies, which completes the proof of the existence of a unique market equilibrium given $P \in \mathcal{P}$.

$\qquad\qquad\square$

Proof of Proposition 2. Because $\mathcal{P}$ is compact, it suffices to show that $\bar{W}$ is a continuous function of P. We have that

$$\bar{W}(P, \delta) = \int_{\Theta_B} \int_{M_{\theta_B}(P,\delta)} \pi_R(\theta_B, \theta_S, \alpha_{\theta_S}(\theta_B)) f_S(\theta_S) f_B(\theta_B) d\theta_S d\theta_B,$$

where, given the results of Proposition 3, $M_{\theta_B}(P, \delta) \subseteq \Theta_S$ is such that $M_{\theta_B}(P, \delta) = \{\theta_S \in \Theta_S; \pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S)) \geq \delta V^\star_{\theta_B}(P, \delta)\}$. Since f_S and f_B are atomless and π_R is continuous, that

$\bar{W}(P, \delta)$ is continuous in (θ_B, P) follows if, for any (θ_B, P), the set $\{\theta_S \in \Theta_S; \pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S)) = \delta V^\star_{\theta_B}(P, \delta)\}$ has zero Lebesgue measure. By quasiconcavity of π_{θ_B} in θ_S and strict quasiconcavity of π_{θ_B} in a, we have that $\pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S))$ is strictly decreasing in $|\theta_S - \sigma(\theta_B)|$.[21] Hence, if there exists $\theta'_S \in \{\theta_S \in \Theta_S; \pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S)) = \delta V^\star_{\theta_B}(P, \delta)\}$, then $\{\theta_S \in \Theta_S; \pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S)) = \delta V^\star_{\theta_B}(P, \delta)\} = \{\theta_S \in \Theta_S; |\theta_S - \sigma(\theta_B)| = |\theta'_S - \sigma(\theta_B)|\}$, which has zero Lebesgue measure. Otherwise, $\{\theta_S \in \Theta_S; \pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S)) = \delta V^\star_{\theta_B}(P, \delta)\}$ is empty and there is nothing to prove. Continuity of $\bar{W}$ then follows from the bounded convergence theorem. $\square$

Proof of Proposition 3.

1. We first show that $M_{\theta_B}(P, \delta) \neq \emptyset$. Suppose, towards a contradiction, that $M_{\theta_B}(P, \delta) = \emptyset$. Then it must be that $V^\star_{\theta_B}(P, \delta) = 0$. But since $\pi_{\theta_B}, \pi_{\theta_S} \geq 0$, our definition of equilibrium implies that $M_{\theta_B}(P, \delta) \neq \emptyset$, yielding the desired contradiction. Next, note that by Assumption 4, for any match (θ_B, θ_S), $\{a \in (\theta_B, \theta_S); \pi_{\theta_B}(\theta_S, a) \geq \delta V^\star_{\theta_B}(P, \delta)\} \neq \emptyset$ if and only if $\pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S)) \geq \delta V^\star_{\theta_B}(P, \delta)$. Our definition of equilibrium and the fact that $\pi_{\theta_S} \geq 0$ implies that $M_{\theta_B}(P, \delta) = \{\theta_S \in \Theta_S; \pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S)) \geq \delta V^\star_{\theta_B}(P, \delta)\}$. Suppose $\theta_S \in M_{\theta_B}(P, \delta)$ and that θ'_S is such that $|\theta'_S - \sigma(\theta_B)| \leq |\theta_S - \sigma(\theta_B)|$. Then

$$
\begin{aligned}
\pi_{\theta_B}(\theta'_S, \hat{\alpha}_{\theta_B}(\theta'_S)) &> \pi_{\theta_B}(\theta'_S, \hat{\alpha}_{\theta_B}(\theta_S)) \\
&\geq \pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S)) \\
&\geq \delta V^\star_{\theta_B}(P, \delta),
\end{aligned}
\tag{9}
$$

   where the first inequality follows from the definition of $\hat{\alpha}_{\theta_B}$ and the second by quasiconcavity. Hence $\theta'_S \in M_{\theta_B}(P, \delta)$, and since $M_{\theta_B}(P, \delta)$ is nonempty, this argument shows that it contains $\sigma(\theta_B)$.

2. Fix $\theta_S \in M_{\theta_B}(P, \delta)$ and note that if $\pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_S}(\theta_B, P)) \geq \delta V^\star_{\theta_B}(P, \delta)$, then by our definition of equilibrium $\mathbb{I}^\star_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_S}(\theta_B, P)) = 1$, and hence $\alpha^\star_{\theta_S}(\theta_B, P) = \hat{\alpha}_{\theta_S}(\theta_B, P)$. If instead $\pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_S}(\theta_B, P)) < \delta V^\star_{\theta_B}(P, \delta)$, then by part 1, $\pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S)) \geq \delta V^\star_{\theta_B}(P, \delta)$. Since, by Assumptions 2 and 4, $\hat{\alpha}_{\theta_S}(\theta_B) \geq \hat{\alpha}_{\theta_S}(\theta_B, P) > \hat{\alpha}_{\theta_B}(\theta_S)$, quasiconcavity and the optimality of sellers' activity choices ensures that $\alpha^\star_{\theta_S}(\theta_B, P)$ must equal the unique value of $a \in [\hat{\alpha}_{\theta_B}(\theta_S), \hat{\alpha}_{\theta_S}(\theta_B, P))$ such that $\pi_{\theta_B}(\theta_S, a) = \delta V^\star_{\theta_B}(P, \delta)$.

3. We first establish the following lemma.

   Lemma 3. *For any $V \in BC(\Theta_B)$ and any θ_B, $T(V, \delta)_{\theta_B} \geq T(V, \delta')_{\theta_B}$.*

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[21]This is shown in the inequalities in (9)

Proof. It suffices to show that

$$\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P; \delta, V)) \geq \pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P; \delta', V)) \tag{10}$$

for all θ_S. First, consider any θ_S such that $\pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S)) < \delta V_{\theta_B}$. Since $\delta V_{\theta_B} > \delta' V_{\theta_B}$,

$$\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P; \delta, V)) = \pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S))$$
$$\geq \pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P; \delta', V)).$$

Second, consider any θ_S such that $\pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S)) \geq \delta V_{\theta_B}$. Equation (10) holds because for $\delta > \delta'$

$$\tilde{\alpha}_{\theta_S}(\theta_B, P; \tilde{\delta}, V) \in \underset{\{a \in P(\theta_B, \theta_S): \pi_{\theta_B}(\theta_S, a) \geq \tilde{\delta} V_{\theta_B}\}}{\arg\max} \pi_{\theta_S}(\theta_B, a)$$

for any $\tilde{\delta} \in \{\delta, \delta'\}$. $\qquad\square$

Suppose, towards a contradiction, that for some θ_B, $V_{\theta_B}^\star(P, \delta) < V_{\theta_B}^\star(P, \delta')$. Then

$$T(V^\star(P, \delta), \delta')_{\theta_B} \leq T(V^\star(P, \delta), \delta)_{\theta_B}$$
$$= V_{\theta_B}^\star(P, \delta)$$
$$< V_{\theta_B}^\star(P, \delta'),$$

where the first inequality follows from Lemma 3. Fix $\bar{V} \in BC(\Theta)$ such that $\bar{V}_{\theta_B} - V_{\theta_B}^\star(P, \delta)$ and, for all θ_B', $|\bar{V}_{\theta_B'} - V_{\theta_B'}^\star(P, \delta')| \leq |V_{\theta_B}^\star(P, \delta) - V_{\theta_B}^\star(P, \delta')|$. Such a $\bar{V}$ exists by the continuity of $V^\star$. It follows that $\sup_{\theta_B} |\bar{V}_{\theta_B} - V_{\theta_B}^\star(P, \delta')| = |V_{\theta_B}^\star(P, \delta) - V_{\theta_B}^\star(P, \delta')|$. Furthermore, $T(\bar{V}, \delta')_{\theta_B} = T(V^\star(P, \delta), \delta')_{\theta_B}$ since sellers' best responses $\hat{\alpha}$ to buyer θ_B depend only on $V_{\theta_B}^\star$. Hence

$$||T(\bar{V}, \delta') - V^\star(P, \delta')||_\infty = \sup_{\theta_B'} |T(\bar{V}, \delta')_{\theta_B'} - V_{\theta_B'}^\star(P, \delta')|$$
$$\geq |V_{\theta_B}^\star(P, \delta') - T(V^\star(P, \delta), \delta')_{\theta_B}|$$
$$\geq |V_{\theta_B}^\star(P, \delta') - V_{\theta_B}^\star(P, \delta)|$$
$$> \delta ||\bar{V} - V^\star(P, \delta')||_\infty,$$

which contradicts the fact that T is a contraction.

4. We mimic the proof of part 3. It suffices to show that for any $V \in BC(\Theta_B)$, any θ_B and $P' \subseteq P$, $T(P', V)_{\theta_B} \geq T(P, V)_{\theta_B}$. Hence, it suffices to show that

$$\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P'; \delta, V)) \geq \pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P; \delta, V))$$

for all θ_S. First, consider any θ_S such that $\pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S)) < \delta V_{\theta_B}$. Then

$$\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P; V, \delta)) = \pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S))$$
$$= \pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P'; V, \delta)).$$

Second, consider any θ_S such that $\pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S)) \geq \delta V_{\theta_B}$. By our results from part 2, for those θ_S such that $\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P'; \delta, V)) = \delta V_{\theta_B}$, then $\tilde{\alpha}_{\theta_S}(\theta_B, P; \delta, V) = \tilde{\alpha}_{\theta_S}(\theta_B, P'; \delta, V)$. For those θ_S such that $\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P'; \delta, V)) > \delta V_{\theta_B}$, quasiconcavity yields $\pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P; \delta, V)) \leq \pi_{\theta_B}(\theta_S, \tilde{\alpha}_{\theta_S}(\theta_B, P'; \delta, V))$ since our results from part 2 imply that

$$\tilde{\alpha}_{\theta_S}(\theta_B, P; \delta, V) \geq \hat{\alpha}_{\theta_S}(\theta_B, P')$$
$$= \tilde{\alpha}_{\theta_S}(\theta_B, P'; \delta, V)$$
$$\geq \hat{\alpha}_{\theta_B}(\theta_S).$$

This completes the proof.

$\square$

Proof of Proposition 4. Fix θ_B, P' and $\delta' < \delta$.

1. Note that by part 4 of Proposition 3, $V^{\star}_{\theta_B}(P', \delta) \geq V^{\star}_{\theta_B}(P', \delta')$ and $M_{\theta_B}(P', \delta) \subseteq M_{\theta_B}(P', \delta')$. By part 2 of Proposition 3, $\alpha^{\star}_{\theta_S}(\theta_B, P, \delta') \geq \alpha^{\star}_{\theta_S}(\theta_B, P, \delta)$ for all $\theta_S \in M_{\theta_B}(P, \delta')$. Hence, if $\alpha^{\star}_{\theta_S}(\theta_B, P, \delta) \geq \hat{\alpha}_R(\theta_B, \theta_S)$, then $\alpha^{\star}_{\theta_S}(\theta_B, P, \delta') \geq \hat{\alpha}_R(\theta_B, \theta_S)$. This implies $R_{\theta_B}(P, \delta) \subseteq R_{\theta_B}(P, \delta')$.

2.

   (a) Let P' be an optimal regulatory under δ' and suppose that $\alpha^{\star}_{\sigma(\theta_B)}(\theta_B, P', \delta') < \hat{\alpha}_R(\theta_B, \sigma(\theta_B))$. Fix $P \supset P'$ such that $\delta' V^{\star}(P', \delta') = \delta V^{\star}(P, \delta)$, which ensures that $M_{\theta_B}(P', \delta') = M_{\theta_B}(P, \delta)$. For $\delta > \delta'$ sufficiently close to δ', such a P exists by parts 3 and 4 of proposition 3 since $\hat{\alpha}_{\sigma(\theta_B)}(\theta_B, P') = \alpha^{\star}_{\sigma(\theta_B)}(\theta_B, P', \delta') <$

$\hat{\alpha}_R(\theta_B, \sigma(\theta_B)) \le \hat{\alpha}_{\sigma(\theta_B)}(\theta_B)$. First, since $\delta' V^\star(P', \delta') = \delta V^\star(P, \delta)$ and $P \supset P'$, we have that $\alpha^\star_{\theta_S}(\theta_B, P, \delta) \ge \alpha^\star_{\theta_S}(\theta_B, P', \delta')$. Second, by part 2 of proposition 3, we have that $\alpha^\star_{\theta_S}(\theta_B, P, \delta) = \alpha^\star_{\theta_S}(\theta_B, P', \delta')$ for all θ_S such that $\pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_S}(\theta_B, P')) \le \delta V^\star_{\theta_B}(P, \delta) = \delta' V^\star_{\theta_B}(P', \delta')$. Third, since $P \supset P'$, $\alpha^\star_{\theta_S}(\theta_B, P, \delta) > \alpha^\star_{\theta_S}(\theta_B, P', \delta')$ for those θ_S sufficiently close to $\sigma(\theta_B)$. We can choose δ sufficiently close to δ' such that for all such θ_S, $\alpha^\star_{\theta_S}(\theta_B, P, \delta) < \hat{\alpha}_R(\theta_B, \theta_S)$, which ensures that

$$\int_{M_{\theta_B}(P,\delta)} \pi_R(\theta_S, \alpha^\star_{\theta_S}(\theta_B, P, \delta)) f_S(\theta_S) d\theta_S$$
$$> \int_{M_{\theta_B}(P',\delta')} \pi_R(\theta_S, \alpha^\star_{\theta_S}(\theta_B, P', \delta')) f_S(\theta_S) d\theta_S.$$

Since under either match or buyer-specific regulation a different regulatory environment P can be found in the same way as above for each θ_B, we have that $W^\star(\delta) \ge W^\star(\delta')$.

(b) Let P' be an optimal regulation under δ'. We show that if δ is sufficiently large, $R_{\theta_B}(P', \delta) = \emptyset$ and hence, for all $\theta_S \in M_{\theta_B}(P', \delta)$, $\alpha^\star_{\theta_S}(\theta_B, P', \delta) \le \hat{\alpha}_R(\theta_B, \theta_S)$. This implies that if P is an optimal regulation under δ, then $P \supset P'$. First, we show that

$$\lim_{\delta \to 1} M_{\theta_B}(P', \delta) \subseteq \{\theta_S \in \Theta_S; \pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S)) \ge \pi_{\theta_B}(\sigma(\theta_B), \hat{\alpha}_{\sigma(\theta_B)}(\theta_B, P'))\}.$$

To do so, we establish that $\lim_{\delta \to 1} V^\star_{\theta_B}(P', \delta) \ge \pi_{\theta_B}(\sigma(\theta_B), \hat{\alpha}_{\theta_S}(\theta_B, P'))$. Towards a contradiction, suppose that instead

$$\lim_{\delta \to 1} \delta V^\star_{\theta_B}(P', \delta) = \lim_{\delta \to 1} V^\star_{\theta_B}(P', \delta)$$
$$< \pi_{\theta_B}(\sigma(\theta_B), \hat{\alpha}_{\theta_S}(\theta_B, P')). \tag{11}$$

Then it follows by (11) and part 2 of Proposition 3 that for all δ sufficiently close to 1, $\alpha^\star_{\theta_S}(\theta_B, P', \delta) = \hat{\alpha}_{\theta_S}(\theta_B, P')$ for some neighborhood $N(\delta)$ of $\sigma(\theta_B)$, with $\lim_{\delta \to 1} N(\delta)$ containing a positive mass of seller types. Consider the buyer's strategy of accepting transactions only with those sellers $\theta_S \in N(\delta)$ such that $\pi_{\theta_B}(\sigma(\theta_B), \hat{\alpha}_{\theta_S}(\theta_B, P')) > \lim_{\delta \to 1} V^\star_{\theta_B}(P', \delta)$. For sufficiently large δ, the payoff to this strategy exceeds $\lim_{\delta \to 1} V^\star_{\theta_B}(P', \delta)$, yielding the desired contradiction. Hence,

since

$$\lim_{\delta \to 1} \delta V_{\theta_B}^{\star}(P', \delta) = \lim_{\delta \to 1} V_{\theta_B}^{\star}(P', \delta)$$

$$\geq \pi_{\theta_B}(\sigma(\theta_B), \hat{\alpha}_{\theta_S}(\theta_B, P')),$$

part 1 of Proposition 3 implies that for any $\theta_S \in \lim_{\delta \to 1} M_{\theta_B}(P', \delta)$,

$$\pi_{\theta_B}(\theta_S, \hat{\alpha}_{\theta_B}(\theta_S)) \geq \lim_{\delta \to 1} \delta V_{\theta_B}^{\star}(P', \delta)$$

$$\geq \pi_{\theta_B}(\sigma(\theta_B), \hat{\alpha}_{\sigma(\theta_B)}(\theta_B, P')), \tag{12}$$

as desired.

By equation (12), we have that $\lim_{\delta \to 1} \alpha_{\theta_S}^{\star}(\theta_B, P', \delta) \leq \overline{\alpha}_{\theta_S}(\theta_B, P')$, where $\overline{\alpha}_{\theta_S}(\theta_B, P')$ is defined such that $\pi_{\theta_B}(\theta_S, \overline{\alpha}_{\theta_S}(\theta_B, P')) = \pi_{\theta_B}(\sigma(\theta_B), \hat{\alpha}_{\sigma(\theta_B)}(\theta_B, P'))$. By Assumption 3, we have that $\pi_{\theta_B}(\theta_S, \hat{\alpha}_R(\theta_B, \theta_S))$ is decreasing in $|\theta_B - \sigma(\theta_B)|$. On the other hand, we have that $\pi_{\theta_B}(\theta_S, \overline{\alpha}_{\theta_S}(\theta_B, P'))$ is constant in $|\theta_B - \sigma(\theta_B)|$. Hence, since $\alpha_{\sigma(\theta_B)}^{\star}(\theta_B, P', \delta') < \hat{\alpha}_R(\theta_B, \sigma(\theta_B))$, we have that, for all $\theta_S \in \lim_{\delta \to 1} M_{\theta_B}(P', \delta)$,

$$\lim_{\delta \to 1} \alpha_{\theta_S}^{\star}(\theta_B, P', \delta) \leq \overline{\alpha}_{\theta_S}(\theta_B, P')$$

$$< \hat{\alpha}_R(\theta_B, \theta_S),$$

which shows that $\lim_{\delta \to} R_{\theta_B}(P', \delta) = \emptyset$. Since our argument holds for all θ_B, our result is valid under any system of regulation.

$\square$