The Properties of Double-Blind Dutch Auctions in a Clearing House; Some New Results for the Mendelson Model*

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Abstract: In this paper, we re-examine Mendelson’s model for the equilibrium price of a double-blind Dutch auction with Poisson-distributed stochastic demand and supply. We present a number of new results. We focus on the various ways that demand and supply cross. We identify four different categories of crossing, extending Mendelson’s results which are based on a single category of crossing. Secondly, conditioning on quantity, we derive the joint distribution of the relevant demand and supply prices associated with such two-sided markets originally described by Bohm-Bawerk (1891). The distributional result is extended to the case where the limit orders on different sides of the market arrive at different rates. Finally, we derive the distributional properties of the price elasticities.

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1. **Introduction**

This paper provides a detailed analysis of an important micro-structure model based on the analysis of a double-blind Dutch auction. The (single) Dutch Auction is different from a standard auction in that the auction starts with the highest selling price for the asset, and then successively lowers it until a bidder accepts the price. MacAfee and McMillan (1987) attest that “The Dutch Auction is used, for example, for selling cut flowers in the Netherlands, fish in Israel and tobacco in Canada.” Double-blind Dutch auctions are where buying and selling prices ascend or descend and where prices are constructed without prior knowledge of the other prices.

Whilst this structure is not frequently used in financial markets, it has been used as a structure for modelling market fragmentation and consolidation, see Mendelson (1987). Furthermore, it has wider relevance, the notion used here of market-clearing equilibrium is much more in line with how economists think about markets; these two-sided market models have a central role in a great deal of market theory. We can but quote Shapley and Shubik (1971); “Two-sided market models are important, as COURNOT, EDGEWORTH, BOHM-BAWERK and others have observed, not only for the insights they may give into more general economic situations with many types of traders, consumers, and producers, but also for the simple reason that in real life many markets and most actual transactions are in fact bilateral--i.e. bring together a buyer and a seller of a single commodity.”

If we add the additional assumption that the orders on both sides of the market are limit orders, then we have a structure which is very similar to the opening and closing auctions on many of the world’s stock exchanges. A few examples, among many, are the ASX, the HKE and the NYSE. The biggest difference between these auctions and the ones we are going to discuss is that we assume agents are “blind”, in that they have no information about the plans or actions of others whilst in reality agents can observe some information about other orders. A second important difference is order size.

Our starting point is that both demand and supply are generated by buy and sell limit orders for one unit of commodity; the numbers of which arriving in time up to \( T \) are generated by independent Poisson processes with a common parameter \( \lambda \). The framework follows Mendelson (1982). The impact of his assumptions is that these processes become homogeneous spatial Poisson processes (see Cressie, 1993) in price space where the analogue of the inter-arrival times is the non-negative change in price for supply and minus the non-positive price decrease for demand. Equilibrium occurs when the two paths cross. Mendelson solves for the volume (quantity) and the prices in equilibrium.
There are a number of reasons why we revisit his particular model; firstly the original paper provides an accomplished and interesting analysis in its own right. However, it is rather terse and difficult to follow in parts; by extending and spelling out in detail some of the technical arguments, we hope to make it more accessible. Secondly, market fragmentation has become an important topic in recent times and, as mentioned above, one of the few fragmentation models, also by Mendelson (1987), builds directly on this structure. Thirdly, there is an expanding literature on modelling trading processes, which is also based on Poisson processes. This includes research on limit order markets, for example, see Parlour(1998), Goettler et al (2007), Cont et al(2012) inter alia. This aims to capture the nature of contemporary trading venues, whereby traders submit a flow of orders, often cancelling some of them immediately. Trading is determined by the stock of live orders, the book, plus the arrival of further orders. Here the emphasis is on price discovery and strategic equilibrium, rather than market-clearing price equilibrium, as in Mendelson. However, there are numerous similarities of approach, which we shall comment on throughout our paper. Fourthly, there is an emerging area of financial econometrics based on work by Easley et.al.(2008), which uses Poisson trade arrivals in the modelling of the probability of information-based trading(PIN).

In what follows, we assume, as does Mendelson, that we start with a bid (the highest), then we follow with a sell (the lowest) and so on, working our way along the respective curves, until the paths cross. In price space, we have what is called an alternating renewal process; these processes are widely used in reliability analysis. Where we differ is that we look at different ways in which the curves cross. These different crossings lead to different marginal pairs of prices which bound the equilibrium price and in turn these different marginal pairs lead to different clearing – price ranges. In Section 2 we analyse the different crossings. In Section 3, we present results on the joint distribution of the equilibrium prices in the two cases of equal or different arrival rates. Section 4 looks at the distribution of the price elasticities; Section 5 presents some numerical results, and conclusions are presented in Section 6.

2. Demand, Supply and Market Clearing

For those unfamiliar with the Mendelson model we briefly outline it now. Orders, both buy and sell arrive over time, each for one unit of commodity. These orders are accumulated during time and the market cleared at time T. Each order is qualified by a limit price, upper price for a buy and lower price for a sell. The limit prices are independent samples from a uniform distribution on (0, m]. Both demand and supply are generated by the buy and sell orders and over time are generated by independent Poisson processes with a common intensity \( \alpha m \). As a consequence of
these assumptions the demand and supply processes, in price-space are independent spatial Poisson processes (see Cressie(1993)). In particular, the Poisson processes have common intensity $\lambda = \alpha T$.

We consider Poisson realisations of demand and supply; our interest is the range of possible prices where the curves cross. Neither ourselves, nor Mendelson discuss the case where limit bids are to the left of limit asks but the curves do not cross; this warrants further analysis. Assuming that the number of limit orders is $k$, which will be the quantity traded, we can envisage four prices, corresponding to moving up the supply curve from left to right and up the demand curve from right to left. The four prices are: $P_{sk}$, $P_{dk}$, $P_{sk+1}$ and $P_{dk+1}$ satisfying the following inequalities: $P_{sk} < P_{sk+1}$ and $P_{dk} > P_{dk+1}$. In Figure 1(below), we show the four possible orderings of these prices subject to the above constraints.

The situation considered by Mendelson is Figure 1; case (a), see Mendelson (1982, Figure 1, page 1508). However, the other three cases occur with finite probability and have equal economic meaning.

![Fig. 1 case (a)](image1)

![Fig. 1 case (b)](image2)
Using the properties of the Poisson process, we consider the probabilities that randomly-generated demand and supply will have market-clearing of these four kinds. We present these later but first we need the joint pdf of the four prices.

Before proceeding to the joint pdf we note that apart from the fact that Mendelson only considered one crossing there are other shortcomings with his analysis. Firstly, his analysis of the upper and lower limits on the market-clearing price is only relevant to crossing (a) and is not a general result. Secondly, given that prices are restricted to lie in $[0, m]$, he does not explicitly consider the issue as to whether the distribution of prices should be censored or truncated.

Dealing with the first of these issues we note that Mendelson uses the following two relationships to analyse the lower limit, $L$, and the upper limit, $U$, on the market-clearing price:

1. $L > x$ if and only if $S(x) = k$ and $D(m-x) = j$ for $j > k$
2. $U < x$ if and only if $S(x) = j$ and $D(m-x) = k$ for $j > k$

Where $S(x)$ signifies the quantity for the supply curve at $x$ while $D(m-x)$ is the quantity for demand at price $m-x$. The following Lemma proves that his analysis is only relevant to crossing (a).

**Lemma**

The only equilibrium consistent with conditions (i) and (ii) is (a). Equilibrium (b) is consistent with neither, equilibrium (c) is consistent with (ii) but not (i) and the reverse holds for equilibrium (d).

**Proof**

From an inspection of the above diagrams it is clear that, in all four cases, $L = P_{sk}$ and $U = P_{dk}$. Suppose now that in diagram (b) $x$ lies between $P_{sk}$ and $P_{dk+1}$, we then have that $S(x) = k$ and
D(m-x) = k+1 which is consistent with (i) even though x > L. Similar arguments apply to case (c) and (d) and with the other limit U. Hence we have a contradiction and proof.

We now address the second issue dealing with the treatment of prices which are restricted to lie in (0, m) but which are not restricted if we assume inter-arrival times to be negative exponential. Mendelson (1982, footnote 9) alludes to this issue but does not discuss it. It is worth thinking about what this might mean in terms of auctioneer behaviour. Restricting offers in excess of m seems intuitively reasonable as they add nothing. A similar thing could be said for bids that are negative. Precisely how the auctioneer treats these will have statistical implications. If she discards them, we are, in effect, truncating the price distributions. If she scales them back to be m for offers and puts the bids at zero, we will have censoring. Mendelson does not address this point explicitly but we will show, by an examination of one of his results, that at least some of the time, he assumes censoring. This is mentioned in the discussion below figure 5, page 1516, where he refers to the probability mass of the execution price. We shall first provide, in Proposition 1, a calculation for a quantity he refers to as the expected value of the market-clearing price range, E(Δ). In the notation used above this is E(U – L), below we will refer to this as E(Y-y). Proposition 1 shows that assumptions about censoring or truncation are not innocuous and highlights the impact of different assumptions on the expected market-clearing price range.

**Proposition 1**

The expected market-clearing price range under censoring is

$$E_c(\Delta) = (1 - \exp(-2\lambda m)) / 2\lambda$$

The expected market-clearing price range under truncation is

$$E_T(\Delta) = (1 - \exp(-2\lambda m) - 2\lambda m \exp(-\lambda m)) / 2\lambda(1 - \exp(-\lambda m)).$$

Furthermore, $$E_c(\Delta) - E_T(\Delta)$$ is always non-negative.

**Proof:** Under censoring we note that the result is given by Mendelson, see his theorem 4.1.

The impact of censoring is that $$P(L \leq 0) = P(m < U) = \exp(-\lambda m).$$

It therefore follows that

$$E_T(U) = (E_c(U) - m \exp(-\lambda m)) / (1 - \exp(-\lambda m))$$

$$E_T(L) = E_c(L) / (1 - \exp(-\lambda m))$$
\[ E_T(U - L) = (E_c(U) - E_c(L) - m \exp(-\lambda m)) / (1 - \exp(-\lambda m)) \]

The formula follows from substituting in the censoring result and simplifying. The inequality can be easily established by noting that the two terms are equal when \( m = 0 \) but the censoring term has a larger derivative in \( m \) for all positive \( m \).

Further analysis of \( \Delta \) and \( U \) and \( L \) individually can be readily undertaken once we have the marginal distributions of \( U \) and \( L \). These can be found from the joint distribution of the four prices of interest, \( P_{sk}, P_{dk}, P_{sk+1} \) and \( P_{dk+1} \). As we have seen \( L = P_{sk} \) and \( U = P_{dk} \) and both prices will depend upon the quantity traded, \( Q \), both are clearing prices, as are any of the prices between them. The point to bear in mind is that these are limit orders, thus the buyer will buy at \( P_{dh} \) or any price cheaper whilst the seller will sell at \( P_{sk} \) or any price higher. The exact calculations for the clearing price range distribution along with those for \( U \) and \( L \) and further comparisons with Mendelson’s results are very lengthy and we shall leave them for a companion paper. However, in the next section we derive the joint pdf of the four prices of interest in the two cases of equal or different arrival rates. We also derive expressions for the probability of occurrence of each crossing and extend Mendelson’s results for \( U \) and \( L \) to allow for different arrival rates.

3. Joint Distributions of the Four Relevant Prices

Let \( x = P_{sk+1}, X = P_{dk+1}, y = P_{sk}, Y = P_{dk} \). We wish to calculate the joint pdf of the four variables, conditional that the volume traded is \( k \). We call this \( \text{pdf}(x, X, y, Y|k) \); this can be thought of as a truncated density (differing from Mendelson) as any bids in excess of \( m \) or asks less than 0 are ignored and our pdf is re-scaled to integrate to 1.

We now consider the joint pdf of \( X, x, Y \) and \( y \), conditional upon \( Q=k \). We do not consider which of the four cases it might be.

**Proposition 2**

The truncated joint density \( \text{pdf}(x, X, y, Y|k) \) is given by the following, subject to the necessary inequality \( X \leq x, y \leq Y \) corresponding to moving along the curves plus the constraint imposed by \( Q = k, Y > X, y < x \).

\[
\text{pdf}(x, X, y, Y|k) = \lambda^2 c^2 y^{k-1} (m - Y)^{k-1} e^{-\lambda y} e^{-\lambda(m-Y)}
\]

where \( c = \lambda^k / \gamma(k, \lambda m) \) and \( \gamma(k, \lambda m) \) is the incomplete gamma function.
Proof: See Appendix.

We could, of course, provide a censored joint pdf as well. This turns out to be very messy as can seen by Proposition 3.

**Proposition 3**

The censored joint density pdf\((x, X, y, Y|k)\) is given by the following, subject to the necessary inequality \(X \leq x, y \leq Y\) corresponding to moving along the curves plus the constraint imposed by \(Q = k, Y > X, y < x\). Under censoring we have that

\[
\begin{align*}
pdf(x, y|k) &= \frac{\lambda^{k+1}}{\gamma(k, \lambda m)} y^{k-1} e^{-\lambda x}, \quad 0<y<x<m \\
&= \frac{\lambda^k}{\gamma(k, \lambda m)} y^{k-1} e^{-\lambda m}, \quad x = m, 0<y<m
\end{align*}
\]

and

\[
\begin{align*}
pdf(X, Y|k) &= \frac{\lambda^{k+1}}{\gamma(k, \lambda m)} (m - Y)^{k-1} e^{-\lambda (m - X)}, \quad 0<X<Y<m \\
&= \frac{\lambda^k}{\gamma(k, \lambda m)} (m - Y)^{k-1} e^{-\lambda m}, \quad X = 0, 0<Y<m
\end{align*}
\]

Therefore, the joint pdf\((x, X, y, Y|k)\) is given by the following four expressions:

\[
\begin{align*}
pdf(x, X, y, Y|k) &= \lambda^2 c^2 y^{k-1} (m - Y)^{k-1} e^{-\lambda x} e^{-\lambda (m - X)}, \quad 0<y<x<m, 0<X<Y<m \\
&= \lambda c^2 e^{-\lambda m} y^{k-1} (m - Y)^{k-1} e^{-\lambda x}, \quad 0<y<x<m, X = 0, 0<Y<m \\
&= \lambda c^2 e^{-\lambda m} y^{k-1} (m - Y)^{k-1} e^{-\lambda (m - X)}, \quad x = m, 0<y<m, 0<X<Y<m \\
&= c^2 e^{-\lambda m} y^{k-1} (m - Y)^{k-1}, \quad x = m, 0<y<m, X = 0, 0<Y<m
\end{align*}
\]

where \(c = \frac{\lambda^k}{\gamma(k, \lambda m)}\).

Proof: See Appendix.
Remark

It is worth noting that our quadrivariate censored pdf in Proposition 3 has four components. When we consider truncated pdfs in four dimensions, our results simplify, see Proposition 2, and we have only one component. It seems clear that the censored pdf is much more complex than the truncated pdf, with no obvious gains in realism; this suggests to us, at least, that we should work, where appropriate, with truncated distributions. In fact, it seems that Mendelson sometimes does not impose the censoring, for example when discussing the distribution of quantities traded. Rather than explore this even further, we shall make the same assumptions for quantities traded as he does.

Having computed the truncated density in Proposition 2, it is now possible to calculate the probabilities of the different crossings; these are given in Proposition 4.

**Proposition 4.**

The probabilities of state, \(a, b, c,\) and \(d\), all conditional upon quantity \(k\), are given by the following expressions \(P(a|k), P(b|k), P(c|k), P(d|k)\)

\[
P(a \mid k) = e^{-\lambda m} \left[ \sum_{j=0}^{\infty} \frac{(\lambda m)^j}{j!(2k+1)^j} (F_1(k, 2k + 1 + j; \lambda m)) - 2_1F_1(k, 2k + 1; \lambda m) + 1 \right] \left[ 1 + e^{-\lambda m} + \frac{\lambda m}{2k + 1} - 2_1F_1(k, 2k + 1; -\lambda m) \right]
\]

\[
P(b \mid k) = \frac{(\lambda m)^2 e^{-\lambda m}}{(2k + 1)^2} \left[ \sum_{j=0}^{\infty} \frac{(\lambda m)^j (k + 1)^j}{j! (2k + 3)^j} F_1(k + 1, 2k + 3 + j; \lambda m) \right] \left[ 1 + e^{-\lambda m} + \frac{\lambda m}{2k + 1} - 2_1F_1(k, 2k + 1; -\lambda m) \right]
\]

\[
P(c \mid k) = \frac{(\lambda m)e^{-\lambda m}}{(2k + 1)} \left[ \sum_{j=0}^{\infty} \frac{(\lambda m)^j (k + 1)^j}{j! (2k + 2)^j} F_1(k, 2k + 2 + j; \lambda m) - 2_1F_1(k + 1, 2k + 2; \lambda m) \right] \left[ 1 + e^{-\lambda m} + \frac{\lambda m}{2k + 1} - 2_1F_1(k, 2k + 1; -\lambda m) \right]
\]

\[
= P(d|k)
\]
Proof: See Appendix. The \( _1 F_1 \) function above is the confluent hypergeometric function, see Slater (1960).

We recall the expression for \( P(Q=k) \) derived by Mendelson (1982, equation 3.3, p1511). As already noted, he makes no censoring or truncating assumptions for this calculation. We shall follow his procedure without attempting to adjust it.

\[
P(Q = k) = \exp(-\lambda m) \frac{(\lambda m)^k}{\Gamma(2k + 1)} \left(1 + \frac{\lambda m}{2k + 1}\right)
\]

(1)

For \( k>0 \),

\[P(Q=k)=P(Q=k, \text{state}=a)+P(Q=k, \text{state}=b)+P(Q=k, \text{state}=c)+P(Q=k, \text{state}=d).\]

Furthermore, we can employ Bayes’s rule to infer

\[P(\text{state}=a|Q=k)=P(Q=k, \text{state}=a)/P(Q=k),\]

with corresponding results for the other states.

We now discuss extensions of our results when the arrival rates of buy and sell limit orders differ; we include this case because of its added realism. We derive the joint pdf under truncation in Proposition 5.

**Proposition 5.**

Extending the result stated in Proposition 2 to the case where demand and supply have different rates of arrival given by \( \lambda_d \) and \( \lambda_s \) respectively, we have the joint truncated conditional \( pdf(x, X, y, Y|k) \) given by

\[
\text{pdf}(x, X, y, Y) \propto \frac{(\lambda_d \lambda_s)^{k+1}}{\gamma(k, \lambda_d m) \gamma(k, \lambda_s m)} y^{k-1} e^{-\lambda_s y} (m - Y)^{k-1} e^{-\lambda_d (m - x)}
\]

for \( 0 < y < x < m \) and \( 0 < X < Y < m \)

Proof: See Appendix.

To help make comparisons with Mendelson’s results, we present marginal pdfs for the case of different arrival rates. Results for his case are based on arguments presented on page 1521.
**Proposition 6.**

The marginal distributions of L and U (under censoring) which bound the market-clearing price in case (a), for unequal arrival rates are given by

\[ f_L(L = x) = e^{-\lambda_m} \] when \( x = 0 \)

\[ = e^{-\lambda_m} \left[ \lambda_d \sum_{k=0}^{\infty} \frac{(m \lambda_d)^k}{\Gamma(2k+2)} e^{(\lambda_d - \lambda_m)x} \cdot \beta(k+1,k+1; x / m) \right. \]

\[ \left. + \lambda_s \sum_{k=0}^{\infty} \frac{(m \lambda_s)^k (m \lambda_d)^{k+1}}{\Gamma(2k+3)} e^{(\lambda_d - \lambda_s)x} \cdot \beta(k+1;k+2; x / m) \right], \text{ for } 0 < x \leq m \]

\[ f_U(U = x) = e^{-\lambda_m} \left[ \lambda_s \sum_{k=0}^{\infty} \frac{(m \lambda_s)^k (m \lambda_d)^k}{\Gamma(2k+2)} e^{(\lambda_d - \lambda_s)x} \cdot \beta(k+1,k+1; x / m) \right. \]

\[ \left. + \lambda_d \sum_{k=0}^{\infty} \frac{(m \lambda_d)^{k+1}}{\Gamma(2k+3)} e^{(\lambda_d - \lambda_s)x} \cdot \beta(k+2,k+1; (x / m)) \right], \text{ for } 0 < x \leq m \]

\[ = e^{-\lambda_m}, \text{ when } x = m \]

where \( \beta(p,q;w) = \frac{1}{B(p,q)} w^{p-1}(1-w)^{q-1} \), the Beta pdf with \( B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \)

**Proof:** See Appendix.

Once we have these two sets of results, we can, in principle, compare the two pdf’s for U and L in the four crossing case versus the (a) crossing case. It could well be possible to provide some analytic results here; we could easily present some numerical results for our various densities. However the issue of censoring versus truncation makes comparisons more complicated.

**4. The Elasticities of Supply and Demand**

In this section, we provide some new results on the elasticity of supply and demand. Since we are computing elasticities at arbitrary points on the demand and supply curves and we do not consider issues of truncation; we use the symbol \( i \) for an arbitrary quantity (not necessarily traded). Here, we take the quantity, \( i \), as given and the price \( p_i \) as random; the (inverse)
elasticity of supply, $\varepsilon_s$ (with a corresponding definition for the inverse elasticity of demand, $\varepsilon_d$) will be defined as:

$$\varepsilon_s = \frac{\Delta p_i}{p_{i-1}} \frac{1}{(i-1)}$$

Noting the independence of the price increments whose common distribution is negative exponential with parameter $\lambda_s$, which we denote by $NE_i(\lambda_s)$; for the $i$th price change, we see that the distribution of the elasticity of supply is that of the following ratio;

$$\frac{(i-1)NE_i(\lambda_s)}{\sum_{j=1}^{i-1} NE_j(\lambda_s)}$$

From the independence of the elements in the series, we see that the numerator and denominator are independent.

The moment-generating function of the numerator (divided by $(i-1)$) is given by

$$m_n(t) = \frac{1}{(1-\lambda_s t)^{(i-1)}}$$

The moment-generating function of the denominator is given by

$$m_d(t) = \frac{1}{(1-\lambda_s t)^{-1}}$$

On inspection, we see that the distribution of the numerator (divided by $i-1$) is $\frac{\lambda_s}{2}$ times a chi-squared with 2 degrees of freedom. Likewise, the distribution of the denominator is $\frac{\lambda_s}{2}$ times a chi-squared with $2(i-1)$ degrees of freedom. Putting these together, we see that $\varepsilon_s$ is distributed as an F-distribution, $F(2, 2(i-1))$. Note in particular the distribution does not depend on $\lambda_s$, a surprising result but it is conditional upon $i$. The unconditional result could be calculated but it seems to have little economic content, being a value of elasticity averaged over quantity. We state the result as Proposition 7.
Proposition 7.

Denoting by $i$ an arbitrary quantity on the supply curve, the inverse elasticity of supply, $\varepsilon_s$, is distributed as $F(2i, 2(i-1))$ whilst the elasticity of supply is $F(2(i-1), 2i)$.

Proof: See above for the inverse elasticity. The distribution for the elasticity is easily found since the inverse of $F(n, m)$ is distributed as $F(m, n)$.

Using results for the inverse moments of the chi-squared distribution, we see

that $E(\varepsilon_s)=1+\frac{1}{(i-2)}$, so that inverse elasticity of supply will on average be above 1 whilst inverse elasticity of demand will, on average, be below-1. Furthermore, $\text{Var}(\varepsilon_s)$ will be $O\left(\frac{1}{i}\right)$. Thus as we move up the supply curve and back up the demand curve our inverse elasticities converge in mean square (for large $i$), and hence probability, to 1 and -1 respectively.

5. Numerical Results

To assess the contribution of the different crossings, we examine crossing probabilities using numerical values from the original paper. Firstly, we present some illustrative results for the parameter $\lambda = .1$ and $m = 20$, and small values of $k$ from 1 to 5, see Table 1. We note in passing that

$$E(Q) = \frac{\lambda m}{2} - \frac{(1 - \exp(-2\lambda m))}{4}$$

and thus, for this example, $E(Q)=0.754$.

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<th>4</th>
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<td>$P(Q=k</td>
<td>a)$</td>
<td>0.6030067156</td>
<td>0.3365814773</td>
<td>0.05569860961</td>
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</tr>
<tr>
<td>$P(Q=k</td>
<td>b)$</td>
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<td>0.00944194220</td>
<td>0.000434657865</td>
</tr>
<tr>
<td>$P(Q=k</td>
<td>c)$</td>
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<td>0.2155159849</td>
<td>0.02401154626</td>
<td>0.001461167516</td>
</tr>
</tbody>
</table>
We can discern from Table 1 that for $k = 1$, $P(Q = k|a)$ is not the highest conditional but this changes as $k$ increases. This can be further illustrated by using Bayes rule as discussed above and graphing all four functions in terms of $k$, i.e., $P(\text{state}|k)$ see Figure 2, below.

**Figure 2(a)**

**Figure 2(b)**
As $k$ tends to infinity, the probability of state $a$, conditional on $k$ increases whilst the other three states all decrease; thus we would expect Mendelson’s model to be highly accurate for large $k$, even in our general case of four states. However, large $k$, especially relative to small $\lambda m$, is a low probability event. The most pertinent calculation is that of the unconditional probabilities of the four states which can be easily calculated from Proposition 2 and equation 1. For the parameters, $\lambda = .1$ and $m = 20$, we find that $P(a) = 0.122$, $P(b) = 0.183$, $P(c) = P(d) = 0.144$, whilst $P(Q = 0) = .406$. Thus, for this example, state($a$) is the least likely.

To gauge whether the extra three cases matter when there is a reasonable volume of data, we consider the second example considered by Mendelson, where $\lambda = 1$ and $m = 20$, here $P(Q=0)$ is approximately 0, and all four states are approximately equally likely. This equality can be verified numerically and further calculations, not reported here, provide evidence that the four states all have probabilities of 0.25 for large $\lambda$ and/or $m$.

6. Conclusions

We have demonstrated in this paper that Mendelson’s model of double-blind Dutch auctions with Poisson arrivals can be analysed in terms of four different ways in which demand and supply curves cross. We have shown that the probabilities of the three “new” states are non-negligible. Indeed, for the example when $\lambda = .1$ and $m = 20$ we might expect, on average, 2 buy
and 2 sell orders per day (unit time). For this case, state (a) is the event least likely relative to the other three states or the probability of no trade. For the second case considered by Mendelson, where \( \lambda = 1 \) and \( m = 20 \), \( P(Q=0) \) is approximately 0, and all four states are approximately equally likely.

Furthermore analysis will depend upon the assumptions we make as to how the auctioneer treats extremely large limit order offers and extremely small limit order bids. We argue for a truncation approach rather than a censoring one favoured by Mendelson. This leads to considerable simplifications in the joint distribution of prices. We also demonstrate numerically that as volume traded increases the Mendelson state, conditional upon volume, becomes the most likely but the probability of high volume in the context of Poisson demand and supply is likely to be very small. These findings will have an impact on market-clearing prices which we leave for a subsequent paper. Finally we derive the distribution of supply price elasticity and show that, conditional on quantity, it follows an \( F \) distribution.
Bibliography


Appendix

Proof of Proposition 2

Since \( x = P_{sk+1}, y = P_{sk}, X = P_{dk+1}, Y = P_{dk} \) and these prices are the occurrence time of our respective spatial Poisson processes we have:

\[
x = y + s_{k+1} \text{ and } X = Y - d_{k+1}
\]

with \( s_{k+1} \) and \( d_{k+1} \) iid \( \text{EX}(\lambda) \). Therefore

\[
\text{pdf}(x|y) \propto \lambda \exp(-\lambda(x-y)), \quad y < x < m
\]

and

\[
\text{pdf}(y) = \frac{\lambda^k}{\gamma(k, \lambda m)} y^{k-1} e^{-\lambda y}, \quad 0 < y < m
\]

where

\[
\gamma(n, z) = \int_{0}^{z} w^{n-1} e^{-w} dw,
\]

the incomplete Gamma function. Consequently,

\[
\text{pdf}(x, y) = \text{pdf}(x|y) \text{pdf}(y) = \frac{\lambda^{k+1}}{\gamma(k, \lambda m)} y^{k-1} e^{-\lambda y}, \quad 0 < y < x < m
\]

For demand we have

\[
Y = P_{dk} = m - \sum d_j
\]

resulting in

\[
X = P_{dk+1} = Y - d_{k+1}.
\]

Consequently,

\[
\text{pdf}(X|Y) \propto \lambda \exp(-\lambda(Y - X))
\]

and
\[ \text{pdf}(Y) = \frac{\lambda^k}{\gamma(k, \lambda m)} (m - Y)^{k-1} \exp(-\lambda(m - Y)), \quad 0 < Y < m \]

Thus

\[ \text{pdf}(X, Y) \propto \frac{\lambda^{k+1}}{\gamma(k, \lambda m)} (m - Y)^{k-1} \exp(-\lambda(m - X)) \]

and since demand and supply are independent we have

\[ \text{pdf}(x, X, y, Y) \propto \frac{\lambda^{2(k+1)}}{\gamma^2(k, \lambda m)} y^{k-1} e^{-\lambda y} (m - Y)^{k-1} \exp(-\lambda(m - X)), \quad 0 < y < x < m, 0 < X < Y < m \]

**Proof of Proposition 3**

Proceeding as in the proof of Proposition 2 we have

\[ x = y + s_{k+1} \]

giving

\[ \text{pdf}(x|y) = \lambda \exp(-\lambda(x - y)), \quad y < x < m \]

and

\[ \text{pdf}(x = m|y) = \int_m^\infty \lambda \exp(-\lambda(x - y)) dx = \exp(-\lambda(m - y)), \quad 0 < y < m \]

Thus

\[ \text{pdf}(x|y) = \lambda \exp(-\lambda(x - y)), \quad 0 < y < x < m \]

\[ = \exp(-\lambda(m - y)), \quad x = m, 0 < y < m \]

and since

\[ \text{pdf}(y|y < m) = \frac{\lambda^k}{\gamma(k, m\lambda)} y^{k-1} e^{-\lambda y}, \quad 0 < y < m \]

we have
\[ \text{pdf}(x, y) = \text{pdf}(x|y) \text{pdf}(y) \]
\[ = \frac{\lambda^{k+1}}{\gamma(k, m\lambda)} y^{k-1} e^{-\lambda y}, \quad 0 < y < m \]
\[ = \frac{\lambda^k}{\gamma(k, m\lambda)} y^{k-1} e^{-\lambda m}, \quad x = m, 0 < y < m \]

Similarly, since
\[ X = Y - d_{k+1} \]
we have
\[ \text{pdf}(X|Y) = \lambda \exp(-\lambda(Y - X)), \quad 0 < X < Y < m \]
\[ \text{pdf}(X = 0|Y) = \int_Y^\infty \lambda \exp(-\lambda w) dw = \exp(-\lambda Y) \]

Thus
\[ \text{pdf}(X|Y) = \lambda \exp(-\lambda(Y - X)), \quad 0 < X < Y < m \]
\[ = \exp(-\lambda Y), \quad X = 0, 0 < Y < m \]

and since
\[ \text{pdf}(Y, Y < m) = \frac{\lambda^k}{\gamma(k, m\lambda)} (m - y)^{k-1} e^{-\lambda(m - y)} \]
we have
\[ \text{pdf}(X, Y) = \frac{\lambda^{k+1}}{\gamma(k, m\lambda)} (m - y)^{k-1} \exp(-\lambda(m - X)), \quad 0 < X < Y < m \]
\[ = \frac{\lambda^k}{\gamma(k, m\lambda)} e^{-\lambda m}(m - Y)^{k-1}, \quad X = 0, 0 < Y < m \]

Finally
\[ \text{pdf}(x, X, y, Y) = \text{pdf}(x,y) \cdot \text{pdf}(X,Y) \]
giving the four expressions stated in the theorems.
Proof of Proposition 4

The various probabilities are readily found by straightforward but tedious integration of the joint \( pdf(x, X, y, Y) \) as specified in Proposition 2 and subject to the particular ordering of \( X, X, y, Y \) for the respective crossing. Thus

\[
P(a|k) = \int_0^m \int_0^y \int_0^x pdf(x, X, y, Y) dX dy dY dx
\]

\[
P(b|k) = \int_0^m \int_y^x \int_0^x pdf(x, X, y, Y) dY dX dx
\]

\[
P(c|k) = \int_0^m \int_0^x \int_y^x pdf(x, X, y, Y) dy dX dx
\]

\[
P(d|k) = \int_0^m \int_y^x \int_0^x pdf(x, X, y, Y) dy dX dx
\]

Proof of Proposition 5

We proceed as in the proof of Proposition 2 except we now allow \( s_{k+1} \) and \( d_{k+1} \) to be distributed, respectively, as \( EX(\lambda_s) \) and \( EX(\lambda_d) \). Thus

\[
pdf(x|y) \propto \lambda_se^{-\lambda_s(x-y)}, \quad y<x<m
\]

with

\[
pdf(y) = \frac{\lambda_s^k}{\gamma(k, \lambda_s, m)} y^{k-1} e^{-\lambda_s y}, \quad 0<y<m
\]

Thus

\[
pdf(x, y) \propto \frac{\lambda_s^{k+1}}{\gamma(k, \lambda_s, m)} y^{k-1} e^{-\lambda_s x}, \quad 0<y<x<m
\]

and

\[
pdf(X, Y) \propto \frac{\lambda_d^{k+1}}{\gamma(k, \lambda_d, m)} (m-y)^{k-1} \exp(-\lambda_d (m-X)), \quad 0<X<Y<m
\]

giving
\[
pdf(x, X, y, Y) \propto \frac{(\lambda_x \lambda_d)^{k+1}}{\gamma(k, \lambda_d m) \gamma(k, \lambda_x m)} y^{k-1} e^{-\lambda_x y} (m - Y)^{k-1} e^{-\lambda_d (m - X)}
\]

\[0 < y < x < m \text{ and } 0 < X < Y < m\]

**Proof of Proposition 6**

From Figure 1 in Mendelson (1982) it is clear that

\[L > x \iff S(x) = k \quad \text{and } D(m - x) = j, \ j \geq k + 1\]

Consequently

\[
P(L > x) = \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} P(S(x) = k, D(m - x) = j)
\]

\[
= \sum_{k=0}^{\infty} P(S(x) = k) \sum_{j=k+1}^{\infty} P(D(m - x) = j)
\]

\[
= e^{-\lambda_d m + (\lambda_d - \lambda_x) x} \sum_{k=0}^{\infty} \frac{(\lambda_x x)^k}{k!} \sum_{j=k+1}^{\infty} \frac{(\lambda_d (m - x))^j}{j!}
\]

Letting

\[a = \lambda_x x\]

\[b = \lambda_d m - (\lambda_d - \lambda_x) x\]

with \((b - a) = \lambda_d (m - x)\)

we have

\[
P(L > x) = e^{-b} \sum_{k=0}^{\infty} \frac{a^k}{k!} \sum_{j=k+1}^{\infty} \frac{(b-a)^j}{j!}
\]

Now

\[
\sum_{j=k+1}^{\infty} \frac{(b-a)^j}{j!} = e^{b-a} - \sum_{j=0}^{k} \frac{(b-a)^j}{j!}
\]
Thus

\[ P(L > x) = 1 - e^{-b} \sum_{k=0}^{\infty} a^k \sum_{j=0}^{k} \frac{(b-a)^j}{j!} \]

Giving

\[ P(L \leq x) = e^{-b} \sum_{k=0}^{\infty} a^k \sum_{j=0}^{k} \frac{(b-a)^j}{j!} \]

And we notice immediately that

\[ P(L \leq 0) = e^{-\lambda x} \]

and \( P(L \leq m) = 1 \)

and further, since \( 0 \leq L \leq m \) we have a point mass at \( x = 0 \) of \( e^{-\lambda x} \).

The pdf, \( f_l(x) \) is readily found via differentiation

i.e.,

\[ f_l(x) = \frac{\partial}{\partial x} P(L \leq x) \]

and since

\[ \frac{\partial a}{\partial x} = \lambda_x, \quad \frac{\partial b}{\partial x} = - (\lambda_d - \lambda_x) \quad \text{and} \quad \frac{\partial (b-a)}{\partial x} = - \lambda_d \]

We have, after some simplification

\[ f_l(x) = \lambda_d e^{-b} \sum_{k=0}^{\infty} \frac{a^k (b-a)^k}{k! k!} + \lambda_x e^{-b} \sum_{k=0}^{\infty} \frac{a^k (b-a)^{k+1}}{k! (k+1)!} \]

Recalling the definitions of \( a \) and \( b \) we note

\[ \frac{a^k (b-a)^k}{\Gamma(k+1) \Gamma(k+1)} = \frac{\lambda_x^k \lambda_d^k m^{2k}}{\Gamma(2k+2)} \cdot \beta(k+1,k+1;x/m) \]
where \( \beta(p, q; z) = \frac{1}{B(p, q)} \cdot z^{p-1}(1-z)^{q-1}, \) the Beta pdf

with \( B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \)

Using a similar approach for the second sum we get the expression for \( f_l(x) \) given in the theorem.

For the distribution of \( U \) we now note that

\[ U \leq x \Leftrightarrow S(x) = j \quad \text{and} \quad D(m-x) = k, \ j \geq k + 1 \]

Thus

\[ P(U < x) = \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} P(S(x) = j, D(m-x) = k) \]

and with the same definitions of \( a \) and \( b \) we have

\[ P(U < x) = 1 - e^{-\lambda} \sum_{k=0}^{\infty} \frac{(b-a)^k}{k!} \sum_{j=0}^{\infty} \frac{a^j}{j!} \]

giving

\[ P(U < 0) = 0 \]

and \( P(U < m) = 1 - e^{-\lambda m} \).

Further, since \( P(U \leq m) = 1 \) we have a point mass at \( x = m \) of \( e^{-\lambda m} \).

Again we have

\[ f_u(x) = \frac{\partial}{\partial x} P(U < x) \]

\[ = \lambda e^{-b} \sum_{k=0}^{\infty} \frac{(b-a)^k}{k!k!} a^k + \lambda_b e^{-b} \sum_{k=0}^{\infty} \frac{(b-a)^k a^{k+1}}{k!(k+1)!} \]

which upon substitution for \( a \) and \( b \) we get the expression for \( f_u(x) \) given in the theorem.